UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

College of Informatics and Electronics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4407 SEMESTER: Autumn 2006-07

MODULE TITLE: Perturbation Methods DURATION OF EXAMINATION: 2 hrs 30 mins

LECTURER: Prof. S.O'Brien PERCENTAGE OF TOTAL MARKS: 90%

EXTERNAL EXAMINER: Prof. J.R. King

INSTRUCTIONS TO CANDIDATES: Full marks for **5** questions. Number each question carefully **in the margin provided on your script**.

Assume that ε is a small parameter.
There are some useful results at the end of the paper.

1 (a) Define what is meant by $f = O(g), f = o(g), f \ll g, f \sim g,\ (\varepsilon \to 0)$ where $f = f(\varepsilon), g = g(\varepsilon)$.

Are the following statements true or false? (Explain your reasoning using L'Hospital's rule where appropriate).

$$\sin x = O(x), \sin x \sim x, x^2 \ll |\ln x|, x^{1/4} \ll x^{1/3} \text{ as } x \to 0.$$
$$e^y + y \sim e^y, e^y \sim \sinh y, \ln y \ll y \text{ as } y \to \infty.$$

12 %

(b) Determine three terms in the expansion of $\sin(1-\varepsilon)$ for $\varepsilon \ll 1$. 2 %

(c) Prove that $\exp\left(-\frac{1}{\varepsilon}\right) \ll \varepsilon^n, \varepsilon \to 0, n \in Z^+$. 4 %

2 (a) Explain what is meant by a regular and singular approximation in the context of the following two quadratic equations:

$$x^2 + \varepsilon x - 1 = 0;\ \varepsilon x^2 + x - 1 = 0.$$

Deduce two term perturbation expansions for the **two** solutions of each equation and explain why it is necessary to rescale the second equation. 14 %

(b) For the first equation in part (a), find the **exact** solutions and show that their expansions for $\varepsilon \ll 1$ agree with your two term perturbation expansion. 4 %

3 Consider the boundary value problem:

$$\varepsilon y'' + (1+2x)y' + y = x;\ y(0) = 0; y(1) = 0.$$

(a) Using the fact that

$$\int \frac{x}{(1+2x)^{1/2}}dx = \frac{1}{6}(1+2x)^{3/2} - \frac{1}{2}(1+2x)^{1/2},$$

show that a straightforward one term regular perturbation solution cannot satisfy both boundary conditions. 6 %

(b) Assuming that there is a boundary layer at $x = 0$, show that the outer solution is $\frac{1}{3}(x-1)$, find the appropriate rescaling in this layer and deduce the leading order boundary layer solution. 8 %

(c) State the van Dyke matching principle and use it to match the outer and boundary layer solutions. Write down a one term composite solution. 4 %

4 A simple toy model for spin-coating of a liquid on a rotating substrate with evaporation effects included is:

$$\frac{dL^*}{dt^*} = -\alpha L^{*2} - \beta; \; L^*(0) = L_0$$

where $L^*(t^*)$ is the film thickness, t^* is the time, $\alpha(\mathrm{m}^{-1}\mathrm{s}^{-1})$ is a parameter representing the relative importance of spin effects and viscosity and $\beta(\mathrm{ms}^{-1})$ is an evaporation parameter. The terms on the right hand side of (4) can be thought of as representing the tendency of the liquid film to thin out via two processes: **flow** effects (as represented by α) and **evaporation** (as represented by β) respectively.

(a) First use an outer scaling (l, t) of the form

$$L^* = l\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}}; \; t^* = t\frac{1}{(\alpha\beta)^{\frac{1}{2}}}$$

to get

$$\frac{dl}{dt} = -l^2 - 1; \; l(0) = \frac{1}{\varepsilon^{\frac{1}{2}}}$$

with $\varepsilon \equiv \frac{\beta}{\alpha L_0^2} \ll 1$ and, ignoring the initial condition, deduce a one term outer solution $l = \tan(A - t)$, where A is an arbitrary constant. 8 %

(b) Now consider an inner (small time) rescaling (L, T)

$$L = \varepsilon^{\frac{1}{2}} l; \; T = \frac{1}{\varepsilon^{\frac{1}{2}}} t$$

and show that the **two** term inner solution fails when $T = O(\frac{1}{\sqrt{\varepsilon}})$. 8 %

(c) Match the one-term leading order inner and outer solutions and deduce a leading order composite solution. 2 %

5 Consider a mass-spring-dashpot system, with corresponding spring and damping forces $F_s = -kx(t), F_d = -c\frac{dx}{dt}$, where $x(t)$ denotes the displacement of the mass m_0 **upwards** from its position when the spring unextended.

