UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

College of Informatics and Electronics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4613

SEMESTER: Autumn 2003/04

MODULE TITLE: Vector Analysis

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Prof. S. O'Brien

PERCENTAGE OF TOTAL MARKS: 90%

EXTERNAL EXAMINER: Prof. J.D. Gibbon

INSTRUCTIONS TO CANDIDATES: **Answer question 1 and any 4 other questions. Use a separate answer book for question 1.**

Vectors are written in bold print and are expressed in cartesian coordinates unless otherwise noted. Number each question carefully *in the margin provided on your script* **. The subscript notation is used intermittently to denote a partial derivative (e.g.** $u_x \equiv \frac{\partial u}{\partial x}$**). N.B.There are some useful results on the last page.**

1 This question is **obligatory**. Answer any **10** of the following. Each is worth 4 %. 40%

(a) Give a precise technical definition of a vector including the vector transformation law.

(b) Prove that $\delta_{ij} a_i b_j = a_i b_i$ (summation convention).

(c) Define the dot (scalar)product of 2 vectors in two different ways. Prove that two non-zero vectors are orthogonal if and only if $\mathbf{a}.\mathbf{b} = 0$. Are $(1, 2, 3)$ and $(-1, -2, -3)$ orthogonal vectors?

(d) Define the triple scalar product $\mathbf{a}.(\mathbf{b} \times \mathbf{c})$ of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. **Prove** that its magnitude corresponds to the volume of the parallelepiped defined by the vectors. What is the volume of the parallelepiped defined by the vectors with components $(1, 0, 0), (1, 2, 1), (2, 3, 3)$?

(e) Given that three non-zero vectors are not coplanar (so that $\mathbf{a}.(\mathbf{b} \times \mathbf{c}) \neq 0$), show that any other vector $\mathbf{d}$ can be expressed uniquely in the form $\mathbf{d} = \lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}$ where λ, μ, ν are constants.

(f) Define what is meant by the terms smooth, piecewise smooth. Sketch the curve with parametric definition $\mathbf{r}(t) = (t, |t|, 0),) - 1 \leq t \leq 1$. Find a tangent vector at each point and **prove** that the curve is piecewise smooth.

(g) A rigid body is rotating is rotating at constant rate Ω about an axis fixed in the body so that the position vector of any point in the body is given by:

$$\mathbf{r}(t) = R\cos(\Omega t)\mathbf{i} + R\sin(\Omega t)\mathbf{j} + c\mathbf{k}$$

where R, c are known constants. Show that the velocity of any point in the body can be written $\mathbf{v}(t) = \Omega\mathbf{k} \times \mathbf{r}$ where the axis of rotation is taken to be in the direction of the unit vector $\mathbf{k}$.

(h) Let $f(x, y, z) = xy\cos z$ and $\mathbf{v}(x, y, z) = (xyz, x + y + z, yz)$ be a scalar and a vector field respectively. Compute ∇f, div $\mathbf{v}$, $\nabla \times \mathbf{v}$, $\nabla^2 f$.

(i) Prove that the function of two variables $f(x, y) = x^2 + xy + 3x + 2y + 5$ has a saddle point at $(x, y) = (-2, 1)$.

(j) Use Taylor's theorem to find a first order approximation for $f(1.1, 3.7)$ based on quantities estimated at the point $(1, 3.5)$ if $f(x, y) = x^3 y + \sin x$.

(k) Find the inverse matrix, eigenvalues and eigenvectors of the matrix:

$$\begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}$$

(l) Draw level curves (contours) for the following functions of two variables $\Omega(x,y) = 2x - y, \Omega(x,y) = x^2 + y^2$. Demonstrate that $\Omega_{xy} = \Omega_{yx}$ for the second of these functions.

(m) Find the directional derivative of $f(x,y,z) = x^4yz$ at the point $(1,2,3)$ in the direction of $\mathbf{i} + \mathbf{j}$.

(n) Show that the repeated integral $\int_0^1 \int_{y=0}^{y=1-x} (x+y)^2 \, dydx$ evaluates to $1/4$.

(o) Define what is meant by the **circulation** of a vector field. Give a precise statement of Stokes' theorem (relating a particular line and surface integral).

2 (a) Define what is meant by the direction cosines of a line through the origin. If l, m, n and l', m', n' are the direction cosines of two lines though the origin, use the cosine rule to prove that the angle θ between the two lines must satisfy:

$$\cos\theta = ll' + mm' + nn'.$$

Prove that two lines through the origin are perpendicular if and only if $ll' + mm' + nn' = 0$. 6 %

(b) Define what is meant by the orthogonal projection of one line onto another. If OA has direction cosines l, m, n and P has coordinates (x,y,z) then prove that the orthogonal projection of OP on OA is $lx + my + nz$.

Deduce the transformation law for the change in the coordinates of a point under rotation of axes in the form $\mathbf{x}' = \mathbf{L}\mathbf{x}$ or $x'_i = l_{ij}x_j$.

