UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

College of Informatics and Electronics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4404

SEMESTER: Spring 2002

MODULE TITLE: Partial Differential Equations

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Prof. S. O'Brien

PERCENTAGE OF TOTAL MARKS: 100%

EXTERNAL EXAMINER: Prof. J.D. Gibbon

INSTRUCTIONS TO CANDIDATES: **Full marks for 5 questions. Number each question carefully** *in the margin provided on your script* **.**

The subscript notation is used intermittently to denote a partial derivative (e.g. $u_x \equiv \frac{\partial u}{\partial x}$).
N.B.There are some useful results at the end of the paper .

1 (a) Consider the second order PDE

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} = f(x,y,u_x,u_y,u)$$

and related Cauchy boundary conditions $u, \frac{\partial u}{\partial n}$ along a curve $\mathbf{r}(s) = (x(s), y(s))$.

Prove carefully that the characteristics are given by:

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}.$$

and suggest a means of classifying equations (hyperbolic, parabolic, elliptic). 12 %

(b) Classify each of the following PDEs (hyperbolic, parabolic or elliptic) and find and **sketch** the characteristics: $u_{xx} = u_t + 3u;\ u_{xx} = 16u_{tt} + u^2;\ u_{xx} + x^2 u_{yy} = 0$. 4%

(c) Verify that $u = x^3 - 3xy^2$ and $u = 3x^2y - y^3$ are solutions of the Laplace equation $u_{xx} + u_{yy} = 0$. 4%

2 The oscillations of a suspension bridge about its equilibrium position can be approximately modelled using an elastic string model where the bridge has mass per unit length ρ and is stretched to a length l with its two ends fixed. Let $u(x,t)$ be any subsequent vertical deviation from the equilibrium position and T be the tension. The bridge is disturbed so that its initial deviation is $f(x)$ and is then released **from rest** whereupon it undergoes small transverse oscillations.

(a) With the aid of the results on the last page, derive a partial differential equation describing the oscillations of the bridge and show that $u(x,t)$ must satisfy $u_{tt} = c^2 u_{xx}$ where T is the tension and $c^2 = T/\rho$. 6%

(b) *Formulate* the mathematical problem modelling the oscillations in the bridge. 4%

(c) Use separation of variables to find an expression for the subsequent oscillations in the form of an infinite series. 6%

(d) If the initial shape of the string is given by $f(x) = 2\sin\frac{3\pi x}{2l}\cos\frac{\pi x}{2l}$ find an explicit expression for $u(x,t)$ and describe the subsequent oscillations by sketching the string at a series of times. 4 %

3 Consider the flow of heat in a thin rectangular plate (whose lateral faces are insulated) of dimensions $0 \le x \le a, 0 \le y \le b$ with its edges maintained at zero degrees. Assume that the subsequent flow of heat is modelled by

$$u_t = c^2(u_{xx} + u_{yy})$$

where c^2 is the known thermal diffusivity. The plate is heated so that its initial temperature profile is given by $f(x,y)$.

(a) Formulate the initial boundary value problem for the subsequent heat flow problem. 4 %

(b) Solve the initial boundary value problem for the subsequent temperature in the plate in the form:

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} \exp\left(-c^2(m^2+n^2)t\right) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

and give general expressions for the superposition constants. 13%

(c) Formulate **but do not solve** the corresponding steady state heat flow problem for $u = u(x,y)$. From physical considerations what will the steady state solution be? 3%

4 In a thin cylinder of wood of length l the concentration of bacteria is given by $u(x,t)$. Assume that the initial concentration of bacteria is given by $u(x,0) = x^2(l-x), 0 \leq x \leq l$. Assume also that the concentration $u(x,t)$ is maintained at zero at each end of the tube.

(a) Assuming that the bacteria diffuse according to Fick's law :

$$Q = -c^2\frac{\partial u}{\partial x}$$

where $Q(x,t)$ is the concentration flux, c^2 is the diffusion coefficient and $u(x,t)$ is the concentration at position x at time t, show that the transport of bacteria in the tube $0 \leq x \leq l$ is governed by the linear diffusion equation:

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2};\ 0 \leq x \leq l, t > 0$$

6%

(b) *Formulate* an initial boundary value problem describing the diffusion of bacteria in the region $0 \leq x \leq l$. 4%

(c) Solve this problem using separation of variables showing that the concentration in $0 \leq x \leq l$ is given by:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \exp\left(-\lambda_n^2 t\right)\ (\lambda_n = cn\pi/l)$$

and derive an explicit expression for the B_n. 6%

(d) If bacteria are able to diffuse out along the lateral face of the tube, the process is modelled by $u_t = c^2 u_{xx} + \phi(x,t)$ where $\phi(x,t)$ is a known function. For the case $\phi = 1$ outline how to solve this inhomogeneous problem using the same initial conditions as above. 4 %

5 (a) Use a Laplace transform in the t variable to solve the following problem in $0 \leq x < \infty$

$$u_{tt} = c^2 u_{xx};\ u(0,t) = f(t);\ u(x,0) = 0;\ u_t(x,0) = 0;.$$

with $u(x,t)$ bounded as $x \to \infty$. (You may use the results at the end of the paper). 10 %

(b) The Fourier sine transform and inverse transform are defined as:

$$\mathbf{F}_s(f) \equiv \hat{f}_s(\omega) = \int_0^\infty f(x) \sin \omega x dx;\ f(x) = \frac{2}{\pi} \int_0^\infty \hat{f}_s(\omega) \sin \omega x d\omega$$

in the usual notation. Prove that $\mathbf{F}_s(f_{xx}) = -\omega^2 \hat{f}_s(\omega) + \omega f(0)$.

