UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

College of Informatics and Electronics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4407

SEMESTER: Autumn 2007-08

MODULE TITLE: Perturbation Methods

DURATION OF EXAMINATION: 2 hrs 30 mins

LECTURER: Prof. S.O'Brien

PERCENTAGE OF TOTAL MARKS: 90%

EXTERNAL EXAMINER: Prof. J.R. King

INSTRUCTIONS TO CANDIDATES: Full marks for **5** questions. Number each question carefully **in the margin provided on your script**.

Assume that ε is a small parameter.
There are some useful results at the end of the paper.

1 (a) Define what is meant by $f = O(g), f = o(g), f \ll g, f \sim g,\ (\varepsilon \to 0)$ where $f = f(\varepsilon), g = g(\varepsilon)$. 2 %

Are the following statements true or false? (Explain your reasoning).

$$\sin(x) = O(x), x^2 \sim x^3, x^{1/4} \ll x^{1/3} \text{ as } x \to 0.$$
$$e^y + y \sim e^y, y = O(\ln(y)), \ln(y^2) \ll y \text{ as } y \to \infty.$$

10 %

(b) Determine two terms in the expansion of $\sqrt{1 - \varepsilon x}$ for $\varepsilon \ll 1$. 2 %

(c) Prove that $\exp\left(-\frac{1}{\varepsilon}\right) \ll \varepsilon^n, \varepsilon \to 0, n \in Z^+$. 4 %

2 (a) Explain what is meant by a regular and singular approximation in the context of the following two algebraic equations:

(i)
$$x^2 - (3 + 2\varepsilon)x + 2 + \varepsilon = 0,$$

(ii)
$$\varepsilon x^2 + x + 1 = 0.$$

Assuming that $\varepsilon \ll 1$ is known, deduce two term expansions for the solutions x of each equation and explain why it is necessary to rescale the second equation. 14 %

(b) Find a two term expansion for the solution (close to $\pi/2$) x of the equation

$$\varepsilon x \tan x = 1.$$

4 %

3 Consider the following projectile problem, including air resistance:

$$\frac{d^2x^*}{dt^{*2}} = -\frac{gR^2}{(x^* + R)^2} - \frac{k}{R + x^*}\frac{dx^*}{dt^*},$$

where $k(\text{m s}^{-1})$ is a nonnegative constant, $R(\text{m})$ is radius of the earth, $g(\text{m s}^{-2})$ is acceleration due to gravity and $v_0(\text{m s}^{-1})$ is initial velocity. Assume here that $x^*(t^* = 0) = 0$ and $x^{*\prime}(t^* = 0) = v_0$.

(a) Non-dimensionalise this problem (**including boundary conditions**) by scaling $t = t^* g/v_0$ and $x = x^* g/v_0^2$ and show that two dimensionless parameters appear: $\epsilon \equiv \frac{v_0^2}{Rg}$, $\alpha \equiv \frac{kv_0}{Rg}$ and that the scaled equation is:

$$\frac{d^2x}{dt^2} = -\frac{1}{(1 + \varepsilon x)^2} - \frac{\alpha}{(1 + \varepsilon x)}\frac{dx}{dt}.$$

8 %

(b) If $\alpha = 1$ and $\epsilon \ll 1$, show that a leading order perturbation approximation for the solution of the problem is $y_0(t) = 2 - 2\exp(-t) - t$. 8 %

(c) Write down an equation for the flight time of the projectile. 2 %

4 Consider the boundary value problem:

$$\varepsilon y'' + y' + y = 0;\ y(0) = \alpha; y(1) = \beta.$$

(a) Show that a straightforward one term regular perturbation solution cannot satisfy both boundary conditions in general. 6 %

(b) Assuming that there is a boundary layer at $x = 0$, find the appropriate rescaling in this layer and deduce the leading order boundary layer solution. 8 %

(c) Write down a one term composite solution. 4 %

5 Consider a mass-spring-dashpot system, with corresponding spring and damping forces $F_s = -kx(t), F_d = -c\frac{dx}{dt}$, where $x(t)$ denotes the displacement of the mass m_0 **upwards** from its position when the spring unextended.

