UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

College of Informatics and Electronics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4404

SEMESTER: Spring 2005

MODULE TITLE: Partial Differential Equations

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Prof. S. O'Brien

PERCENTAGE OF TOTAL MARKS: 90%

EXTERNAL EXAMINER: Prof. J.R. King

INSTRUCTIONS TO CANDIDATES: **Full marks for 5 questions. Number each question carefully** *in the margin provided on your script* **.**

N.B.There are some useful results at the end of the paper .

1 Consider the longitudinal oscillations of a column of air in a long thin pipe of length l. Let $u(x,t)$ be the displacement of air about its mean position and assume an adiabatic relationship between the pressure, $p(x,t)$ and the density $\rho(x,t)$ of the form:

$$p = k\rho^{\gamma}$$

where ρ, k are given constants.

(a) Prove that the oscillations are governed by the wave equation $u_{tt} = c^2 u_{xx}$, $c^2 = \frac{dp}{d\rho} = \gamma k p^{\gamma-1}$. 6 %

(b) If the two ends of the pipe are closed, the air is initially in its mean position, and the initial velocity is $g(x)$, formulate the mathematical problem for the subsequent displacement $u(x,t)$. 4 %

(c) Use separation of variables to find an expression for $u(x,t)$ in the form of an infinite series. 6 %

(d) If the end $x = l$ is open rather than closed, derive the appropriate boundary condition. 2 %

2 (a) Using the result stated in question 7(b), classify each of the following PDEs (hyperbolic, parabolic or elliptic) and find and **sketch** the characteristics: $u_{xx} = u_t + u$; $u_{xx} = u_{tt} + u^2$; $u_{xx} + y^2 u_{yy} = 0$. 6%

(b) Verify whether or not $u = \sin x \exp(-t)$ and $u = \cos x \exp(-2t)$ are solutions of the diffusion equation $u_{xx} = u_t$. 6%

(c) If u_1 and u_2 are any solutions of a linear homogeneous PDE $L[u] = 0$ in some region, prove that $c_1 u_1 + c_2 u_2$ is also a solution where c_1, c_2 are arbitrary constants. 6 %

3 The transverse oscillations of a suspension bridge about its equilibrium position can be approximately modelled using an elastic string model where the bridge has mass per unit length ρ and is stretched to a length l with its two ends fixed. Let $u(x,t)$ be any subsequent vertical deviation from the equilibrium position and T be the tension. The bridge is disturbed so that its initial deviation is $f(x)$ and is then released **from rest** whereupon it undergoes small transverse oscillations.

(a) With the aid of the results on the last page, derive a partial differential equation describing the oscillations of the bridge and show that $u(x,t)$ must satisfy the wave equation $u_{tt} = c^2 u_{xx}$, $c^2 = T/\rho$. 5 %

(b) *Formulate* the mathematical problem modelling the oscillations of the bridge. 4 %

(c) Use separation of variables to find an expression for the subsequent oscillations in the form of an infinite series. 5 %

(d) Suppose that the effects of the suspending cables is included in the modelling process so that the governing partial differential equation describing the oscillations is $u_{tt} = c^2 u_{xx} - \alpha^2 u$ where T is the tension, $c^2 = T/\rho$, and α is a given constant. Derive a solution to the problem using separation of variables. 4 %

4 A viscous fluid, initially at rest, occupies the region between two horizontal planes $x = 0, x = h$ and is set in motion by the movement of the plane $x = 0$ with known velocity U in the y direction. The plane $x = h$ does not move. The fluid velocity vector is of the form $(u(x,t), 0, 0)$ where $u(x,t)$ satisfies

$$u_t = c^2 u_{xx};\ u(x,0) = 0;\ u(0,t) = U;\ u(x = h, t) = 0$$

where c^2 is the known fluid viscosity.

(a) Formulate the steady state problem and find the steady state velocity profile. 6 %

(b) By writing the velocity field in the form $u(x,t) = v(x) + w(x,t)$ **formulate** and solve the problem for $u(x,t)$ using separation of variables. 10 %

(c) Describe **in words** a heat flow problem which is modelled by the above mathematical problem. 2 %

5 Consider the diffusion of bacteria in a thin rectangular region (whose lateral faces are sealed) of dimensions $0 \le x \le a, 0 \le y \le b$. Assume that the initial concentration is zero and that the edge $x = a, 0 \le y \le b$ is maintained at a concentration $f(y)$ while the other edges are maintained at zero concentration. Assume that the concentration $u(x,y,t)$ is modelled by the diffusion equation

$$u_t = c^2(u_{xx} + u_{yy})$$

where c^2 is a known constant.