A slender rod, whose curved surface is insulated, stretches from $x = 0$ to $x = \infty$. Assume that the flow of heat is governed by $u_t = c^2 u_{xx}$. Use a Fourier sine transform to show that the temperature in the rod as a function of x and t is given by:

$$u(x,t) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(s) \exp\left(-\omega^2 t / c^2\right) \sin \omega s \sin \omega x ds d\omega$$

if the left hand edge of the rod $(x = 0)$ is maintained at the constant temperature zero degrees and if the initial temperature is given by $u(x,0) = f(x)$. 10 %

6 (a) Use the method of characteristics to solve the following first order PDE problems:

$$u_x + u_y = 1,\ u(0,y) = y + 7;\quad u_t + u u_x + u = 0,\ u(x,0) = -\frac{x}{2}$$

10%

(b) Let $c(x,t)$ be the concentration of cars on a road and $q(x,t)$ be the flux (rate of flow). By performing a balance over an infinitesimal stretch of road show that conservation of cars requires that $c_t + q_x = 0$.

For a particular stretch of road the mathematical problem describing the flow of cars has a flux given by: $q(x,t) = 25 - c^2, 0 \leq c \leq 5$. Show that the flow of cars is governed by $c_t - 2cc_x = 0$. If at some particular time $t = 0$ the concentration of cars is given by $c(x,0) = \exp\left(-x^2\right)$ find a general expression for $c(x,t)$ and explain briefly how the concentration will develop at later times. 10%

7 Answer any **three** of the following:

(a) Consider oscillations on an infinite string governed by $u_{tt} = c^2 u_{xx}$ with initial (Cauchy) conditions $u(x,0) = f(x)$, $u_t(x,0) = 0$ where $f(x), g(x)$ are known functions. Show that a general solution can be written in the form $u(x,t) = \phi(x+ct) + \psi(x-ct)$ where ϕ, ψ are arbitrary sufficiently smooth functions. Hence derive D'Alembert's solution to the particular problem by finding ϕ, ψ. $6\frac{2}{3}$ %

(b) Consider the one dimensional diffusion problem $u_t = c^2 u_{xx}$ with prescribed initial and boundary (Dirichlet) conditions. **Prove** that $u(x,t)$ attains its maximum on the boundary of the domain of the independent variables defined by $t = 0 (0 \le x \le l), x = 0 (t > 0), x = l (t > 0)$. $6\frac{2}{3}$ %

(c) (i) If u_1 and u_2 are any solutions of a linear homogeneous PDE $L[u] = 0$ in some region, prove that $c_1 u_1 + c_2 u_2$ is also a solution where c_1, c_2 are arbitrary constants.

(ii) Assuming that $u = u(x,y)$ find the general solution of the following pseudo PDEs:

$$u_{xx} = 2;\ u_{xx} + 3u = x;$$

$6\frac{2}{3}$ %

(d) Find a change of dependent variable which reduces the parabolic PDE: $u_{xx} + 4u_x - 2u_t + 8u = 0$ to a one-dimensional heat equation $u_t = c^2 u_{xx}$. (Hint: $u(x,t) = v(x,t)\exp(\alpha x + \beta t)$). $6\frac{2}{3}$ %

(e) (i) Show that the equation $u_t + 2u_x = 0$ has a travelling wave solution and determine the wave speed.

(ii) Consider the PDE problem:

$$u_{xx} = u_t;\ u(0,t) = 1;\ u(\infty,t) = 0;\ u(x,0) = 0, x > 0.$$

Using a similarity variable $u = f(\eta)$, $\eta = x/\sqrt{t}$ show that the PDE problem reduces to a consistent ODE problem (which you need not solve). $6\frac{2}{3}$ %

(f) A heat flow problem of major importance in geophysics is to determine the effect of climactic environmental changes on the subsurface temperature. For example if a forest or area of thick vegetation is cleared, then the average yearly surface temperature will suddenly increase with the result that the subsurface temperature will increase as well. **Formulate** a mathematical model to investigate these effects assuming the relevant spatial domain is $x \in [0, \infty)$ while gradients in the temperature disappear as $x \to \infty$. Outline briefly a method for solving this problem. $6\frac{2}{3}$ %

Useful equations

Half range Fourier series for odd and even function $f(x)$ of period $2l$.

$$f(x) = \sum_{n=1}^{\infty} A_n \sin n\pi x/l;\ \ A_n = \frac{2}{l}\int_0^l f(x)\sin n\pi x/l\,dx$$

$$f(x) = B_0/2 + \sum_{n=1}^{\infty} B_n \cos n\pi x/l;\ \ B_n = \frac{2}{l}\int_0^l f(x)\cos n\pi x/l\,dx$$

Half range orthogonality relationships

$$\begin{aligned}\int_0^l \sin n\pi x/l \sin m\pi x/l\,dx &= 0,\ (m \neq n)\\ &= l/2,\ (m = n \neq 0)\\ &= 0,\ (m = n = 0)\end{aligned}$$

$$\begin{aligned}\int_0^l \cos n\pi x/l \cos m\pi x/l\,dx &= 0,\ (m \neq n)\\ &= l/2,\ (m = n \neq 0)\\ &= l,\ (m = n = 0)\end{aligned}$$

Laplace transforms

$$\begin{aligned}\mathbf{L}[\frac{df(t)}{dt}] &= s\bar{f}(s) - f(0)\\ \mathbf{L}[\frac{d^2f(t)}{dt^2}] &= s^2\bar{f}(s) - sf(0) - f'(0)\\ \mathbf{L}^{-1}[e^{-as}\bar{f}(s)] &= f(t-a)H(t-a)\end{aligned}$$

where H is the Heaviside step function.

Trigonometric results

$$\cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}}; \sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$$