UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

College of Informatics and Electronics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4407

SEMESTER: Autumn 2005-06

MODULE TITLE: Perturbation Methods

DURATION OF EXAMINATION: 2 hrs 30 mins

LECTURER: Prof. S.O'Brien

PERCENTAGE OF TOTAL MARKS: 90%

EXTERNAL EXAMINER: Prof. J.R. King

INSTRUCTIONS TO CANDIDATES: Full marks for **5** questions. Number each question carefully **in the margin provided on your script**.

Assume that ε is a small parameter.
There are some useful results at the end of the paper.

1 (a) If $\lim_{\varepsilon \to 0} \frac{f}{g}$ exists, define what is meant by

$$f = O(g), f = o(g), f \ll g, f \sim g, \ (\varepsilon \to 0)$$

where $f = f(\varepsilon), g = g(\varepsilon)$.
Are the following statements true or false?

$$\sin(x) = O(x), \ x^2 \sim 3x^2, \ x^{1/4} \ll x^{1/3}, \ x(1-x)^{\frac{1}{2}} \sim \sqrt{x} \ \text{ as } x \to 0,$$
$$\sinh y + \sin y \sim \frac{1}{2}e^y, \ y = O(y^2), \ y \ln y \ll y^2 \ \text{ as } y \to \infty.$$

Show your reasoning in each case. 12 %

(b) Determine two terms (i.e including terms of $O(\varepsilon^2)$) in the expansion of $\ln(1 + \sin\varepsilon)$ for $\varepsilon \ll 1$. 3 %

(c) Prove that $\exp\left(-\frac{1}{\varepsilon}\right) \ll \varepsilon^n, \varepsilon \to 0, n \in Z^+$. 3 %

2 (a) Explain what is meant by a regular and singular approximation in the context of the following two algebraic equations:

$$x^2 - (2 + \varepsilon)x + 1 + \varepsilon = 0; \ \varepsilon x^2 + x + 1.$$

Assuming that $\varepsilon \ll 1$ is known, deduce two term expansions for the solutions x of each equation and explain why it is necessary to rescale the second equation. (**Hint**: for the first equation, it is necessary to expand as far as $O(\varepsilon^2)$.) 14 %

(b) Find a two term expansion for the solution x of the equation

$$x \exp(-x) = \varepsilon.$$

4 %

3 Consider the boundary value problem:

$$\varepsilon y'' - y' = 1; \ y(0) = \alpha; y(1) = \beta.$$

(a) Show that a straightforward one term regular perturbation solution cannot satisfy both boundary conditions. 4 %

(b) Assuming that there is a boundary layer at $x = 1$, find the appropriate rescaling in this layer and deduce the leading (one term) order boundary layer solution. 7 %

(c) Sketch the inner and outer solutions. 2 %

(d) State the van Dyke matching principle and using this write down a one term composite solution. 2 %

(e) In a singular perturbation problem ($x \in [0,1]$) with a boundary layer near $x = 0$, it is found that:

$$y_{outer} \sim \beta \exp(1-x) + \varepsilon\beta(1-x)\exp(1-x),$$
$$y_{inner} \sim \alpha - b_0 + b_0 \exp(-\bar{x})$$
$$+\varepsilon[-b_1 + b_1 \exp(-\bar{x}) - (\alpha - b_0)\bar{x} + b_0\bar{x}\exp(-\bar{x})],$$

where $x \equiv \varepsilon\bar{x}$. Use asymptotic matching to find the constants b_0, b_1 assuming that α, β are known. 3 %

4 Consider the damped oscillator problem:

$$u'' + 2\varepsilon u' + u = 0;\ u(0) = 1; u'(0) = 0, (t > 0).$$

(a) Show that a two term regular perturbation solution can be found which satisfies both initial conditions but which contains secular terms. Explicitly determine the secular terms and indicate for which values of t this solution is valid. 6 %

(b) Use multiple scales ($t_1 = t, t_2 = \varepsilon t$) to obtain a solution valid up to at least $t = O(1/\varepsilon)$. Interpret the solution if the $O(\varepsilon)$ terms are taken to represent small damping effects in a linear oscillator. 12 %

5 Consider the eigenvalue problem:

$$y'' + \frac{1}{\varepsilon^2} f(x) y = 0;\ y(1) = 0; y(2) = 0$$

where the the eigenvalues are $\frac{1}{\varepsilon^2}$.

(a) Solve the boundary value problem if $f(x) \equiv 1$ and write down the eigenvalue equation (without solving it). 4 %

(b) If $f(x) = x^2$, use a WKB approximation to find a leading order approximation for $y(x)$ for $\varepsilon \ll 1$. 11 %

(c) Show that a leading order approximation for the large eigenvalues is given by: $\frac{1}{\varepsilon_n} = \frac{2}{3} n\pi, n \in Z^+$. 3 %

6 (a) Use Laplace's method to show for large x that:

$$\int_0^1 \exp(-xt)\ln(2+t)\,dt \sim \frac{\ln 2}{x}.$$

6 %

(b) Use integration by parts to show for large x and for fixed positive λ that :

$$\int_x^\infty e^{-t}t^{\lambda-1}\,dt \sim x^\lambda e^{-x}\left(\frac{1}{x}+\frac{\lambda-1}{x^2}+O(\frac{1}{x^3})\right)$$