(a) Show that the equilibrium position of the spring is $x_s = -\frac{m_0 g}{k}$. If the mass is disturbed from its equilibrium position with an initial velocity v_0 explain why the subsequent motion is described by:

$$m_0 x'' = -kx(t) - m_0 g - cx'; \; x(0) = x_s, x'(0) = v_0.$$

2 %

(b) Transform $y(t) = x(t) - x_s$ to obtain the problem in the equivalent form:

$$m_0 y'' = -ky - cy';\ y(0) = 0, y'(0) = v_0.$$

4 %

By writing $t = \bar{t}\sqrt{\frac{m_0}{k}}$ and choosing a suitable scale for y, write the problem in the dimensionless form:

$$\frac{d^2\bar{y}}{d\bar{t}^2} + \bar{y}(\bar{t}) + \varepsilon\frac{d\bar{y}}{d\bar{t}} = 0;\ \bar{y}(\bar{t} = 0) = 0;\ \frac{d\bar{y}}{d\bar{t}}(\bar{t} = 0) = 1,$$

where $\frac{c}{\sqrt{km_0}} \equiv \varepsilon$. 4 %

(c) In the context of the equation

$$\frac{d^2y}{dt^2} + \alpha^2 y = A\cos t + B\sin t,$$

explain what is meant by resonant or secular terms. 2 %

(d) Assuming that $\varepsilon \ll 1$, show that an attempt to solve the problem in part (b) using a regular perturbation approach $\bar{y} \sim \bar{y}_0 + \varepsilon\bar{y}_1$ gives rise to secular terms at $O(\varepsilon)$. 4 %

(e) Explicitly derive the secular terms which arise in part (c). 2 %

6 Consider the initial value problem;

$$y'' + 2\varepsilon y' + y = 0;\ y(0) = 0; y'(0) = 1, (t > 0).$$

(a) Show that a 2 term regular perturbation solution can be found which satisfies both initial conditions but which contains secular terms. Indicate clearly for which values of t this solution is valid. 5 %

(b) Use multiple scales ($t_1 = t, t_2 = \varepsilon t$) to obtain an improved solution valid up to at least $t = O(1/\varepsilon)$. 11 %

(c) In a multiple scales problem, if one wished to find a solution valid up to $t = O(1/\varepsilon^2)$ how would one proceed, in principle? 2 %

7 Consider the linear problem:

$$y'' - \frac{1}{\varepsilon^2}f(x)y = 0;\ y(\frac{1}{2}) = 0; y(1) = 1.$$

(a) Solve the boundary value problem if $f(x) \equiv -1$. 3 %

(b) If $f(x) = x(2-x)$, use a WKB approximation, ignoring the boundary conditions, to find a leading order approximation for $y(x)$ for $\varepsilon \ll 1, \frac{1}{2} \leq x \leq 1$. 11 %

N.B.

$$\int \sqrt{x(2-x)} = \frac{1}{2}(x-1)(2x-x^2)^{1/2} + \frac{1}{2}\arcsin(x-1), 0 < x \leq 1.$$

(c) If $f(x) = x(2-x), -1 \leq x \leq 1$ and the boundary conditions are $y(-1) = y(1) = 1$, describe the basic features of the solution for $-1 \leq x < 0, 0 < x \leq 1$. Write down an appropriate rescaling of the equation in the vicinity of the turning point $x = 0$. 4 %

8 Answer any **two** of the following:

(a) Consider the projectile problem:

$$\frac{d^2x}{dt^2} = -\frac{gR^2}{(x+R)^2},\ x(0) = 0, x'(0) = v_0$$

where x, t are the distance from the earth's surface and the time, while the constant parameters g, R, v_0 are acceleration due to gravity, the radius of the earth and the initial velocity of an object thrown into the air.

Non-dimensionalise this problem using v_0/g as time scale and v_0^2/g as length scale and show that the parameter $\varepsilon = v_0^2/Rg$ appears in this formulation.

Show that any reasonable estimate of the parameters indicates that $\varepsilon \ll 1$. Hence find a one term perturbation solution of the scaled problem.

