UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

College of Informatics and Electronics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4407

SEMESTER: Autumn 2004-05

MODULE TITLE: Perturbation Methods

DURATION OF EXAMINATION: 2 hrs 30 mins

LECTURER: Prof. S.O'Brien

PERCENTAGE OF TOTAL MARKS: 90%

EXTERNAL EXAMINER: Prof. J.R. King

INSTRUCTIONS TO CANDIDATES: Full marks for **5** questions. Number each question carefully **in the margin provided on your script**.

Assume that ε is a small parameter.
There are some useful results at the end of the paper.

1 (a) Define what is meant by $f = O(g), f = o(g), f \ll g, f \sim g,\ (\varepsilon \to 0)$ where $f = f(\varepsilon), g = g(\varepsilon)$.
Are the following statements true or false? (Explain your reasoning).

$$\cos(x) = O(x), x^2 \sim x, x^{1/4} \ll x^{1/3} \text{ as } x \to 0.$$
$$e^y + y^5 \sim e^y, y = O(1/y), \ln(y^2) \ll y \text{ as } y \to \infty.$$

12 %

(b) Determine two terms in the expansion of $\sqrt{1 - \varepsilon/2 + \varepsilon^2}$ for $\varepsilon \ll 1$. 2 %

(c) Prove that $\exp\left(-\frac{1}{\varepsilon}\right) \ll \varepsilon^n, \varepsilon \to 0, n \in Z^+$. 4 %

2 (a) Explain what is meant by a regular and singular approximation in the context of the following two quadratic equations:

$$x^2 + (3 + 2\varepsilon)x + 2 + \varepsilon = 0;\ \varepsilon x^2 + x + 1 = 0.$$

Deduce two term expansions for the solutions of each equation and explain why it is necessary to rescale the second equation. 14 %

(b) Draw a rough graph of $\tan x$ and use it to help find a two term expansion for the large roots of

$$x \tan x = 1.$$

4 %

3 Consider the boundary value problem:

$$\varepsilon y'' - y' + y = 0;\ y(0) = \alpha; y(1) = \beta.$$

(a) Show that a straightforward one term regular perturbation solution cannot satisfy both boundary conditions. 4 %

(b) Assuming that there is a boundary layer at $x = 1$, find the appropriate rescaling in this layer and deduce the leading order boundary layer solution. 8 %

(c) Write down a one term composite solution. 3 %

(d) State the van Dyke matching principle. In a singular perturbation problem ($x \in [0, 1]$) there are two outer solutions $y_L(x), y_R(x)$ with a corner layer $y_{cor}(\bar{x})$ near $x = \frac{1}{2}$ as follows:

$$y_L = 1 - x + 0.\varepsilon,\ 0 \le x < \frac{1}{2},$$
$$y_{cor} = \frac{1}{2} + \varepsilon(2\ln(1 + b\exp(\bar{x})) - \bar{x} + c),$$
$$y_R = x + 0.\varepsilon,\ \frac{1}{2} < x \le 1.$$

where $x \equiv \frac{1}{2} + \varepsilon\bar{x}$. Use asymptotic matching to find the constants b, c. 3 %

4 Consider the initial value problem;

$$y'' + 2\varepsilon y' + (1+\varepsilon)y = 0;\ y(0) = \alpha; y'(0) = 0, (t > 0).$$

(a) Show that a 2 term regular perturbation solution can be found which satisfies both initial conditions but which contains secular terms. Indicate for which values of t this solution is valid. 6 %

(b) Use multiple scales ($t_1 = t, t_2 = \varepsilon t$) to obtain an improved solution valid up to at least $t = O(1/\varepsilon)$. 12 %

5 Consider the eigenvalue problem:

$$y'' + \frac{1}{\varepsilon^2} f(x)y = 0;\ y(1) = 0; y(2) = 0$$

where the the eigenvalues are $\frac{1}{\varepsilon^2}$.

(a) Solve the boundary value problem if $f(x) \equiv 1$ and write down the eigenvalue equation (without solving it). 4 %

(b) If $f(x) = x^2$, use a WKB approximation to find a leading order approximation for $y(x)$ for $\varepsilon \ll 1$. 11 %

(c) Show that a leading order approximation for the large eigenvalues is given by: $\frac{1}{\varepsilon_n} = \frac{2}{3}n\pi, n \in Z^+$. 3 %

6 (a) Use integration by parts to show for large x that :

$$\int_x^\infty \frac{e^{-t}}{t}\,dt = e^{-x}\left(\frac{1}{x} - \frac{1!}{x^2} + O(\frac{1}{x^3})\right)$$

6 %

(b) Use Laplace's method to show for large x that:

$$\int_0^\pi e^{x\sin t}\,dt \sim e^x\sqrt{\frac{2\pi}{x}}$$

(Note: $\int_{-\infty}^{\infty} \exp(-as^2)ds = \sqrt{\frac{\pi}{a}}$.) 6 %

(c) Consider for $b > 0, x \gg 1$

$$I(x) = \int_0^b f(t)e^{-xt}dt,\quad f(t) \sim t^\alpha \sum_{n=0}^{\infty} a_n t^{\beta n}, (t \to 0^+)$$

where $\alpha > -1, \beta > 0$.

