UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

College of Informatics and Electronics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4404

SEMESTER: Spring 2003-04

MODULE TITLE: Partial Differential Equations

DURATION OF EXAMINATION: 2 hours 30 mins

LECTURER: Prof. S. O'Brien

PERCENTAGE OF TOTAL MARKS: 90%

EXTERNAL EXAMINER: Prof. J.D. Gibbon

INSTRUCTIONS TO CANDIDATES: **Full marks for 5 questions. Number each question carefully** *in the margin provided on your script* **.**

The subscript notation is used intermittently to denote a partial derivative (e.g. $u_x \equiv \frac{\partial u}{\partial x}$**).**
N.B.There are some useful results at the end of the paper .

1 Answer any **three** of the following:

(a) Consider the transverse oscillations of a long elastic string $-\infty < x < \infty$. Assume that the oscillations are governed by $u_{tt} = c^2 u_{xx}$ where the configuration of the string at $t = 0$ is given by $\sin x$ with zero initial velocity .

Formulate the mathematical problem and using the change of variables $\eta = x + ct$; $\xi = x - ct$ or otherwise, show that a general solution can be written in the form $u(x,t) = \phi(x + ct) + \psi(x - ct)$ where ϕ, ψ are arbitrary sufficiently smooth functions. Hence derive D'Alembert's solution to the particular problem by finding ϕ, ψ. 6 %

(b) Air undergoes longitudinal oscillations in a pipe of length l. If the oscillations are governed by the linear wave equation $u_{tt} = c^2 u_{xx}$ with both ends of the tube *open*, **formulate and solve** the problem for the subsequent oscillations if the air is initially in its equilibrium position with initial velocity $u_t(x, 0) = g(x)$. 6 %

(c) Assuming that $u = u(x, y)$ find the general solution of the following pseudo PDEs:

$$u_{xx} = 2;\ u_{xx} + 4u = 0;\ u_{xx} = \sin x.$$

6 %

(d) Show that

$$u = \frac{A}{t^{1/2}} \exp \frac{-x^2}{4c^2 t}$$

is a solution of the linear diffusion equation $u_t = c^2 u_{xx}$ where A is a constant. Give a physical interpretation of this solution paying particular attention to $x \neq 0, t \to 0$ and $x = 0, t \to 0$. 6 %

(e) Consider the Laplace problem with **Cauchy** boundary conditions:

$$u_{xx} + u_{yy} = 0\ (y \geq 0);\ u(x, 0) = 0;\ u_y(x, 0) = \frac{\sin nx}{n}.$$

Show that the solution is given by $u(x, y) = \frac{1}{n^2} \sinh ny \sin nx$.

By consideration of the limiting case $n \to \infty$, show that the above problem is not well posed. 6 %

2 (a) Consider the second order PDE

$$A(x, y)u_{xx} + 2B(x, y)u_{xy} + C(x, y)u_{yy} = f(x, y, u_x, u_y, u)$$

and related Cauchy boundary conditions $u, \frac{\partial u}{\partial n}$ along a curve $\mathbf{r}(s) = (x(s), y(s))$. **Prove** carefully that the characteristics are given by:

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A},$$

and suggest a means of classifying equations (hyperbolic, parabolic, elliptic). 10 %

(b) Classify each of the following PDEs (hyperbolic, parabolic or elliptic). Find and **sketch** the characteristics (if any):

$$u_{xx} = u_t + \sin u;\; u_{xx} = u_{tt};\; u_{xx} + 4u_{yy} = 0.$$

4%

(c) If u_1 and u_2 are any solutions of a linear homogeneous PDE $L[u] = 0$ in some region, prove that $c_1u_1 + c_2u_2$ is also a solution where c_1, c_2 are arbitrary constants. 4 %

3 The oscillations of a suspension bridge about its equilibrium position can be approximately modelled using an elastic string model where the bridge, of length l, has mass per unit length ρ with its two ends fixed. Let $u(x,t)$ be any subsequent vertical deviation from the equilibrium position and T be the tension. The bridge is disturbed so that its initial deviation is given by a continuous function $f(x), 0 \leq x \leq l$ and is then released **from rest** whereupon it undergoes small transverse oscillations. Gravitational effects are ignored.

