



UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

College of Informatics and Electronics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4613

SEMESTER: Autumn 2002/03

MODULE TITLE: Vector Analysis

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Prof. S. O'Brien

PERCENTAGE OF TOTAL MARKS: 90%

EXTERNAL EXAMINER: Prof. J.D. Gibbon

INSTRUCTIONS TO CANDIDATES: Answer question 1 and any 4 other questions. Use a separate answer book for question 1.

Vectors are written in bold print and are expressed in cartesian coordinates unless otherwise noted. Number each question carefully *in the margin provided on your script*. The subscript notation is used intermittently to denote a partial derivative (e.g. $u_x \equiv \frac{\partial u}{\partial x}$).
N.B. There are some useful results on the last page.

1 This question is **obligatory**. Answer any **10** of the following. Each is worth 4 %.

40%

- (a) Give a precise technical definition of a vector including the vector transformation law.
- (b) Define what is meant by the direction cosines l, m, n of a line through the origin. Find the direction cosines of the line joining the origin to the point $(1, 1, 1)$.
- (c) Prove that $\delta_{ij}a_i b_j = a_i b_i$.
- (d) Define the vector (cross) product of 2 vectors in two different ways. Prove that two non-zero vectors are parallel if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.
- (e) Define the triple scalar product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. What is the volume of the parallelepiped defined by the vectors with components $(1, 1, 2), (1, 2, 1), (2, 3, 3)$? Explain your answer.
- (f) Given that three non-zero vectors are not coplanar (so that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \neq 0$), show that any other vector \mathbf{d} can be expressed uniquely in the form $\mathbf{d} = \lambda \mathbf{a} + \mu \mathbf{b} + \nu \mathbf{c}$ where λ, μ, ν are constants.
- (g) Sketch the curve with parametric definition $\mathbf{r}(t) = (t, t^2, 0), -1 \leq t \leq 1$. Find a tangent vector at each point and **prove** that the curve is smooth.
- (h) Let $f(x, y, z) = xy + z$ and $\mathbf{v}(x, y, z) = (x + y + z, yz, xyz)$ be a scalar and a vector field respectively. Compute $\nabla f, \operatorname{div} \mathbf{v}, \nabla \times \mathbf{v}, \nabla^2 f$.
- (i) Prove that the function of two variables $f(x, y) = x^2 + y^2 - 1$ has a minimum at $(x, y) = (0, 0)$.
- (j) Find the inverse matrix, eigenvalues and eigenvectors of the matrix:

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

- (k) Draw level curves (contours) for the following functions of two variables $\Omega(x, y) = x + y, \Omega(x, y) = x^2 + y^2$.
- (l) Find the directional derivative of $f(x, y, z) = x^2 y z$ at the point $(1, 2, 3)$ in the direction of $\mathbf{i} + \mathbf{j}$.
- (m) Show that the repeated integral $\int_0^2 \int_{y=x/2}^{y=1/2} xy \, dy \, dx$ evaluates to $-1/4$.
- (n) Define what is meant by the **flux** of a vector field through a surface S . Give a precise statement of the divergence theorem (Gauss's theorem) relating a particular surface and volume integral.

- (o) Write down the tensor transformation law for a rank 2 tensor (in matrix or index notation). Two sets of axes $Oxyz$ and $Ox'y'z'$ are such that the first set may be placed in the position of the second set by a rotation of 180° about the z axis (i.e. the z axis is held fixed during this rotation). Write down the corresponding matrix of direction cosines and show how the tensor

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

transforms under this rotation of axes.

- 2 (a) If l, m, n and l', m', n' are the direction cosines of two lines through the origin, use the cosine rule to prove that the angle θ between the two lines must satisfy:

$$\cos \theta = ll' + mm' + nn'.$$

Find the angle between the lines OA and OB where $A = (1, 2, 1)$, $B = (1, 0, 3)$.

6 %

- (b) Define what is meant by the orthogonal projection of one line onto another. If OA has direction cosines l, m, n and P has coordinates (x, y, z) then prove that the orthogonal projection of OP on OA is $lx + my + nz$.

Deduce the transformation law for the change in the coordinates of a point under rotation of axes in the form $\mathbf{x}' = \mathbf{L}\mathbf{x}$ or $x'_i = l_{ij}x_j$.

Consider a rotation whereby the $x_3(z)$ axis is held fixed and the x_1 and x_2 axes are rotated through 45° in an anticlockwise direction. If a point has coordinates $(1, 0, 1)$ before rotation, what are its coordinates in the rotated system?

6.5 %

- 3 Suppose that $\mathbf{r}(t) = (x(t), y(t), z(t))$ is the parametric definition of a curve in space.

