UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4407

SEMESTER: Autumn 2012-13

MODULE TITLE: Perturbation Methods

DURATION OF EXAMINATION: 2 hrs 30 mins

LECTURER: Prof. S.O'Brien

PERCENTAGE OF TOTAL MARKS: 90%

EXTERNAL EXAMINER: Prof. T. Myers

INSTRUCTIONS TO CANDIDATES: Full marks for **5** questions.
There are some useful results at the end of the paper.

1 (a) Assuming $\lim \frac{f}{g}$ exists as $\varepsilon \to \varepsilon_0$, define what is meant by

$$f = O(g),\ f = o(g),\ f \ll g,\ f \sim g,\ (\varepsilon \to \varepsilon_0)$$

where $f = f(\varepsilon), g = g(\varepsilon)$.
Are the following statements true or false?

as $x \to 0: \quad \sin(x) \sim x,\ x^2 \ll x^2 + e^x,\ x^{1/4} \ll x^{1/3},$
$\ln(1+x) \sim x, x(1-x)^{\frac{1}{2}} \sim \sqrt{x};$

as $y \to \infty: \quad \sinh y + \sin y \sim \dfrac{1}{2}e^y,\ \ln y \sim \ln y^2,\ y^2 \ln y \gg y^3.$

Show your reasoning. 8 %

(b) Including terms of $O(\varepsilon^2)$, use Taylor series and the binomial theorem to determine expansions for

$$\cos e^{\varepsilon} \text{ and } (1+\varepsilon)^{-4}$$

for $\varepsilon \ll 1$. 4 %

(c) Prove that $\exp(-\frac{1}{\varepsilon}) \ll \varepsilon^n, \varepsilon \to 0, n \in \mathbb{Z}^+$. Hence show that the result holds for $n \in \mathbb{R}^+$. 4 %

(d) Find a two-term asymptotic expansion, for small ε, for the point (x_m, y_m) at which the function $f(x,y) = x^2 + 2\varepsilon \sin(x + e^y) + y^2$ attains a minimum. 2%

2 (a) Find a regular 2 term asymptotic expansion for the solution of the boundary value problem:

$$y'' + \varepsilon y' - y = 1;\ y(0) = y(1) = 1.$$

5 %

(b) Find a regular expansion including terms of $O(\varepsilon)$ of each solution of:

$$x^2 + x - \varepsilon = 0.$$

5 %

(c) Explain what is meant by a singular approximation in the context of the following algebraic equation:

$$\varepsilon x^2 + 2x - 1 = 0.$$

Assuming that $\varepsilon \ll 1$ is known, deduce two term expansions for the solutions x and explain why it is necessary to rescale to find the second root. 8 %

3 A 'small' mass m hangs from a weightless spring with internal damping proportional to the speed. A known vertical impulse I (instantaneous momentum change) is imparted to the mass by striking it with a hammer. Initial conditions on the vertical deflection $y^*(t^*)$ can thus be taken to be:

$$y^*(t^* = 0) = 0;\ m\frac{dy^*}{dt^*}(0) = I.$$

The governing differential equation for the motion of the mass is:

$$m\frac{d^2y^*}{dt^{*2}} + \mu\frac{dy^*}{dt^*} + ky^* = 0,$$

where μ, k are the damping and string constants respectively. Assume that the mass is sufficiently small that there is strong overdamping and the mass will quickly return to rest after the impulse is expended in stretching the spring.

Find an appropriate choice of dimensionless variables $y^* = yL$, $t^* = tT$ to formulate the dimensionless initial value problem:

$$\varepsilon\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 0;\ y(0) = 0,\ \varepsilon\frac{dy}{dt}(0) = 1,\ \varepsilon \ll 1.$$

Find a leading order composite solution using the method of matched expansions. (Hint: Do not impose *any* initial condition on the outer problem. Then find an inner solution ($t = \varepsilon T$) which satisfies *both* initial conditions).