(a) Show that the equilibrium position of the spring is $x_s = -\frac{m_0 g}{k}$. If the mass is disturbed from its equilibrium position with an initial velocity v_0 explain why the subsequent motion is described by:

$$m_0 x'' = -kx(t) - m_0 g - cx';\ x(0) = x_s, x'(0) = v_0.$$

2 %

(b) Transform $y(t) = x(t) - x_s$ to obtain the problem in the equivalent form:

$$m_0 y'' = -ky - cy';\ y(0) = 0, y'(0) = v_0.$$

By writing $t = \bar{t}\sqrt{\frac{m_0}{k}}$ and choosing a suitable scale for y, write the problem in the dimensionless form:

$$\frac{d^2\bar{y}}{d\bar{t}^2} + \bar{y}(\bar{t}) + \varepsilon\frac{d\bar{y}}{d\bar{t}} = 0;\ \bar{y}(\bar{t} = 0) = 0; \frac{d\bar{y}}{d\bar{t}}(\bar{t} = 0) = 1,$$

where $\frac{c}{\sqrt{km_0}} \equiv \varepsilon$. 4 %

(c) In the context of the equation

$$\frac{d^2 y}{dt^2} + \alpha^2 y = A\cos t + B\sin t,$$

explain what is meant by resonant or secular terms. 4 %

(d) Assuming that $\varepsilon \ll 1$, show that an attempt to solve the problem in part (b) using a regular perturbation approach $\bar{y} \sim \bar{y}_0 + \varepsilon \bar{y}_1$ gives rise to secular terms at $O(\varepsilon)$. 8 %

6 Consider the damped oscillator problem:

$$y'' + \varepsilon y'^3 + y = 0;\ y(0) = 0; y'(0) = 1, (t > 0).$$

(a) Show that a two term regular perturbation solution can be found which satisfies both initial conditions but which contains secular terms. Indicate for which values of t this solution is valid. 8 %

(b) Use multiple scales ($t_1 = t, t_2 = \varepsilon t$) to show how to obtain the solution $y(t) \sim \frac{2\sin t}{\sqrt{(4+3\varepsilon t)}}$ valid up to at least $t = O(1/\varepsilon)$.
Hint: You may use the results

$$\cos^3 t = \frac{1}{4}\cos 3t + \frac{3}{4}\cos t;\ \sin^3 t = -\frac{1}{4}\sin 3t + \frac{3}{4}\cos t.$$

10 %

7 Consider the eigenvalue problem:

$$y'' + \frac{x^2}{\varepsilon^2}y = 0;\ y(x=1) = 0; y(x=2) = 0$$

where the the eigenvalues are $\frac{1}{\varepsilon^2}$.

(a) Use a WKB approximation to find a leading order approximation for $y(x)$ for $\varepsilon \ll 1$. 12 %

(b) Show that a leading order approximation for the eigenvalue equation is

$$\tan\frac{2}{\varepsilon} - \tan\frac{1}{2\varepsilon} = 0.$$

Deduce that the large eigenvalues are given by: $\frac{1}{\varepsilon_n} = \frac{2}{3}n\pi, n \in Z^+$.
You may use the result $\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}$. 6 %

8 Answer any **two** of the following:

(a) Use integration by parts to show for large x that :

$$\int_x^\infty \frac{e^{-t}}{t}\,dt = e^{-x}\left(\frac{1}{x} - \frac{1!}{x^2} + O(\frac{1}{x^3})\right)$$

(b) Use Laplace's method to show for large x that:

$$\int_0^\pi e^{x\sin t}\,dt \sim e^x\sqrt{\frac{2\pi}{x}}$$

(Note: $\int_{-\infty}^{\infty}\exp(-as^2)ds = \sqrt{\frac{\pi}{a}}$.)

(c) Solve the initial value problem in question 5(b) for $\bar{y}(\bar{t})$ using multiple scales.

(d) A simple toy model for spin-coating of a liquid on a rotating substrate with evaporation effects included is:

$$\frac{dL^*}{dt^*} = -\alpha L^{*2} - \beta;\ L^*(0) = L_0$$

where $L^*(t^*)$ is the film thickness, t^* is the time, $\alpha(\mathrm{m}^{-1}\mathrm{s}^{-1})$ is a parameter representing the relative importance of spin effects and viscosity and $\beta(\mathrm{ms}^{-1})$ is an evaporation parameter. The terms on the right hand side of (8(d)) can be thought of as representing the tendency of the liquid film to thin out via two processes: **flow** effects (as represented by α) and **evaporation** (as represented by β) respectively.

(i) Consider an inner (small time) rescaling (L, T)

$$L = \varepsilon^{\frac{1}{2}} l;\ T = \frac{1}{\varepsilon^{\frac{1}{2}}} t$$

and show that the **two** term inner solution fails when $T = O(\frac{1}{\sqrt{\varepsilon}})$.