Consider a rotation whereby the $x_3(z)$ axis is held fixed and the x_1 and x_2 axes are rotated though 90^0 in an anticlockwise direction when viewed from above the x_1x_2 plane. If a point has coordinates $(1,0,1)$ before rotation, what are its coordinates in the rotated system? 6.5 %

3 (a) Suppose that $\mathbf{r}(t) = (x(t), y(t), z(t))$ is the parametric definition of a curve in space. Find a parametric representation $\mathbf{r}(\theta) = (x(\theta), y(\theta), z(\theta))$ for the circle in the xy plane ($x^2 + y^2 = 1, z = 0$). Hence find an expression for the arclength along this line measured from $(1,0,0)$. Find a tangent vector to this curve at any point. Using the expression for the arclength, write down the intrinsic equation of this curve and find its curvature. 7.5 %

(b) The position vector of a particle moving in a circle of radius R is given by $\mathbf{r}(t) = R\cos(\omega t)\mathbf{i} + R\sin(\omega t)\mathbf{j}$ where ω is a known constant and t

is the time. Find the velocity and acceleration of the particle and show that the acceleration is towards the centre of the circle. 5 %

4 (a) If $\Omega(x, y, z)$ is a scalar field, define what is meant by $\nabla\Omega$ and the level surfaces of Ω. Prove that the vector $\nabla\Omega$ is perpendicular to the level surfaces of Ω. 6 %

(b) If $F(x, y, z) = x^2y^2z$ and $x = t, y = t^2, z = 2t$, use the chain rule to compute $\frac{dF}{dt}$. Check your answer by writing F explicitly as a function of t and differentiating. 6.5 %

5 (a) Give a limiting definition of the area integral $\int\int_R f(x, y)dA$ over the two dimensional region R. Show how such an integral can be used to evaluate area of the region of R. Write down a double integral which represents the area of the circle $x^2+y^2 = a^2, z = 0$. By transformation to polar coordinates or otherwise evaluate this integral.

(b) If $\mathbf{f} = (3x^2+6y)\mathbf{i}-(14yz)\mathbf{j}+(20xz^2)\mathbf{k}$ find the work done in moving a particle in a force field given by $\mathbf{f}$ along a curve C given by $y = x^2, z = x$ from $(0, 0, 0)$ to $(1, 1, 1)$. (Work done = $\int \mathbf{f}.d\mathbf{r}$). 12.5 %

6 Do **either** (a) **or** (b)

(a) Calculate the surface area of the cylindrical surface defined parametrically by $\mathbf{r}(u, v) = (\cos u, \sin u, v), 0 \le u \le 2\pi, 0 \le v \le 2$. 12.5 %

(b) For a double integral, prove the change of variables rule $dxdy = |J|dudv$ where the Jacobian determinant $|J| = |\frac{\partial(x,y)}{\partial(u,v)}|$ in the usual notation.

7 Do any **2** of the following:

(a) Define what is meant by the flux of a vector field $\mathbf{u}$ through a surface S. Give a precise statement of the divergence theorem. Use the divergence theorem to write the surface integral

$$\int\int_S \mathbf{u}.d\mathbf{S}$$

as a volume integral if S is the sphere $x^2 + y^2 + z^2 = a^2$, and $\mathbf{u} = (x^3, y^3, z^3)$. Show how the volume integral can be written in terms of spherical coordinates using the Jacobian determinant (see last page) and evaluate this integral.

(b) Define what is meant by a conservative vector field and prove that every irrotational vector is conservative.

Show that the vector field $\mathbf{u} = (y^2 + z\exp(xz), 2xy, x\exp(xz))$ is irrotational and find a corresponding scalar potential $\Omega(x, y, z)$ such that $\mathbf{u} = \nabla\Omega(x, y, z)$.

(c) Write down the tensor transformation law for a rank 2 tensor (in matrix or index notation). Two sets of axes $Oxyz$ and $Ox'y'z'$ are such that the first set may be placed in the position of the second set by a rotation of 180^0 about the z axis (i.e. the z axis is held fixed during this rotation). Write down the corresponding matrix of direction cosines and show how the tensor

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

transforms under this rotation of axes.

(d) Give a physical interpretation of the components of the stress tensor T_{ij} (or $\mathbf{T}$) and the stress vector for a continuously deforming three dimensional medium and write down the relationship between the stress tensor and the stress vector $\mathbf{v}$ in terms of the stress exerted on a small surface element with unit normal $\hat{\mathbf{n}}$. 12.5 %

Useful results

Cosine rule: $\cos C = \frac{a^2+b^2-c^2}{2ab}$
Plane polar coordinates: $x = r\cos\theta,\ y = r\sin\theta$.
Jacobian determinant for polars: $J = r$.
Cylindrical polars: $x = r\cos\theta,\ y = r\sin\theta,\ z = z$.
Spherical polars: $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta;\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi$.
Jacobian determinant for sphericals: $r^2\sin\theta$.
Scalar differential surface element: $dS = |\mathbf{r}_u \times \mathbf{r}_v|\,dudv$ if $\mathbf{r}(u,v)$ defines the surface parametrically.
Vector differential surface element: $d\mathbf{S} = \mathbf{r}_u \times \mathbf{r}_v\,dudv$ if $\mathbf{r}(u,v)$ defines the surface parametrically.
Indefinite integral: $\int \sin^3 u du = -\cos u + 1/3\cos^3 u$.
Arclength for the curve $\mathbf{r}(t) = (x(t), y(t), z(t))$ is $s(t) = \int_{t_0}^{t} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2} dt$.
Taylor series: $f(x+h, y+k) = f(x,y) + hf_x(x,y) + kf_y(x,y) + O(h^2+k^2)$.