UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MA4607

SEMESTER: Autumn 2012-13

MODULE TITLE: Introduction to Fluids

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Prof. S. O'Brien

PERCENTAGE OF TOTAL MARKS: 90%

EXTERNAL EXAMINER: Prof. T. Myers

INSTRUCTIONS TO CANDIDATES: **Full marks for 5 questions. There are some useful results at the end of the paper.**

1 **Streamlines, conservation of mass, rates of change**

(a) A glider, drifting at the local air velocity $\mathbf{u}$, is measuring $c(x, y, z, t)$, the concentration of a pollutant in the air. What is meant by the ordinary partial derivative $\frac{\partial C}{\partial t}$ and the convective rate of change $\frac{DC}{Dt}$? Use the chain rule to *prove* that

$$\frac{DC}{Dt} = \frac{\partial C}{\partial t} + \mathbf{u} \cdot (\nabla C).$$

If, instead of drifting, the velocity of the glider is $\mathbf{v}$, what derivative measures the rate of change of pollutant as measured from the glider? 5 %

(b) The velocity vector for a particular flow $\mathbf{u}(x, y, z, t) = (u, v, w)$ has been calculated for all values of x, y, z, t. Explain what is meant by the streamlines of the flow and show that at any particular time they are given by: $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$.
Sketch the streamlines (with direction arrows) for the flow

$$u = \alpha x, v = -\alpha y, w = 0,$$

where α is a positive constant. Let the concentration of some pollutant in the fluid be

$$c(x, y, t) = \beta x^2 y e^{-\alpha t},$$

for $y > 0$, where β is a constant. Does the pollutant concentration following a fluid element change with time?
Sketch the velocity profile for the shear flow $\mathbf{u} = (3y, 0, 0), 0 \leq y \leq 1$. 7 %

(c) Using the principle of conservation of mass and the divergence theorem, derive the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

in the usual notation. If the flow is incompressible, deduce that $\nabla \cdot \mathbf{u} = 0$.
Under what conditions does the velocity field

$$\mathbf{u} = (a_1 x + b_1 y + c_1 z)\mathbf{i} + (a_2 x + b_2 y + c_2 z)\mathbf{j} + (a_3 x + b_3 y + c_3 z)\mathbf{k}$$

represent an incompressible flow where a_i, b_i, c_i are constants? 6 %

2 **Vorticity, Bernoulli**

(a) Define vorticity. For a two dimensional flow $\mathbf{u} = (u(x, y, t), v(x, y, t), 0)$, show that the vorticity is given by $(0, 0, v_x - u_y)$. Discuss briefly how vorticity is created in viscous flows.

A body of fluid is in rigid rotation with velocity vector $\mathbf{u} = (-\Omega y, -\Omega x, 0)$ where Ω is a constant. Compute the vorticity and comment on your result. 6 %

(b) Beginning with the Euler equations in the form:

$$\frac{\partial \mathbf{u}}{\partial t} + (\nabla \times \mathbf{u}) \times \mathbf{u} = -\nabla H; \quad H \equiv p/\rho + \frac{1}{2}\mathbf{u}^2 + \chi,$$

prove the following theorems:

(i) the Bernoulli streamline theorem that if an ideal fluid is in steady flow, then H is constant along a streamline,

(ii) the second Bernoulli theorem that if an ideal fluid is in steady *irrotational* flow then H is constant throughout the whole flow field,

(iii) the vorticity equation for ideal flow in the form:

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega}.\nabla)\mathbf{u},$$

(iv) the vorticity equation for two dimensional ideal flow:

$$\frac{D\omega}{Dt} = 0.$$

12 %

3 **Non-dimensionalisation**

(a) Starting with the two dimensional steady Navier-Stokes equations, non-dimensionalise velocities, distances and the pressure using the scales $L, U, \rho U^2$ and demonstrate the occurrence of a Reynolds number $R \equiv UL/\nu$. Explain the physical significance of the Reynolds' number. What approximations might you make if

(i) $R \gg 1$,

(ii) $R \ll 1$. 7 %

(b) Prove that for flow at small Reynolds number (Stokes flow)

$$\mathbf{0} = -\nabla p + \mu \nabla^2 \mathbf{u}$$

with boundary condition $\mathbf{u} = \mathbf{f}_b(\mathbf{r})$, the flow is reversible.

Deduce that a small sphere moving parallel to a wall in Stokes flow will experience no force in the direction parallel to the wall.

Describe an experiment (involving a mechanical fish, for example) to verify the reversibility property. 7 %

(c) A two dimensional flow with velocity vector $\mathbf{u} = (u(x, y), v(x, y), 0)$ in $y \geq 0$ occurs in the vicinity of a stationary wall located along the x axis $y = 0$. Name the simplest equations of motion suitable for the flow (Euler or Navier-Stokes) and the boundary conditions for u and v on $y = 0$ if

(i) the liquid is inviscid.