(a) Formulate the initial-boundary value problem for $u(x,y,t)$. 4 %

(b) Explain why the appropriate steady state equation is Laplace's equation. Hence formulate the boundary value problem for the **steady state** concentration $u(x,y)$. 4 %

(c) Solve the boundary value problem for the concentration in the form:

$$u(x,y) = \sum_{n=1}^{\infty} E_n \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}$$

and give a general expression for the superposition constants E_n. 6%

(d) Consider the problem where the boundary conditions are $u = f(y), x = a, 0 \leq y \leq b$ and $u = g(x), 0 \leq x \leq a, y = b$ with the other edges maintained at zero concentration. Obtain a general solution to this problem. 4%

6 (a) Use the method of characteristics to solve the following first order PDE problems:

$$\begin{aligned} xu_x + yu_y &= u, \ u(1,y) = y^2; \\ u_x + u^3 u_y &= 0, \ u(x,0) = \cos x. \end{aligned}$$

9%

(b) In a one dimensional transport problem, let $c(x,t)$ be the concentration and $q(x,t)$ be the flux (rate of flow). By performing a balance over an infinitesimal element of length dx derive the conservation law $c_t + q_x = 0$.

Suppose that the flow of ice in a glacier is described by such a conservation law where $c(x,t)$ is the ice thickness and $q(x,t) = c^2$. Show that the flow of the glacier is governed by $c_t + 2cc_x = 0$. If at some particular time $t = 0$ the ice thickness is given by $c(x,0) = \exp(-x^2)$ find a general expression for $c(x,t)$ and explain briefly how the profile of the glacier will develop at later times. 9%

7 Answer any **three** of the following:

(a) Give a physical argument to show that for steady state heat flow in a region, both the maximum and minimum must occur on the boundary.

Consider the steady state heat flow problem $u_{xx} + u_{yy} = 0, 0 < x < 1, 0 < y < 1$ with boundary conditions $u(x,0) = x(x-1), 0 \leq x \leq 1$ with $u = 0$ on the other boundaries. What is the hottest point on the plate and at what point (x,y) does it occur? 6%

(b) Consider the second order PDE

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} = f(x,y,u_x,u_y,u)$$

and related Cauchy boundary conditions $u, \frac{\partial u}{\partial n}$ along a curve $\mathbf{r}(s) = (x(s), y(s))$.

Prove that the characteristics are given by:

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}.$$

6 %

(c) Assuming that $u = u(x, y), v = v(x, y)$ find the general solution of the following pseudo PDEs:

$$u_{xx} = 2;\ v_{xx} + 3v = x;$$

6 %

(d) Find a change of dependent variable which reduces the parabolic PDE: $u_{xx} + 4u_x - 2u_t + 8u = 0$ to a one-dimensional heat equation $u_t = c^2 u_{xx}$. (Hint: $u(x,t) = v(x,t)\exp(\alpha x + \beta t)$). 6 %

(e) Consider oscillations on an infinite string governed by $u_{tt} = c^2 u_{xx}$ with initial (Cauchy) conditions $u(x, 0) = f(x)$, $u_t(x, 0) = 0$ where $f(x)$ is a known function. Show that a general solution can be written in the form $u(x,t) = \phi(x + ct) + \psi(x - ct)$ where ϕ, ψ are arbitrary sufficiently smooth functions. Hence derive D'Alembert's solution to the particular problem by finding ϕ, ψ. 6 %

(f) Use a Laplace transform in the t variable to solve the following problem in $0 \leq x < \infty$

$$u_t + u_x = 0;\ u(0,t) = \exp t;\ u(x, 0) = \exp(-x).$$

(You may use the results at the end of the paper). 6 %

Useful equations

Half range Fourier series for odd and even function $f(x)$ of period $2l$.

$$f(x) = \sum_{n=1}^{\infty} A_n \sin n\pi x/l;\;\; A_n = \frac{2}{l}\int_0^l f(x)\sin n\pi x/l\, dx$$

$$f(x) = B_0/2 + \sum_{n=1}^{\infty} B_n \cos n\pi x/l;\;\; B_n = \frac{2}{l}\int_0^l f(x)\cos n\pi x/l\, dx$$

Half range orthogonality relationships

$$\begin{aligned}\int_0^l \sin n\pi x/l \sin m\pi x/l\, dx &= 0,\ (m \neq n)\\ &= l/2,\ (m = n \neq 0)\\ &= 0,\ (m = n = 0)\end{aligned}$$

$$\begin{aligned}\int_0^l \cos n\pi x/l \cos m\pi x/l\, dx &= 0,\ (m \neq n)\\ &= l/2,\ (m = n \neq 0)\\ &= l,\ (m = n = 0)\end{aligned}$$

Laplace transforms

$$\begin{aligned}\mathbf{L}[f(t)] &\equiv \int_0^\infty f(t)\exp(-st)\, dt\\ \mathbf{L}[\frac{df(t)}{dt}] &= s\bar{f}(s) - f(0)\\ \mathbf{L}[\exp ct] &= \frac{1}{s-c}.\end{aligned}$$

Trigonometric results

$$\cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}}; \sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$$