18 %

4 Consider the boundary value problem:

$$\varepsilon y'' + y' + y = 0;\ y(0) = \alpha,\ y(1) = \beta.$$

(a) State the van Dyke matching principle. Assuming there is a boundary layer at $x = 0$, use the method of matched expansions to determine a leading order composite expansion. 14 %

(b) Find the exact solution and compare it with your approximate answer. 4 %

5 Consider Rayleigh's equation:

$$\ddot{y} + \varepsilon(\frac{1}{3}\dot{y}^3 - \dot{y}) + y = 0.$$

(a) Obtain a regular perturbation solution and show that this contains secular terms. 4 %

(b) Using multiple scales ($t_1 = t, t_2 = \varepsilon t$) show that the leading order solution is

$$y = 2\cos(t+\beta)(1+ke^{-\varepsilon t})^{-\frac{1}{2}},$$

where k, β are constants of integration.

Why is this solution an improvement on the regular perturbation solution? 14 %

You may use the trigonometric identities at the end of the paper and the result

$$2\frac{dA}{d\tau} + \frac{1}{4}A(\tau)^3 - A(\tau) = 0 \text{ has solution } A = \frac{2}{\sqrt{1+ke^{-\tau}}}, k \text{ constant.}$$

6 (a) Apply the WKB method directly to the problem:

$$xy'' + y' + \frac{1}{\varepsilon^2}x(1-x^2)y = 0, \ -1 < x < 1.$$

6 %

(b) Use the WKB method to show that the problem

$$\varepsilon^2 y'' - (1+x^2)^2 y = 0, \ y(0) = 0, \ y'(0) = 1$$

has the approximate solution:

$$y \sim \varepsilon(1+x^2)^{-\frac{1}{2}} \sinh \frac{x^3/3 + x}{\varepsilon}.$$

12 %

7 (a) Prove the result:

$$\int_{-\infty}^{\infty} e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}}, \ a > 0.$$

5 %

(b) By splitting the range of integration, show that for $\varepsilon \ll 1$,

$$\int_0^1 \frac{dx}{(x+\varepsilon)^{1/2}} \sim 2$$

and

$$I = \int_0^{\pi/4} \frac{d\theta}{\varepsilon^2 + \sin^2\theta} \sim \frac{\pi}{2\varepsilon}.$$

5 %

(c) Show that $\sin^2 t$ has a local maximum at $t = 3\pi/2$ and that the Taylor series for $\cos^2 t$ about his point is:

$$\cos^2 t \sim \left(t - \frac{3\pi}{2}\right)^2 + o\left(t - \frac{3\pi}{2}\right)^2.$$

Hence use Laplace's method to deduce that:

$$\int_{\pi}^{2\pi} \cos^2 t\, e^{x \sin^2 t}\, dt \sim \frac{\sqrt{\pi} e^x}{2x^{3/2}}, \quad x \gg 1.$$

8 %

8 Answer either part (a) or any **two** other parts of the following:

(a) Consider the problem

$$\varepsilon y''' - y' = 1; \quad y(0) = \alpha, y'(0) = \beta, y(1) = \gamma.$$

Determine the exact solution and use it to show there is a boundary layer at each end. Determine a two term uniform expansion and compare your answer with the exact solution. (Hint: look for expansions of the form $y = y_0 + \sqrt{\varepsilon} y_1$ in each region). 18 %

(b) Solve the advection-diffusion problem

$$u_t + u_x = \varepsilon u_{xx}; \; u(0, t) = 1; \; u(1, t) = 2, \; u(x, 0) = 0; \; \varepsilon \ll 1,$$

by exploiting the smallness of ε to solve an advection problem on most of the domain.

Show in detail how to resolve the boundary layer near $x = 1$. The advective part of the solution should be in the form of a moving step function. Show how this too can be smoothed by insertion of a boundary layer.

(c) Consider Burger's equation

$$u_t + uu_x = \varepsilon u_{xx}, \; -\infty < x < \infty, t > 0.$$

(i) Use the method of characteristics to solve the "outer" problem with $\varepsilon = 0$ subject to the initial condition $u(x, 0) = f(x)$. Explain why shocks will generally form.