(ii) the liquid is viscous. 4 %

4 Problems involving the Navier-Stokes equations

(a) Viscous fluid flows between two stationary rigid boundaries $y = \pm h$ under a constant known pressure gradient $P = -dp/dx$ i.e., $\nabla p = (-P, 0, 0)$. Show that the Navier-Stokes equations reduce to a single ordinary differential equation for $\mathbf{u} = (u(y), 0, 0)$ and that

$$u = \frac{P}{2\mu}(h^2 - y^2), v = w = 0.$$

9 %

(b) An incompressible fluid, with kinematic viscosity ν and uniform density ρ, is restricted to the region between the planes $y = 0$ and $y = b$. The fluid motion is steady and the velocity vector takes the form $\mathbf{u} = U(y)\mathbf{i} + V_0\mathbf{j}$ where V_0 is a constant.

There are no body forces and the pressure gradient is specified to take the value $\nabla p = (-G, 0)$ throughout the fluid, where G is a positive constant.

Show that $U(y)$ satisfies the equation:

$$\frac{d^2U}{dy^2} - \frac{V_0}{\nu}\frac{dU}{dy} = -\frac{G}{\rho\nu}.$$

If there is no slip imposed at $y = 0$ and $y = b$, solve the ordinary differential equation to find $U(y)$. 9 %

5 Navier-Stokes equations and stress tensor

(a) Explain what is meant by the stress vector with typical component v_i and the stress tensor with typical component T_{ij}. *Explain* the significance of the relationship $v_i = T_{ij}n_j$.
For the two dimensional flow $\mathbf{u} = (y^2, 0, 0), p = 2\rho\nu x$, calculate the stress tensor at any point in the flow and the stress vector on an infinitesimal element located at $(1, 0, 0)$ with unit normal $(0, 1, 0)$. 6 %

(b) *Derive* Cauchy's equation of motion (see page 7) for a deforming continuous medium. (A statement of the tensor divergence theorem is given on the last page). 6 %

(c) Using Cauchy's equation of motion and the constitutive relation for a Newtonian viscous fluid (see page 7), *derive* the Navier-Stokes equations. 6 %

6 Thin film approximation

(a) Starting with the two dimensional steady Navier-Stokes equations, with a velocity field of the form $\mathbf{u} = (u, 0, w)$ (using U as the main velocity scale and $\frac{\mu UL}{H^2}$ as the pressure scale) derive the *thin film approximation*:

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial z^2},\ \frac{\partial p}{\partial z} = 0,\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

assuming that the length scales are L, H with $H \ll L$. 9 %

(b) A thin splash of honey on a window pane (with known liquid properties μ, ρ, ν) has an initial configuration given by $z = h(x, 0) = f(x)$ where $f(x)$ is a known function defined on $-\infty < x < \infty$. The honey trickles under gravity down the window pane (with the x direction pointing downwards). Assume the problem is uniform in the y direction ($\partial/\partial y = 0$) and that the air next to the free surface exerts zero shear stress ($\mu \frac{\partial u}{\partial z} = 0$) and zero pressure on the liquid film.

- *Formulate* the problem modelling the flow in the film of honey, writing down the governing equations and all the relevant boundary conditions. *Assume first that the free surface described by* $z = h(x, t)$ *is known.* Solve this problem for $\mathbf{u} = (u, 0, w)$ and the pressure $p(x, z, t)$ using the thin film approximation.
- If the free surface of the honey film is described by $z = h(x, t)$ then explain why $D/Dt(z - h(x, t)) = 0$ on $z = h(x, t)$ for all time. If for a particular flow $\mathbf{u} = (u, 0, w)$ show that the above

so-called "kinematic" condition reduces to: $h_t + uh_x = w$ on $z = h(x,t)$.
Hence write down an evolution equation for the free surface $h(x,t)$. 9 %

7 Answer any **2** of the following:

(a) Consider two parallel plane walls of length $2a$ ($-a \leq x \leq a$), of infinite extent in the y direction, which are separated by a thin film of viscous liquid with one of the walls moving away from the other at speed dh/dt where $z = h(t)$ is the separation at any time. Assuming that the velocity field takes the form:

$$\mathbf{u} = (u(x,z,t), 0, w(x,z,t)),$$

use the thin film approximation of 6(a) to find the velocity and show that $p = \frac{6\mu}{h^3}\frac{dh}{dt}(x^2 - a^2)$. Deduce that the force per unit length in the y direction required to maintain the motion is:

$$F = \frac{8\mu a^3 \frac{dh}{dt}}{h^3}.$$

You may assume that the pressure at $x = \pm a$ is zero. What is the significance of $F \sim O(h^{-3})$ as $h \to 0$?

(b) Explain briefly what is meant by the continuum hypothesis. At a recent numerical analysis conference, there was much discussion when one author, while modelling a fluid flow problem, used finite differences to approximate the governing Navier-Stokes equations. For example he took:

$$\frac{\partial p}{\partial x} = \frac{p(x+h,y,z,t) - p(x,y,z,t)}{h}$$

where $p(x,y,z,t)$ is the fluid pressure. This approximation is obviously only reasonable if h is "small". In one particular case he took $h = 10^{-12}$m which is smaller than the molecular length scale. In the light of the continuum hypothesis discuss whether this is a reasonable thing to do?

