UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

College of Informatics and Electronics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MA4607

SEMESTER: Autumn 2004-05

MODULE TITLE: Fluid mechanics

DURATION OF EXAMINATION: 2 hrs 30 mins

LECTURER: Prof. S.O'Brien

PERCENTAGE OF TOTAL MARKS: 90%

EXTERNAL EXAMINER: Prof. J.R. King

INSTRUCTIONS TO CANDIDATES: Full marks for **5** questions. Number each question carefully **in the margin provided on your script**.

Vectors are written in bold print and are expressed in Cartesian coordinates unless otherwise noted. The subscript notation is used intermittently to denote a partial derivative (e.g. $u_x \equiv \frac{\partial u}{\partial x}$).
There are some useful results at the end of the paper.

1 (a) Let $C(x, y, z, t)$ be the concentration of a nuclear pollutant in a region in the sea relative to a Cartesian system. A submarine is active in the polluted region. Its sensors can detect the concentration of pollutant $C(x, y, z, t)$ in the water.

(i) What is meant by the usual partial derivative $\frac{\partial C}{\partial t}$ and the convective rate of change $\frac{DC}{Dt}$ of $C(x, y, z, t)$ in a moving fluid with velocity vector $\mathbf{u}(x, y, z, t)$?

(ii) Use the chain rule to prove that

$$\frac{DC}{Dt} \equiv \frac{\partial C}{\partial t} + \mathbf{u}.(\nabla C) = \frac{\partial C}{\partial t} + (\mathbf{u}.\nabla)C.$$

(iii) Write down a mathematical expression for the rate of change of C

A. if the submarine is stationary (w.r.t. the ocean bottom) at some point (a, b, c),

B. if the submarine is moving at a velocity $\mathbf{v}(x, y, z, t)$ relative to the ocean bottom,

C. if the submarine is drifting at the local sea velocity $\mathbf{u}(x, y, z, t)$.

D. Calculate each of these derivatives if $C(x, y, z, t) = x^2yt$, $\mathbf{v} = (xy, x^2y, 0)$, $\mathbf{u} = (x - y, x + y, 0)$.

10%

(b) The velocity vector for a particular flow $\mathbf{u}(x, y, z, t) = (u, v, w)$ has been calculated for all values of x, y, z, t.

(i) Explain what is meant by the streamlines of the flow and show that at any particular time they are given by: $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$.

(ii) Sketch the streamlines (with direction arrows) for the flows: $\mathbf{u} = (4y, -4x, 0)$ and $\mathbf{u} = (4x, -4y, 0)$.

8%

2 (a) By consideration of the net mass flow into an arbitrary volume V of fluid of density $\rho(x, y, z, t)$, use the vector divergence theorem to derive the continuity equation in the form:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{u}) = 0$$

in the usual notation. Deduce the continuity condition for incompressible flow ($\nabla.\mathbf{u} = 0$).

6 %

(b) Define vorticity. For two dimensional flow $\mathbf{u} = (u(x, y, t), v(x, y, t), 0)$, show that the vorticity is given by $(0, 0, v_x - u_y)$. Given the vorticity equation for inviscid flow in the form:

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega}.\nabla)\mathbf{u},$$

show that this reduces to $\frac{D\boldsymbol{\omega}}{Dt} = \mathbf{0}$ in the case of two dimensional flow. 4%

(c) Given Euler's equation in the form:

$$\frac{\partial \mathbf{u}}{\partial t} + (\nabla \times \mathbf{u}) \times \mathbf{u} = -\nabla H(x, y, z, t); \; H \equiv \nabla(\frac{p}{\rho} + \frac{1}{2}\mathbf{u}^2).$$

(i) Prove the Bernoulli streamline theorem that if an inviscid incompressible fluid is in steady flow, then H is constant along a streamline. 4%

(ii) Define what is meant by irrotational flow. Deduce that if the flow is also irrotational then H is a constant throughout the flow. 4%