6 %

(c) Consider for $b > 0, x \gg 1$

$$I(x) = \int_0^b f(t)e^{-xt}dt, \quad f(t) \sim t^\alpha \sum_{n=0}^{\infty} a_n t^{\beta n}, (t \to 0^+)$$

where $\alpha > -1, \beta > 0$. **Prove** Watson's lemma that

$$I(x) \sim \sum_{n=0}^{\infty} \frac{a_n \Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}}.$$

(You may use the result: $\int_0^\infty t^a e^{-xt}dt = x^{-1-a}\Gamma(a+1)$). 6 %

7 Answer any **two** of the following:

(a) Consider the boundary value problem:

$$\varepsilon^2 y'' + \varepsilon x y' - y = -e^{-x};\; y(0) = 2, y(1) = 1.$$

Assuming that there are boundary layers of $O(\varepsilon)$ at both $x = 0$ and $x = 1$, use matched asymptotics to find a leading order composite solution.

(b) Consider Burger's equation

$$u_t + uu_x = \epsilon u_{xx},\; -\infty < x < \infty, t > 0.$$

(i) Use the method of characteristics to solve the "outer" problem with $\epsilon = 0$ subject to the initial condition $u(x, 0) = f(x)$. Explain why shocks will generally form.

(ii) Assuming that the shock speed is $s(t)$, use the rescaling

$$u(x,t) \to U(\bar{x},\tau);\; \bar{x} = \frac{x - s(t)}{\varepsilon}, \tau \equiv t.$$

to obtain a boundary layer equation valid in the region of the shock.

(c) Using a multiple scales approach, show that a leading order approximation to the solution of the PDE problem (which models oscillations in a string with weak damping):

$$u_{xx} = u_{tt} + \varepsilon u_t,$$
$$u(x = 0, t) = u(x = 1, t) = 0;\ u(x, 0) = g(x); u_t(x, 0) = 0$$

for $0 \leq x \leq 1, t > 0$ is given by

$$u(x,t) \sim \sum_{n=1}^{\infty} \beta_n \exp\left(\frac{\varepsilon t}{2}\right) \cos(\lambda_n t) \sin(\lambda_n x)$$

with $\lambda_n = n\pi, \beta_n = 2\int_0^1 g(x) \sin(\lambda_n x)\, dx$.

(d) Define what is meant by an asymptotic series and discuss the differences between a convergent series and a divergent asymptotic one.

Suppose that $f(x,\varepsilon), \phi(x,\varepsilon)$ are continuous functions for $x \in I, 0 \leq \varepsilon < \varepsilon_1$. Explain what is meant by the statement "ϕ is a uniformly valid asymptotic approximation of f for any $x \in I$ as $\varepsilon \to 0$."

In terms of x, find the regions in which the following asymptotic series become non-uniform:

(i) $\varepsilon \cos x + \varepsilon^2 \frac{1+x^2}{1-2x^2} + ...$

(ii) $1 - \varepsilon x + \varepsilon^2 x^2 ...$

(iii) $x^{3/2} + \varepsilon \ln \frac{1+\sqrt{x}}{1-\sqrt{x}} + ...$

(e) Consider the projectile problem:

$$\frac{d^2x}{dt^2} = -\frac{gR^2}{(x+R)^2},\ x(0) = 0, x'(0) = v_0$$

where x, t are the distance from the earth's surface and the time, while the constant parameters g, R, v_0 are acceleration due to gravity, the radius of the earth and the initial velocity of an object thrown into the air.

Non-dimensionalise this problem using v_0/g as time scale and v_0^2/g as length scale and show that the parameter $\varepsilon = v_0^2/Rg$ appears in this formulation.

Assuming that $\varepsilon \ll 1$, find a one term perturbation solution of the scaled problem.

(f) Find a two term expansion for the solutions x of the cubic equation:

$$\varepsilon x^3 - 3x + 1 = 0.$$

2×9 %

Useful Taylor (Maclaurin) series

$$
\begin{aligned}
f(0+x) &= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + O(x^3) \\
\exp(x) &= 1 + x + x^2/2! + x^3/3! + O(x^4) \\
\ln(1+x) &= x - x^2/2 + x^3/3 - x^4/4 + O(x^5)(-1 < x \leq 1) \\
\sin(x) &= x - x^3/3! + x^5/5! + O(x^7) \\
\cos(x) &= 1 - x^2/2! + x^4/4! + O(x^6) \\
\tan(x) &= x + x^3/3 + 2x^5/15 + O(x^7)(\mid x \mid < \pi/2) \\
\sinh(x) &= x + x^3/3! + x^5/5! + O(x^7) \\
\cosh(x) &= 1 + x^2/2! + x^4/4! + O(x^6) \\
\tanh(x) &= x - x^3/3 + 2x^5/15 + O(x^7)(\mid x \mid < \pi/2) \\
\mathrm{erf}(x) &= \frac{2}{\sqrt{\pi}}(x - \frac{x^3}{3.1!} + \frac{x^5}{5.2!} - \frac{x^7}{7.3!} + ...)
\end{aligned}
$$