(b) Use integration by parts to show for large x that :

$$\int_x^\infty \frac{e^{-t}}{t}\,dt = e^{-x}\left(\frac{1}{x} - \frac{1!}{x^2} + O(\frac{1}{x^3})\right)$$

(c) Use Laplace's method to show for large x that:

$$\int_0^\pi e^{x\sin t}\,dt \sim e^x\sqrt{\frac{2\pi}{x}}$$

(Note: $\int_{-\infty}^{\infty} \exp(-as^2)ds = \sqrt{\frac{\pi}{a}}$.)

(d) Consider for $b > 0, x \gg 1$

$$I(x) = \int_0^b f(t)e^{-xt}dt, \quad f(t) \sim t^\alpha \sum_{n=0}^{\infty} a_n t^{\beta n}, (t \to 0^+)$$

where $\alpha > -1, \beta > 0$.
Prove Watson's lemma that

$$I(x) \sim \sum_{n=0}^{\infty} \frac{a_n \Gamma(\alpha + \beta n + 1)}{x^{\alpha+\beta n+1}}.$$

(You may use the result: $\int_0^\infty t^a e^{-xt}dt = x^{-1-a}\Gamma(a+1)$).

(e) Consider the boundary value problem:

$$\varepsilon^2 y'' + \varepsilon x y' - y = -e^{-x}; \; y(0) = 2, y(1) = 1.$$

Assuming that there are boundary layers of $O(\varepsilon)$ at both $x = 0$ and $x = 1$, use matched asymptotics to find a leading order composite solution.

(f) Consider Burger's equation

$$u_t + uu_x = \epsilon u_{xx}, \; -\infty < x < \infty, t > 0.$$

(i) Use the method of characteristics to solve the "outer" problem with $\epsilon = 0$ subject to the initial condition $u(x, 0) = f(x)$. Explain why shocks will generally form.

(ii) Assuming that the shock speed is $s(t)$, use the rescaling

$$u(x,t) \to U(\bar{x}, \tau); \; \bar{x} = \frac{x - s(t)}{\varepsilon}, \tau \equiv t.$$

to obtain a boundary layer equation valid in the region of the shock.

(g) Suppose that $f(x, \varepsilon), \phi(x, \varepsilon)$ are continuous functions for $x \in I, 0 \leq \varepsilon < \varepsilon_1$. Explain what is meant by the statement "ϕ is a uniformly valid asymptotic approximation of f for any $x \in I$ as $\varepsilon \to 0$."

In terms of x, find the regions in which the following asymptotic series become non-uniform:

(i) $\varepsilon \cos x + \varepsilon^2 \frac{1+x^2}{1-2x^2} + \ldots$

(ii) $1 - \varepsilon x + \varepsilon^2 x^2 \ldots$

(iii) $x^{3/2} + \varepsilon \ln \frac{1+\sqrt{x}}{1-\sqrt{x}} + \ldots$

2×9 %

Useful series

$$
\begin{aligned}
f(0+x) &= f(0) + xf'(0) + \tfrac{x^2}{2!}f''(0) + O(x^3) \\
\exp(x) &= 1 + x + x^2/2! + x^3/3! + O(x^4) \\
\ln(1+x) &= x - x^2/2 + x^3/3 - x^4/4 + O(x^5)(-1 < x \leq 1) \\
\sin(x) &= x - x^3/3! + x^5/5! + O(x^7) \\
\cos(x) &= 1 - x^2/2! + x^4/4! + O(x^6) \\
\tan(x) &= x + x^3/3 + 2x^5/15 + O(x^7)(\mid x \mid < \pi/2) \\
\cot(x) &= 1/x - x/3 - x^3/45 + O(x^5)(0 < \mid x \mid < \pi) \\
\sinh(x) &= x + x^3/3! + x^5/5! + O(x^7) \\
\cosh(x) &= 1 + x^2/2! + x^4/4! + O(x^6) \\
\tanh(x) &= x - x^3/3 + 2x^5/15 + O(x^7)(\mid x \mid < \pi/2) \\
\mathrm{erf}(x) &= \tfrac{2}{\sqrt{\pi}}\left(x - \tfrac{x^3}{3.1!} + \tfrac{x^5}{5.2!} - \tfrac{x^7}{7.3!} + \ldots\right)
\end{aligned}
$$