UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

College of Informatics and Electronics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4404

SEMESTER: Spring 2003

MODULE TITLE: Partial Differential Equations

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Prof. S. O'Brien

PERCENTAGE OF TOTAL MARKS: 90%

EXTERNAL EXAMINER: Prof. J.D. Gibbon

INSTRUCTIONS TO CANDIDATES: **Full marks for 5 questions. Number each question carefully** *in the margin provided on your script* **.**

The subscript notation is used intermittently to denote a partial derivative (e.g. $u_x \equiv \frac{\partial u}{\partial x}$**).**
N.B.There are some useful results at the end of the paper .

1 (a) Consider the second order PDE

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} = f(x,y,u_x,u_y,u)$$

and related Cauchy boundary conditions $u, \frac{\partial u}{\partial n}$ along a curve $\mathbf{r}(s) = (x(s), y(s))$.

Prove carefully that the characteristics are given by:

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}.$$

and suggest a means of classifying equations (hyperbolic, parabolic, elliptic). 10 %

(b) Classify each of the following PDEs (hyperbolic, parabolic or elliptic) and find and **sketch** the characteristics:

$$u_{xx} = u_t + u^3;\ u_{xx} = 4u_{tt} + u^2;\ u_{xx} + y^2 u_{yy} = 0.$$

4%

(c) If u_1 and u_2 are any solutions of a linear homogeneous PDE $L[u] = 0$ in some region, prove that $c_1u_1 + c_2u_2$ is also a solution where c_1, c_2 are arbitrary constants. 4 %

2 The oscillations of a guitar string about its equilibrium position can be approximately modelled using an elastic string model where the string has mass per unit length ρ and is stretched to a length l with its two ends fixed. Let $u(x,t)$ be any subsequent vertical deviation from the equilibrium position and T be the tension. The string is disturbed so that its initial deviation is $f(x)$ and is then released **from rest** whereupon it undergoes small transverse oscillations. Gravitational effects are ignored.

(a) With the aid of the results on the last page, derive a partial differential equation describing the oscillations of the string and show that $u(x,t)$ must satisfy $u_{tt} = c^2 u_{xx}$ where T is the tension and $c^2 = T/\rho$. 5%

(b) **Formulate** the mathematical problem modelling the oscillations in the string. 4%

(c) Use separation of variables to find an expression for the subsequent oscillations in the form of an infinite series. 6%

(d) If the initial shape of the string is given by $f(x) = 2\sin\frac{5\pi x}{2l}\cos\frac{5\pi x}{2l}$ find an explicit expression for $u(x,t)$ and describe the subsequent oscillations by sketching the string at a series of times. 3 %

3 (a) Heat diffuses according to the linear diffusion equation $u_t = c^2 u_{xx}$ along an infinite bar whose initial temperature is given by $f(x), -\infty \leq x < \infty$. **Formulate** the mathematical problem and use separation of variables and the results on the last page to obtain an expression for the subsequent temperature at any point along the bar in the form:

$$u(x,t) = \int_0^\infty \left[A(p)\cos px + B(p)\sin px\right] dp,$$

and derive expressions for $A(p), B(p)$. 9 %

(b) Consider the longitudinal oscillations of air in a thin infinite tube $0 \leq x < \infty$. Assume that the oscillations are governed by $u_{tt} = c^2 u_{xx}$ where the air at $t = 0$ is in the position $f(x)$ relative to its equilibrium position with initial velocity $g(x)$. **Formulate** the mathematical problem and using the change of variables $\eta = x + ct$; $\xi = x - ct$ or otherwise, show that a general solution can be written in the form $u(x,t) = \phi(x+ct) + \psi(x-ct)$ where ϕ, ψ are arbitrary sufficiently smooth functions. Hence derive D'Alembert's solution to the particular problem by finding ϕ, ψ. 9 %

4 Consider the diffusion of bacteria in a thin rectangular plate (whose lateral faces are sealed) of dimensions $0 \leq x \leq a, 0 \leq y \leq b$ with its edges maintained at zero concentration. Assume that the subsequent diffusion of bacteria is modelled by

$$u_t = c^2(u_{xx} + u_{yy})$$

where c^2 is the known diffusion constant. The initial concentration profile in the plate is given by $f(x,y)$.