(c) Non-dimensionalise the stress tensor as in part 6(a) and show that $T_{ij} \approx -p\delta_{ij}$ for a thin film flow. What consequences does this have for lubricating films?

(d) A two dimensional irrotational flow occupies the half-space $y < 0$ and is given by the velocity potential $\phi = \exp(ky)\sin(kx)(k > 0)$. Calculate the velocity field. Show that the flow is incompressible and deduce that ϕ must satisfy Laplace's equation $\nabla^2\phi = 0$.
Calculate the stream function $\psi(x, y)$ such that $u = \frac{\partial\psi}{\partial y}$, $v = -\frac{\partial\psi}{\partial x}$.

(e) For the 2D flow field $\mathbf{u} = (\alpha x, -\alpha y)$, show by computing the line integral that the circulation along the circle $x^2 + y^2 = 1$ is zero.
Verify that his flow is incompressible and irrotational. Use the latter fact and Stokes' theorem to verify that the circulation in the first part of the question must be zero.

(f) Incompressible viscous fluid of kinematic viscosity ν lies at rest in the region $0 < y < \infty$. At $t = 0$ the rigid boundary at $y = 0$ is suddenly set into motion in the $x-$direction with constant speed U.

(i) Show that a parallel flow of the form $\mathbf{u} = u(y, t)\mathbf{i}$ develops and is governed by the equation

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}.$$

(ii) State the initial and boundary conditions for $u(y, t)$.

(iii) Using a Laplace transform on the time variable, or otherwise, show that

$$u(y, t) = U\left(1 - \text{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)\right).$$

You may use the following results:
$\mathbf{L}[df/dt] = s\bar{f}(s) - f(0)$; $\mathbf{L}\left[1 - \text{erf}\left(\frac{a}{2\sqrt{t}}\right)\right] = \frac{\exp(-a\sqrt{s})}{s}$.

(g) Find an approximate solution using matched asymptotic expansions to the singularly perturbed ordinary differential equation problem:

$$\epsilon\frac{d^2u}{dy^2} + 2\frac{du}{dy} + u = 0,\ u(0) = 0, u(1) = 1, \epsilon << 1.$$

(Hint: Use $y = \epsilon Y$ as an "inner" rescaling.) 2×9%

Useful equations

Navier-Stokes equation:

$$\frac{D\mathbf{u}}{Dt}(=\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}.\nabla)\mathbf{u})=-\frac{1}{\rho}\nabla p+\nu\nabla^2\mathbf{u}+\mathbf{g};\ (\nu=\mu/\rho)$$

Navier-Stokes in index notation

$$\frac{Du_i}{Dt}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 u_i}{\partial x_j\partial x_j}+g_i \text{ (summation convention)}$$

Navier-Stokes in component form:

$$\begin{aligned}
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2})+g_x\\
\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2})+g_y\\
\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2})+g_z\\
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} &= 0 \qquad \text{(incompressibility)}
\end{aligned}$$

Euler equations: put $\nu=0$ in the Navier-Stokes equations.

Cauchy's equation of motion:

$$\rho\frac{Du_i}{Dt}=\frac{\partial T_{ij}}{\partial x_j}+\rho g_i$$

Constitutive equation for a Newtonian Fluid:

$$T_{ij}=-p\delta_{ij}+\mu(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i})(i,j=1,2,3)$$

Useful parameters

Reynolds number $R=\rho UL/\mu\equiv UL/\nu$.
Air: $\nu=15.10^{-6}\text{m}^2\text{s}^{-1}$; $\rho=1.205\text{Kg m}^{-3}$; $(\nu=\mu/\rho)$

Useful results

$$(\mathbf{u}.\nabla)\mathbf{u}\equiv(\nabla\times\mathbf{u})\times\mathbf{u}+\nabla(\mathbf{u^2}/2),\ \mathbf{u^2}\equiv\mathbf{u}.\mathbf{u}\equiv|\mathbf{u}|^2$$

$$\nabla\times(\boldsymbol{a}\times\mathrm{b})\equiv(\mathrm{b}.\nabla)\boldsymbol{a}-(\boldsymbol{a}.\nabla)\mathrm{b}+\boldsymbol{a}\nabla.\mathrm{b}-\mathrm{b}\nabla.\boldsymbol{a}$$

$$\frac{Df}{Dt}\equiv\frac{\partial f}{\partial t}+\mathbf{u}.(\nabla f)$$

Tensor divergence theorem

$$\iint_S T_{ij} n_j \, dS = \iiint_V \frac{\partial T_{ij}}{\partial x_j} \, dV.$$

Error function

$$\text{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} \, ds, \;\; \text{erf}(0) = 0, \; \text{erf}(\infty) = 1.$$