(a) With the aid of the results on the last page, derive a partial differential equation describing the oscillations of the bridge and show that $u(x,t)$ must satisfy $u_{tt} = c^2u_{xx}$ where T is the tension and $c^2 = T/\rho$. 5%

(b) **Formulate** the mathematical problem modelling the oscillations. 4%

(c) Use separation of variables to find an expression for the subsequent oscillations in the form of an infinite series. 6%

(d) If the initial shape of the string is given by $f(x) = 2\sin\frac{7\pi x}{2l}\cos\frac{7\pi x}{2l}$ find an explicit expression for $u(x,t)$ and describe the subsequent oscillations by sketching the string at a series of times. 3 %

4 (a) Heat diffuses according to the linear diffusion equation $u_t = c^2u_{xx}$ along an infinite bar whose initial temperature is given by $f(x), -\infty \leq x < \infty$.

(i) Using the Fourier heat law $Q = -\kappa\frac{\partial u}{\partial x}$, where Q is the heat flux and κ is a material constant, derive the diffusion equation. 4 %

(ii) **Formulate** the mathematical problem for the flow of heat and use separation of variables and the results on the last page to obtain an expression for the subsequent temperature at any point along the bar in the form:

$$u(x,t) = \int_0^\infty [A(p)\cos px + B(p)\sin px]\, dp,$$

and derive expressions for $A(p), B(p)$. 9 %

(b) Consider the one dimensional diffusion problem $u_t = c^2 u_{xx}, 0 \leq x \leq l$ with prescribed initial and boundary (Dirichlet) conditions. **Prove** that $u(x,t)$ attains its maximum on the boundary of the domain of the independent variables defined by $t = 0 (0 \leq x \leq l), x = 0 (t > 0), x = l (t > 0)$. 5 %

5 Consider the diffusion of bacteria in a thin rectangular plate (whose lateral faces are sealed) of dimensions $0 \leq x \leq a, 0 \leq y \leq b$ with three of its edges maintained at zero concentration while on the boundary $x = a, 0 \leq y \leq b$ there is a source of bacteria modelled by the condition $c(a,y) = f(y)$. Assume that the subsequent diffusion of bacteria is modelled by

$$u_t = c^2(u_{xx} + u_{yy})$$

where c^2 is the known diffusion constant. The initial concentration profile in the plate is given by $f(x,y)$.

(a) **Formulate** the initial boundary value problem for the subsequent diffusion problem for $u(x,y,t)$. 4 %

(b) Suppose the plate is left undisturbed for a period of time until a steady state is reached so that $\partial/\partial t = 0$. **Formulate** the boundary value problem for the steady state problem for $u(x,y)$. 4 %

(c) Solve the steady state problem and show that

$$u(x,y) = \sum_{n=1}^{\infty} E_n \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}$$

and deduce an expression for the constants E_n for the case $f(y) = 100$.

6 %

(d) Suppose the boundary conditions of the problem were changed so that the concentration of bacteria on each edge was given by:

$$u(x,0) = 100;\ u(x,b) = 400;\ u(0,y) = 100;\ \ u(a,y) = 200.$$

Solve the resulting problem for the steady state concentration $u(x,y)$.

4%

6 A circular compact disk (of radius R) whose upper and lower faces are insulated is so thin that heat flow in the disk can be considered two dimensional. In terms of polar coordinates, the flow of heat is governed by:

$$u_t = c^2(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}), \tag{1}$$

where $u(r, \theta)$ is the temperature at any point and c^2 is the thermal diffusivity of the disk.

The initial temperature in the disk is $f(r), 0 \leq r \leq R$ and the edges of the disc are maintained at zero degrees.

(a) Explain how the governing equation can be simplified using symmetry and formulate the initial boundary value problem modelling the flow of heat in the disk with $u = u(r, t)$. 5 %

(b) Show that Bessel's equation (see end of paper) arises on using separation of variables and solve the resulting problem. 9 %

(c) Starting with the heat equation in Cartesians: $u_t = c^2(u_{xx} + u_{yy})$, derive the polar coordinate form (1) of the heat equation. 4 %

7 (a) Let $c(x, t)$ be the density of cars on a road where $q(x, t)$ is the flux (rate of flow). By performing a balance over an infinitesimal portion of the road show that car conservation requires that $c_t + q_x = 0$.