(a) **Formulate** the initial boundary value problem for the subsequent diffusion problem. 4 %

(b) Solve the initial boundary value problem for the subsequent concentration profile in the plate in the form:

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} \exp\left(-c^2(m^2+n^2)t\right) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

and give general expressions for the superposition constants. 10%

(c) Consider the same problem, with one change in the boundary conditions so that the right hand edge $x = a, 0 \leq y \leq b$ is sealed (so the flux is zero) rather than maintained at zero concentration. Using symmetry arguments or otherwise, sketch a technique for solving this problem. 4%

5 In a thin cylinder of length l containing stagnant water the concentration of coloured dye in the water is given by $u(x,t)$. Assume that the initial concentration is given by $u(x,0) = x^2(l-x), 0 \le x \le l$. Assume also that the concentration $u(x,t)$ is maintained at zero at each end of the tube.

(a) Assuming that the dye diffuses according to Fick's law :

$$Q = -c^2 \frac{\partial u}{\partial x}$$

where $Q(x,t)$ is the concentration flux, c^2 is the diffusion coefficient and $u(x,t)$ is the concentration at position x at time t, show that the transport of dye through the stagnant water $0 \le x \le l$ is governed by the linear diffusion equation:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2};\ 0 \le x \le l, t > 0$$

5 %

(b) **Formulate** an initial boundary value problem describing the diffusion problem in the region $0 \le x \le l$. 4%

(c) Solve this problem using separation of variables showing that the concentration in $0 \le x \le l$ is given by:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \exp\left(-\lambda_n^2 t\right)\ (\lambda_n = cn\pi/l)$$

and derive an explicit expression for the B_n. 5 %

(d) If dye is able to diffuse out along the lateral face of the tube, the process is modelled by $u_t = c^2 u_{xx} + \phi(x,t)$ where $\phi(x,t)$ is a known function. For the case $\phi = 1$ outline how to solve this inhomogeneous problem using the same initial and boundary conditions as above. 4 %

6 (a) Use the method of characteristics to solve the following first order PDE problems:

$$2u_x + u_y = 1,\ u(0,y) = y + 5;\quad u_t + uu_x + u = 0,\ u(x,0) = -\frac{x}{2}$$

9 %

(b) Let $h(x,t)$ be the thickness of ice in a flowing glacier of infinite extent in the lateral y direction and let $q(x,t)$ be the flux (rate of flow). By performing a balance over an infinitesimal portion of the glacier show that mass conservation requires that $h_t + q_x = 0$.

For a particular glacier the flux is given by: $q(x,t) = h^3$. Show that the flow of the glacier is governed by $h_t + 3h^2 h_x = 0$. If at some particular time $t = 0$ the glacier profile is described by $h(x,0) = \exp(-x^2)$ find a general expression for $h(x,t)$ and explain briefly how the profile will develop at later times. How might the modelling of this problem change if the effect of snow fall was to be taken into account? 9 %

7 Answer any **three** of the following:

(a) Air undergoes longitudinal oscillations in a pipe of length l. If the oscillations are governed by the linear wave equation $u_{tt} = c^2 u_{xx}$ with both ends of the tube **open**, formulate and solve the problem for the subsequent oscillations if the air is initially in its equilibrium position with initial velocity $u_t(x,0) = g(x)$. 6 %

(b) Show that the Laplace transform of $\exp ct$ is $\frac{1}{s-c}$ where c is a constant. Use a Laplace transform in the t variable to solve the following problem in $0 \leq x < \infty$

$$u_t + u_x = 0;\ u(0,t) = \exp ct;\ u(x,0) = \exp(-x).$$

(You may use the results at the end of the paper). 6 %

(c) Laplace's equation $u_{xx} + u_{yy} = 0$ holds in a rectangular region $-\infty < x < \infty, 0 \leq y < \infty$ with the boundary condition $u(x,0) = f(x)$ while on the other boundaries $u = 0$. Solve this problem using a Fourier transform with respect to the x variable and show that the solution is given by:

$$u(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp i\omega x \exp(-\omega y) \left[\int_{-\infty}^{\infty} \exp(-i\omega x) f(x) dx \right] d\omega.$$

