UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

College of Informatics and Electronics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4613

SEMESTER: Autumn 2001-02

MODULE TITLE: Vector Analysis

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Prof. S. O'Brien

PERCENTAGE OF TOTAL MARKS: 96%

EXTERNAL EXAMINER: Prof. J.D. Gibbon

INSTRUCTIONS TO CANDIDATES: **Answer question 1 and any 4 other questions. Use a separate answer book for question 1.**

Vectors are written in bold print and are expressed in cartesian coordinates unless otherwise noted. Number each question carefully *in the margin provided on your script* **. The subscript notation is used intermittently to denote a partial derivative (e.g.** $u_x \equiv \frac{\partial u}{\partial x}$**).**

Please return the exam paper with your answer book.
N.B.There are some useful results on the last page.

Marks

1 This question is **obligatory**. Answer any **9** of the following. Each is worth 4 %. 9×4 %

(a) Give a precise technical definition of a vector including the vector transformation law.

(b) Define what is meant by the direction cosines l, m, n of a line through the origin and prove that $l^2 + m^2 + n^2 = 1$. Find the direction cosines of the line joining the origin to the point $(6, 2, 5)$.

(c) If two lines passing through the origin have direction cosines l, m, n and l', m', n' respectively, use the cosine rule to prove that the angle between the lines must satisfy $\cos\theta = ll' + mm' + nn'$.

(d) Explain with the aid of a diagram what is meant by the orthogonal projection of a point P onto a line OA. Using the result in 1(c) show that if OA has direction cosines l, m, n and P has coordinates (x, y, z) then the orthogonal projection of OP on OA is $lx + my + nz$.

(e) Define the **vector** product of 2 vectors in two different ways. Are the vectors with components $(1, 2, 1), (3, 4, 2)$ parallel (or anti-parallel) or otherwise?

(f) Define the triple scalar product of three vectors. Hence find the volume of the parallelepiped defined by the vectors with components $(1, 2, 3), (2, 4, 1), (1, 5, 7)$.

(g) Let $f(x, y, z) = xy + z$ and $\mathbf{v}(x, y, z) = (xyz, z, x + y + z)$ be a scalar and a vector field respectively. Compute ∇f, div $\mathbf{v}$, $\nabla \times \mathbf{v}$, $\nabla^2 f$.

(h) A particle moves in a circle of radius R with constant angular velocity ω such that its position at any time t is given by: $\mathbf{r}(t) = R\cos\omega t\,\mathbf{i} + R\sin\omega t\,\mathbf{j}$. Find the velocity of the particle at any time t and show that its acceleration is towards the centre of the circle.

(i) Draw level curves (contours) for the following functions of two variables $\Omega = x^2 + y^2$, $\Omega(x, y) = x + y$.

(j) Find the directional derivative of $f(x, y, z) = x^2yz$ at the point $(1, 2, 3)$ in the direction of $\mathbf{i} + \mathbf{j}$.

(k) Show that $f(x, y) = x^2 + xy + 3x + 2y + 5$ has a saddle point at $(-2, 1)$.

(l) Evaluate the repeated integral $\int_{y=0}^{y=1}\int_{x=1}^{x=y} xy\,dx\,dy$.

(m) Define what is meant by the **circulation** of a vector field about a closed curve C. Give a precise statement of Stokes' theorem relating a particular surface and line integral.

2 (a) If l, m, n and l', m', n' are the direction cosines of two lines though the origin, the angle θ between the two lines satisfies

$$\cos\theta = ll' + mm' + nn'.$$

Prove that two such lines are perpendicular **if and only if** $ll' + mm' + nn' = 0$.

Find the angle between the lines OA and OB where $A = (1, 3, 1)$ and $B = (1, 2, 3)$. 7 %

(b) Derive the transformation law for the components of a point (x, y, z) after rotation of axes in the usual notation: $\mathbf{x}' = \mathbf{L}\mathbf{x}$ where $\mathbf{x}'$ are the new coordinates and $\mathbf{L}$ is the matrix of direction cosines.

(N.B. You may use the fact that if the line OA has direction cosines l, m, n and P has coordinates x, y, z then the orthogonal projection of OP on OA is $lx + my + nz$). 4 %

(c) Consider a rotation of axes whereby the $x_3(z)$ axis is held fixed and the x_1 and x_2 axes are rotated through 90^0 in an anticlockwise direction. What are the coordinates of the point $(-1, 2, 0)$ relative to the rotated axes? 4 %

3 Let $\mathbf{r}(t) = (x(t), y(t), z(t))$ be the parametric definition of a curve in space.

(a) Define carefully *smooth, piecewise smooth*. Sketch the curve with parametric definition $\mathbf{r}(t) = (t, |t|, 0),) - 1 \le t \le 1$. Find a tangent vector at each point and **prove** that the curve is piecewise smooth. 9 %

(b) Write the circle $x^2 + y^2 = a^2, z = 0$, a constant, in parametric form $\mathbf{r}(t) = (x(t), y(t), z(t))$ and hence find an expression for arclength s along the circle measured from $(0, 0, 0)$ in terms of the parameter t. Hence write the curve in intrinsic form using s as parameter. 6 %