(ii) Assuming that the shock location is $s(t)$, use the rescaling

$$u(x, t) \to U(\bar{x}, \tau); \; \bar{x} = \frac{x - s(t)}{\varepsilon}, \tau \equiv t.$$

to obtain a boundary layer equation valid in the region of the shock.

(d) Find a two term expansion for the solutions x of the cubic equation $\varepsilon x^3 - 3x + 1 = 0$.

(e) Consider the turning point problem

$$\varepsilon^2 y'' = x(2-x)y, y(-1) = y(1) = 1.$$

Describe (using a rough sketch) the main features of the WKB solution for $-1 \leq x < 0$ and $0 < x \leq 1$. Outline how you would find the solution in each case. (It is not necessary to show all the details).

Obtain a suitable rescaling of the equation near $x = 0$ and show that it can be approximated by Airy's equation $Z'' - sZ(s) = 0$. (You do not have to solve this equation).

(f) Show that $\tan(\pi/2 + \alpha) \sim -1/\alpha, \alpha \ll 1$. Hence find a two term expansion for the root of $\varepsilon x \tan x - 1 = 0$ which is near $\pi/2$.

The natural frequencies of an elastic string are governed by the transcendental equation $\tan\lambda = \lambda$. Sketch the two functions and show there are an infinite number of solutions.

Assuming that

$$\lambda \sim \varepsilon^{-1} + \varepsilon^{\alpha} a_1$$

and *determining* $\varepsilon \ll 1$ and the $O(1)$ constants α, a_1, or otherwise, find a two term asymptotic expansion for the large solutions of the equation.

(g) Show graphically that $x\exp(-x) = \varepsilon$ has a small and large root. Hence show that the two approximate roots are

$$x_1 \sim \varepsilon + \varepsilon^2, \quad x_2 \sim \ln(1/\varepsilon) + \ln(\ln(1/\varepsilon)).$$

Hint: If $\varepsilon \ll 1$, then $\ln(1/\varepsilon) \gg 1$.

2×9 %

Useful Results

$$
\begin{aligned}
(a+b)^{\alpha} &= a^{\alpha} + \alpha a^{\alpha-1}b + \frac{1}{2!}\alpha(\alpha-1)a^{\alpha-2}b^{2}\ldots \ \ \alpha \in \mathbb{R} \\
f(0+x) &= f(0) + xf^{'}(0) + \frac{x^2}{2!}f^{''}(0) + O(x^3) \\
\exp(x) &= 1 + x + x^2/2! + x^3/3! + O(x^4) \\
\ln(1+x) &= x - x^2/2 + x^3/3 - x^4/4 + O(x^5) \ (-1 < x \leq 1) \\
\sin(x) &= x - x^3/3! + x^5/5! + O(x^7) \\
\cos(x) &= 1 - x^2/2! + x^4/4! + O(x^6) \\
\tan(x) &= x + x^3/3 + 2x^5/15 + O(x^7) \ (\mid x \mid < \pi/2) \\
\sinh(x) &= x + x^3/3! + x^5/5! + O(x^7) \\
\cosh(x) &= 1 + x^2/2! + x^4/4! + O(x^6) \\
\tanh(x) &= x - x^3/3 + 2x^5/15 + O(x^7) \ (\mid x \mid < \pi/2) \\
\text{erf}(x) &= \frac{2}{\sqrt{\pi}}(x - \frac{x^3}{3.1!} + \frac{x^5}{5.2!} - \frac{x^7}{7.3!} + O(x^9)) \\
\sinh(x) &\equiv \frac{e^x - e^{-x}}{2} \\
\cosh(x) &\equiv \frac{e^x + e^{-x}}{2} \\
\int \frac{1}{1+x^2}\,dx &= \tan^{-1} x \\
\int \frac{1}{\sin^2 x}\,dx &= -\cot x \\
\sin^3 x &= \frac{1}{4}(3\sin x - \sin 3x) \\
\cos^3 x &= \frac{1}{4}(\cos 3x + 3\cos x)
\end{aligned}
$$