6 %

(d) Consider the one dimensional diffusion problem $u_t = c^2 u_{xx}$ with prescribed initial and boundary (Dirichlet) conditions. **Prove** that $u(x,t)$ attains its maximum on the boundary of the domain of the independent variables defined by $t = 0 (0 \leq x \leq l), x = 0 (t > 0), x = l (t > 0)$. 6 %

(e) Show that $u(x,t) = \frac{a}{\sqrt{t}} \exp \frac{-x^2}{4c^2 t}$ is a solution of the linear diffusion equation $u_t = c^2 u_{xx}$. Examine the behaviour of this solution as $t \to 0$ and explain why it represents a point source solution. 6 %

(f) Assuming that $u = u(x, y)$ find the general solution of the following pseudo PDEs:

$$u_{xx} = 2;\ u_{xx} + 4u = 0;\ u_{xx} = \sin x.$$

6 %

(g) Find a change of dependent variable which reduces the parabolic PDE: $u_{xx} + 4u_x - 2u_t + 8u = 0$ to a one-dimensional heat equation $u_t = c^2 u_{xx}$. (Hint: $u(x,t) = v(x,t)\exp(\alpha x + \beta t)$). 6 %

Useful equations

Half range Fourier series for odd and even function $f(x)$ of period $2l$.

$$f(x) = \sum_{n=1}^{\infty} A_n \sin n\pi x/l; \;\; A_n = \frac{2}{l}\int_0^l f(x)\sin n\pi x/l\, dx$$

$$f(x) = B_0/2 + \sum_{n=1}^{\infty} B_n \cos n\pi x/l; \;\; B_n = \frac{2}{l}\int_0^l f(x)\cos n\pi x/l\, dx$$

Half range orthogonality relationships

$$\begin{aligned}\int_0^l \sin n\pi x/l \sin m\pi x/l\, dx &= 0,\ (m \neq n)\\ &= l/2,\ (m = n \neq 0)\\ &= 0,\ (m = n = 0)\end{aligned}$$

$$\begin{aligned}\int_0^l \cos n\pi x/l \cos m\pi x/l\, dx &= 0,\ (m \neq n)\\ &= l/2,\ (m = n \neq 0)\\ &= l,\ (m = n = 0)\end{aligned}$$

Fourier integral

$$\begin{aligned}f(x) &= \int_0^\infty [A(p)\cos px + B(p)\sin px]\, dp\\ A(p) &= \frac{1}{\pi}\int_0^\infty f(v)\cos pv\, dv, B(p) = \frac{1}{\pi}\int_0^\infty f(v)\sin pv\, dv.\end{aligned}$$

Fourier transform and inverse transform

$$\mathbf{F}[g] = \hat{g}(\omega) = \int_{-\infty}^{\infty} g(x)\exp(-i\omega x)dx; \;\; \mathbf{F}^{-1}[\hat{g}(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(\omega)\exp(i\omega x)d\omega$$

$$F[\frac{d^2 f}{dx^2}] = (i\omega)^2 F[f(x)].$$

Laplace transform

$$L[f(t)] \equiv \bar{f}(s) \equiv \int_0^\infty f(t)\exp(-st)dt$$

$$\mathbf{L}[c] = \frac{c}{s},\ \text{c, a constant}$$

$$
\begin{aligned}
\mathbf{L}[\frac{df(t)}{dt}] &= s\bar{f}(s) - f(0) \\
\mathbf{L}[\frac{d^2f(t)}{dt^2}] &= s^2\bar{f}(s) - sf(0) - f'(0)
\end{aligned}
$$

Trigonometric results

$$
\cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}}; \sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}
$$