(ii) Now use an outer scaling (l, t) of the form

$$L^* = l\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}};\ t^* = t\frac{1}{(\alpha\beta)^{\frac{1}{2}}}$$

to write the outer problem (do **not** solve this problem) in the form:

$$\frac{dl}{dt} = -l^2 - 1;\ l(0) = \frac{1}{\varepsilon^{\frac{1}{2}}}$$

with $\varepsilon \equiv \frac{\beta}{\alpha L_0^2} \ll 1$

(e) Consider the boundary value problem:

$$\varepsilon^2 y'' + \varepsilon x y' - y = -e^{-x};\ y(0) = 2, y(1) = 1.$$

Assuming that there are boundary layers of $O(\varepsilon)$ at both $x = 0$ and $x = 1$, use matched asymptotics to find a leading order composite solution.

(f) Consider Burger's equation

$$u_t + uu_x = \epsilon u_{xx},\ -\infty < x < \infty, t > 0.$$

(i) Use the method of characteristics to solve the "outer" problem with $\epsilon = 0$ subject to the initial condition $u(x,0) = f(x)$. Explain why shocks will generally form.

(ii) Assuming that the shock speed is $s(t)$, use the rescaling

$$u(x,t) \to U(\bar{x},\tau);\ \bar{x} = \frac{x - s(t)}{\varepsilon}, \tau \equiv t.$$

to obtain a boundary layer equation valid in the region of the shock.

(g) Using a multiple scales approach, show that a leading order approximation to the solution of the PDE problem (which models oscillations in a string with weak damping):

$$u_{xx} = u_{tt} + \varepsilon u_t,$$
$$u(x=0,t) = u(x=1,t) = 0;\ u(x,0) = g(x); u_t(x,0) = 0$$

for $0 \le x \le 1, t > 0$ is given by

$$u(x,t) \sim \sum_{n=1}^{\infty} \beta_n \exp\left(\frac{\varepsilon t}{2}\right) \cos(\lambda_n t) \sin(\lambda_n x)$$

with $\lambda_n = n\pi, \beta_n = 2\int_0^1 g(x)\sin(\lambda_n x)\,dx$.

(h) Define what is meant by an asymptotic series and discuss the differences between a convergent series and a divergent asymptotic one.

Suppose that $f(x,\varepsilon), \phi(x,\varepsilon)$ are continuous functions for $x \in I, 0 \le \varepsilon < \varepsilon_1$. Explain what is meant by the statement "ϕ is a uniformly valid asymptotic approximation of f for any $x \in I$ as $\varepsilon \to 0$."

In terms of x, find the regions in which the following asymptotic series become non-uniform:

(i) $\varepsilon \cos x + \varepsilon^2 \frac{1+x^2}{1-2x^2} + \ldots$

(ii) $1 - \varepsilon x + \varepsilon^2 x^2 \ldots$

(iii) $x^{3/2} + \varepsilon \ln \frac{1+\sqrt{x}}{1-\sqrt{x}} + \ldots$

(i) Find a two term expansion for the solutions x of the cubic equation:

$$\varepsilon x^3 - 3x + 1 = 0.$$

2×9 %

Useful Taylor (Maclaurin) series

$$
\begin{aligned}
f(0+x) &= f(0) + xf^{'}(0) + \frac{x^2}{2!}f^{''}(0) + O(x^3) \\
\exp(x) &= 1 + x + x^2/2! + x^3/3! + O(x^4) \\
\ln(1+x) &= x - x^2/2 + x^3/3 - x^4/4 + O(x^5)(-1 < x \leq 1) \\
\sin(x) &= x - x^3/3! + x^5/5! + O(x^7) \\
\cos(x) &= 1 - x^2/2! + x^4/4! + O(x^6) \\
\tan(x) &= x + x^3/3 + 2x^5/15 + O(x^7)(\mid x \mid < \pi/2) \\
\sinh(x) &= x + x^3/3! + x^5/5! + O(x^7) \\
\cosh(x) &= 1 + x^2/2! + x^4/4! + O(x^6) \\
\tanh(x) &= x - x^3/3 + 2x^5/15 + O(x^7)(\mid x \mid < \pi/2) \\
\mathrm{erf}(x) &= \frac{2}{\sqrt{\pi}}(x - \frac{x^3}{3.1!} + \frac{x^5}{5.2!} - \frac{x^7}{7.3!} + ...)
\end{aligned}
$$