For a particular road the flux is given by: $q(x, t) = 10^2 - (c - 10)^2$. Show that the flow of cars is governed by $c_t - 2(c - 10)c_x = 0$. If at some particular time $t = 0$ the traffic density is described by $c(x, 0) = \exp(-x^2)$ find a general expression for $c(x, t)$. Explain briefly how shocks can develop. What determines the direction of the basic wave motion? 9 %

(b) Use the method of characteristics to solve the following first order PDE problems:

$$2u_x + u_y = 1,\ u(0, y) = y + 5;\ \ u_t + uu_x + u = 0,\ u(x, 0) = -\frac{x}{2}.$$

9 %

8 (a) The temperature profile $T(x, t)$ of a semi-infinite thin metal bar is initially at $T(x, 0) = 0, 0 \leq x < \infty$. At time $t = 0$ the temperature at $x = 0$ is suddenly increased and maintained at a constant temperature of 50 degrees. Assuming that the heat flow is governed by the linear diffusion equation $T_t = c^2T_{xx}, 0 \leq x < \infty$ and that the temperature

satisfies $T_x = 0 (x \to \infty)$ **formulate** the mathematical problem. Use a Fourier sine transform (see end of paper) to find an expression for the subsequent temperature profile $T(x, t)$ in the form of an integral. 9%

(b) Laplace's equation $u_{xx} + u_{yy} = 0$ holds in a rectangular region $-\infty < x < \infty, 0 \leq y < \infty$ with the boundary condition $u(x, 0) = \sin x$ while on the other boundaries $u = 0$. Formulate and solve this problem using a full Fourier transform (see end of paper) with respect to the x variable and show that the solution is given by:

$$u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp i\omega x \exp(-\omega y) \left[\int_{-\infty}^{\infty} \exp(-i\omega x) \sin x dx \right] d\omega.$$

9 %

Useful results

Half range Fourier series for odd and even function $f(x)$ of period $2l$.

$$f(x) = \sum_{n=1}^{\infty} A_n \sin n\pi x/l; \;\; A_n = \frac{2}{l}\int_0^l f(x) \sin n\pi x/l \, dx$$

$$f(x) = B_0/2 + \sum_{n=1}^{\infty} B_n \cos n\pi x/l; \;\; B_n = \frac{2}{l}\int_0^l f(x) \cos n\pi x/l \, dx$$

Half range orthogonality relationships

$$\begin{aligned}\int_0^l \sin n\pi x/l \sin m\pi x/l \, dx &= 0, \, (m \neq n) \\ &= l/2, \, (m = n \neq 0) \\ &= 0, \, (m = n = 0)\end{aligned}$$

$$\begin{aligned}\int_0^l \cos n\pi x/l \cos m\pi x/l \, dx &= 0, \, (m \neq n) \\ &= l/2, \, (m = n \neq 0) \\ &= l, \, (m = n = 0)\end{aligned}$$

Bessel's equation; orthogonality of Bessel functions

$$\begin{aligned}\frac{d^2W}{ds^2} + \frac{1}{s}\frac{dW}{ds} + W &= 0; \;\; W = AJ_0(s) + BY_0(s). \\ \int_0^R r J_0(\frac{\alpha_m r}{R}) J_0(\frac{\alpha_p r}{R}) dr &= 0; \;\; m \neq p, \\ &= \frac{R^2}{2[J_1(\alpha_m)]^2}; \;\; m = p.\end{aligned}$$

Fourier integral

$$\begin{aligned}f(x) &= \int_0^\infty [A(p) \cos px + B(p) \sin px] \, dp \\ A(p) &= \frac{1}{\pi}\int_0^\infty f(v) \cos pv \, dv, B(p) = \frac{1}{\pi}\int_0^\infty f(v) \sin pv \, dv.\end{aligned}$$

Fourier sine transform

$$\mathbf{F}_s(f) \equiv \hat{f}_s(\omega) = \int_0^\infty f(x) \sin \omega x dx; \;\; f(x) = \frac{2}{\pi}\int_0^\infty \hat{f}_s(\omega) \sin \omega x d\omega$$

$$\mathbf{F}_s(f_{xx}) = -\omega^2 \hat{f}_s(\omega) + \omega f(0).$$

Fourier transform and inverse transform

$$\mathbf{F}[g] = \hat{g}(\omega) = \int_{-\infty}^{\infty} g(x) \exp(-i\omega x) dx; \;\; \mathbf{F}^{-1}[\hat{g}(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(\omega) \exp(i\omega x) d\omega$$

$$\mathbf{F}(f_{xx}) = -\omega^2 \hat{f}(\omega)$$

Trigonometric results

$$\cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}}; \sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$$