4 (a) If $F(x, y) = x + \sin y$ with $x = t\cos u, y = t\sin u$ then use the **chain rule** to compute $\frac{\partial F}{\partial t}$. Verify your answer by writing F explicitly as a function of t, u and differentiating. 4 %

(b) If $\Omega(x, y, z)$ is a scalar field **prove** that the vector grad Ω is perpendicular to the level surfaces of Ω. Use this result to compute a **unit** normal vector to the surface $z - 3x^2y - x = 0$ at the point $(1, 1, 4)$. 6 %

(c) Use Taylor's theorem to find a first order approximation for $f(1.1, 3.7)$ based on quantities estimated at the point $(1, 3.5)$ if $f(x, y) = x^3y + \sin x$. 5 %

5 Do any **2** of the following:

(a) If $\mathbf{f} = (3x^2 + 6y)\mathbf{i} - 14yz\mathbf{j} + 20xz^2\mathbf{k}$ find the work done in moving a particle in a force field given by $\mathbf{f}$ along a curve C given by $x = t, y = t^2, z = t^3$, from $(0, 0, 0)$ to $(1, 1, 1)$. (Work done = $\int \mathbf{f}.d\mathbf{r}$). 7.5 %

(b) If C is the segment of the line $y = 2x, z = 0$ in the xy plane from $(-1, -2, 0)$ to $(1, 2, 0)$, find a parametric representation for this line. Evaluate the scalar line integral $\int_C \Omega(x, y, z)ds$ where $\Omega(x, y, z) = xy^3$. 7.5 %

(c) Sketch the region in the xy plane bounded by $x^2 + y^2 = 4, (z = 0)$. Write down a repeated integral representing the area of this region and using polar coordinates or otherwise evaluate this integral. 7.5 %

6 Do **either** (a) **or** (b)

(a) Let S be the the part of the paraboloid $z = 2-x^2-y^2$ located above the xy plane ($z = 0$) with parametric representation $\mathbf{r}(u, v) = (u, v, 2 - u^2 - v^2)$. Show that $\int\int_S \mathbf{r}.d\mathbf{S} = 6\pi$.

(b) Evaluate the scalar surface integral $\int\int_S \Omega dS$ if $\Omega(x, y, z) = x^2 + y^2$ and S is the surface $x^2 + y^2 + z^2 = a^2$. (Spherical coordinates and a useful indefinite integral are given on the last page). 15 %

7 Do any **2** of the following:

(a) Define what is meant by the flux of a vector field $\mathbf{u}$ through a surface S. Give a precise statement of the divergence theorem. Use the divergence theorem to write the surface integral

$$\int\int_S \mathbf{u}.d\mathbf{S}$$

as a volume integral if S is the sphere $x^2 + y^2 + z^2 = a^2$ and $\mathbf{u} = (x^3, y^3, z^3)$. By transforming to spherical coordinates or otherwise include the limits on the volume integral but do **not** evaluate it. (Spherical coordinates and the relevant Jacobian determinant are given on the last page). 7.5 %

(b) Write down the tensor transformation law for a rank 2 tensor (in matrix or index notation). Two sets of axes $Oxyz$ and $Ox'y'z'$ are such that the first set may be placed in the position of the second set by a rotation of 180^0 about the z axis (i.e. the z axis is held fixed during this rotation). Write down the corresponding matrix of direction cosines and show how the tensor

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

transforms under this rotation of axes. 7.5 %

(c) Prove the identity $\nabla \times \nabla\phi = \mathbf{0}$ for any scalar field $\phi(x, y, z)$.

Define what is meant by a conservative vector field and prove that every irrotational vector is conservative. Show that the vector field $\mathbf{g} = (0, 0, -g)$ where g is the constant scalar acceleration due to gravity is irrotational and find a corresponding scalar potential $\Omega(x, y, z)$ such that $\mathbf{g} = \nabla\Omega(x, y, z)$. 7.5 %

Useful results

Cosine rule: $\cos C = \frac{a^2+b^2-c^2}{2ab}$
Plane polar coordinates: $x = r\cos\theta,\ y = r\sin\theta$.
Jacobian determinant for Cartesians to plane polars: J=r.
Jacobian determinant for Cartesians to spherical polars: $J = r^2\sin\theta$.
Cylindrical polars: $x = r\cos\theta,\ y = r\sin\theta,\ z = z$.
Spherical polars: $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta;\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi$.
Scalar differential surface element: $dS = |\mathbf{r}_u \times \mathbf{r}_v|\, dudv$ if $\mathbf{r}(u,v)$ defines the surface parametrically.
Vector differential surface element: $d\mathbf{S} = \mathbf{r}_u \times \mathbf{r}_v\, dudv$ if $\mathbf{r}(u,v)$ defines the surface parametrically.
Indefinite integral: $\int \sin^3 u du = -\cos u + 1/3\cos^3 u$.
Arclength for the curve $\mathbf{r}(t) = (x(t), y(t), z(t))$ is $s(t) = \int_{t_0}^{t} \sqrt{(\frac{dx}{d\tau})^2 + (\frac{dy}{d\tau})^2 + (\frac{dz}{d\tau})^2} d\tau$.
Taylor series: $f(x+h, y+k) = f(x,y) + hf_x(x,y) + kf_y(x,y) + O(h^2+k^2)$.