Prove Watson's lemma that

$$I(x) \sim \sum_{n=0}^{\infty} \frac{a_n \Gamma(\alpha + \beta n + 1)}{x^{\alpha+\beta n+1}}.$$

(You may use the result: $\int_0^\infty t^a e^{-xt} dt = x^{-1-a}\Gamma(a+1)$). 6 %

7 Answer any **two** of the following:

(a) Consider the projectile problem:

$$\frac{d^2x}{dt^2} = -\frac{gR^2}{(x+R)^2},\ x(0) = 0, x'(0) = v_0$$

where x, t are the distance from the earth's surface and the time, while the constant parameters g, R, v_0 are acceleration due to gravity, the radius of the earth and the initial velocity of an object thrown into the air.

Non-dimensionalise this problem using v_0/g as time scale and v_0^2/g as length scale and show that the parameter $\varepsilon = v_0^2/Rg$ appears in this formulation.

Assuming that $\varepsilon \ll 1$, find a one term perturbation solution of the scaled problem.

(b) Consider the boundary value problem:

$$\varepsilon^2 y'' + \varepsilon x y' - y = e^{-x};\ y(0) = 2, y(1) = 1.$$

Assuming that there are boundary layers of $O(\varepsilon)$ at both $x = 0$ and $x = 1$, use matched asymptotics to find a leading order composite solution.

(c) Consider Burger's equation

$$u_t + uu_x = \epsilon u_{xx},\ -\infty < x < \infty, t > 0.$$

(i) Use the method of characteristics to solve the "outer" problem with $\epsilon = 0$ subject to the initial condition $u(x, 0) = f(x)$. Explain why shocks will generally form.

(ii) Assuming that the shock speed is $s(t)$, use the rescaling

$$u(x,t) \to U(\bar{x}, \tau);\ \bar{x} = \frac{x - s(t)}{\varepsilon}, \tau \equiv t.$$

to obtain a boundary layer equation valid in the region of the shock.

(d) Using a multiple scales approach, find a leading order approximation to the solution of the initial value problem:

$$y'' + \varepsilon y' + y = 0;\ y(0) = 0, y'(0) = 1.$$

(e) Suppose that $f(x,\varepsilon), \phi(x,\varepsilon)$ are continuous functions for $x \in I, 0 \leq \varepsilon < \varepsilon_1$. Explain what is meant by the statement "ϕ is a uniformly valid asymptotic approximation of f for any $x \in I$ as $\varepsilon \to 0$."

In terms of x, find the regions in which the following asymptotic series become non-uniform:

(i) $\varepsilon \cos x + \varepsilon^2 \frac{1+x^2}{1-2x^2} + \ldots$

(ii) $1 - \varepsilon x + \varepsilon^2 x^2 \ldots$

(iii) $x^{3/2} + \varepsilon \ln \frac{1+\sqrt{x}}{1-\sqrt{x}} + \ldots$

2×9 %

Useful Taylor (Maclaurin) series

$$\begin{aligned}
f(0+x) &= f(0) + xf'(0) + \tfrac{x^2}{2!}f''(0) + O(x^3) \\
\exp(x) &= 1 + x + x^2/2! + x^3/3! + O(x^4) \\
\ln(1+x) &= x - x^2/2 + x^3/3 - x^4/4 + O(x^5)(-1 < x \leq 1) \\
\sin(x) &= x - x^3/3! + x^5/5! + O(x^7) \\
\cos(x) &= 1 - x^2/2! + x^4/4! + O(x^6) \\
\tan(x) &= x + x^3/3 + 2x^5/15 + O(x^7)(\mid x \mid < \pi/2) \\
\sinh(x) &= x + x^3/3! + x^5/5! + O(x^7) \\
\cosh(x) &= 1 + x^2/2! + x^4/4! + O(x^6) \\
\tanh(x) &= x - x^3/3 + 2x^5/15 + O(x^7)(\mid x \mid < \pi/2) \\
\operatorname{erf}(x) &= \tfrac{2}{\sqrt{\pi}}(x - \tfrac{x^3}{3.1!} + \tfrac{x^5}{5.2!} - \tfrac{x^7}{7.3!} + \ldots)
\end{aligned}$$