- (a) Define the terms *smooth*, *piecewise smooth*, *arclength*. Find a parametric representation $\mathbf{r} = \mathbf{r}(\theta) = (x(\theta), y(\theta), z(\theta))$ for the line in the xy plane ($z = 0$) through the points $(0, 0, 0)$ and $(1, 2, 0)$. Hence find an expression for the arclength along this line measured from $(0, 0, 0)$.

7 %

- (b) Find a tangent vector to the curve at any point. Define and find the curvature of the curve.

5.5 %

- 4 (a) If $F(x, y, z) = x^2 + y^2 + z$ and $x = t, y = t^2, z = t^3$, use the chain rule to compute $\frac{dF}{dt}$. Check your answer by writing F explicitly as a function of t and differentiating. 6.5 %
- (b) Use Taylor's theorem to find a first order approximation for $f(1.1, 2.9)$ based on quantities estimated at the point $(1, 3)$ if $f(x, y) = x^2 + y^2$. 6 %
- 5 Do any 2 of the following:
- (a) If $\mathbf{f} = (3x^2 + 6y)\mathbf{i} - (14yz)\mathbf{j} + (20xz^2)\mathbf{k}$ find the work done in moving a particle in a force field given by \mathbf{f} along a curve C given by $y = x^2, z = x$ from $(0, 0, 0)$ to $(1, 1, 1)$. (Work done = $\int \mathbf{f} \cdot d\mathbf{r}$). 12.5 %
- (b) Sketch the region in the xy plane bounded by $y = x^2, x = 2, y = 1$. Write down a repeated integral representing the area of this region and evaluate this integral.
- (c) Write down a double integral which represents the area of the circle $x^2 + y^2 = a^2, z = 0$. By transformation to polar coordinates or otherwise evaluate this integral.
- 6 Do either (a) or (b)
- (a) Calculate the surface area of the cylindrical surface defined parametrically by $\mathbf{r}(u, v) = (\cos u, \sin u, v), 0 \leq u \leq 2\pi, 0 \leq v \leq 2$. 12.5 %
- (b) Evaluate the scalar surface integral $\int \int_S \Omega dS$ if $\Omega(x, y, z) = x^2 + y^2$ and S is the surface $x^2 + y^2 + z^2 = a^2, z \geq 0$. (Spherical coordinates and a useful indefinite integral are given on the last page). 12.5 %
- 7 Do any 2 of the following:
- (a) Define what is meant by a conservative vector field and prove that every irrotational vector is conservative.
Show that the vector field $\mathbf{u} = (y^2 + z \exp(xz), 2xy, x \exp(xz))$ is irrotational and find a corresponding scalar potential $\Omega(x, y, z)$ such that $\mathbf{u} = \nabla\Omega(x, y, z)$.
- (b) Define what is meant by the circulation of a vector field \mathbf{u} about a closed curve C . Give a precise statement of Stokes' theorem. Use Stokes' theorem to write the surface integral
- $$\int \int_S (\nabla \times \mathbf{u}) \cdot d\mathbf{S}$$
- as a line integral if S is the hemisphere $x^2 + y^2 + z^2 = a^2, z \geq 0$ and $\mathbf{u} = (xyz, xy, 0)$. Include the limits on the line integral but do **not** evaluate it.

- (c) Give a physical interpretation of the components of the stress tensor T_{ij} (or \mathbf{T}) and the stress vector for a continuously deforming three dimensional medium and write down the relationship between the stress tensor and the stress vector \mathbf{v} in terms of the stress exerted on a small surface element with unit normal $\hat{\mathbf{n}}$.

12.5 %

Useful results

Cosine rule: $\cos C = \frac{a^2+b^2-c^2}{2ab}$

Plane polar coordinates: $x = r \cos \theta$, $y = r \sin \theta$.

Jacobian determinant for polars: $J = r$.

Cylindrical polars: $x = r \cos \theta$, $y = r \sin \theta$, $z = z$.

Spherical polars: $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$; $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$.

Scalar differential surface element: $dS = |\mathbf{r}_u \times \mathbf{r}_v| du dv$ if $\mathbf{r}(u, v)$ defines the surface parametrically.

Vector differential surface element: $d\mathbf{S} = \mathbf{r}_u \times \mathbf{r}_v du dv$ if $\mathbf{r}(u, v)$ defines the surface parametrically.

Indefinite integral: $\int \sin^3 u du = -\cos u + 1/3 \cos^3 u$.

Arclength for the curve $\mathbf{r}(t) = (x(t), y(t), z(t))$ is $s(t) = \int_{t_0}^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$.

Taylor series: $f(x+h, y+k) = f(x, y) + hf_x(x, y) + kf_y(x, y) + O(h^2 + k^2)$.