3 (a) Explain the physical significance of the Reynolds number $R = \frac{UL}{\nu}$ depending on whether it is large, small or $O(1)$. Steady two dimensional flow ($\mathbf{u} = (u(x, y), v(x, y), 0)$) occurs in the vicinity of a stationary flat wall located at $y = 0$. Write down the different approximations to the Navier Stokes equations and discuss appropriate boundary conditions for the velocity field on $y = 0$ for the cases $R \gg 1, R \ll 1, R = 1, R = \infty$. 9 %

(b) Consider steady two dimensional flow ($\mathbf{u} = (u(x, y, t), v(x, y, t), 0)$) of an infinite plane of viscous liquid past an impermeable square of side L defined by $-L/2 \leq x, y \leq L/2$ relative to a Cartesian system fixed in the square. Assume that the velocity far from the square takes the form $\mathbf{u} \to (U, 0, 0)$ as $x^2 + y^2 \to \infty$ where U is a known speed. No slip and no flow conditions apply on the sides of the square.

Non-dimensionalise this problem (using ρU^2 as the pressure scale). Include all the boundary conditions in your non-dimensionalisation and explicitly demonstrate the occurrence Reynolds number. 9 %

4 A viscous incompressible liquid film of uniform constant thickness H is flowing under gravity down a plane $y = 0$ of infinite extent inclined at 45^o to the horizontal. Assume that the flow is steady, that there is no slip on the plane $y = 0$ and that the conditions $p = 0, \mu\frac{du}{dy} = 0$ hold on the free surface $y = H$. Simplify the Navier Stokes equations accordingly and find an expression for the velocity field. Sketch the velocity profile and find the net flux down the plane. 18 %

5 Incompressible viscous fluid occupies the half space $y > 0$ above a plane rigid boundary $y = 0$ which oscillates back and forth in the x direction with velocity $U \cos \omega t, \omega > 0$. Assuming that $\mathbf{u} = (u(y,t), 0, 0))$ satisfies:

$$u_t = \nu u_{yy}$$

show, by writing $u = \Re(f(y) \exp{(i\omega t)})$, that

$$u = U \exp{(-\sqrt{\frac{\omega}{2\nu}} y)} \cos{(\sqrt{\frac{\omega}{2\nu}} y - \omega t)}.$$

(The roots of $z^2 = i$ are $z = \pm(1+i)/\sqrt{2}$.) 18 %

6 Do any 3 of the following:

(a) Explain what is meant by the stress vector with typical component v_i and the stress tensor with typical component T_{ij}. **Prove** the relationship $v_i = T_{ij} n_j$ in the usual notation.

For the two dimensional flow of a Newtonian viscous fluid with $\mathbf{u} = (y, 0, 0), p = x + y$, calculate the stress tensor at any point in the flow and the stress vector on an infinitesimal element located at $(1, 0, 0)$ with unit normal $(0, 1, 0)$. 6 %

(b) **Derive** Cauchy's equation of motion (see last pages) for a deforming continuous medium. (A statement of the tensor divergence theorem is given on the last page). 6 %

(c) Using Cauchy's equation of motion and the constitutive relation for a Newtonian viscous fluid (see last pages), **derive** the Navier Stokes equations. 6 %

(d) By a consideration of the kinetic energy of a volume V of Newtonian viscous fluid:

$$T = \frac{1}{2} \int_V \rho u_i^2 \, dV$$

show that the rate of energy dissipation in a viscous flow is $2\mu e_{ij}^2$ in the usual notation. 6%

7 (a) Starting with the steady two dimensional Stokes equations for flow of an incompressible liquid in a **thin** film, derive the thin film equations for the velocity field $\mathbf{u} = (u(x,y), v(x,y))$ in the form:

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2}, \ \frac{\partial p}{\partial y} = 0, \ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

You may assume that U the characteristic velocity scale is known and the film thickness $H \ll L$, the basic length scale in the direction of the flow. By non-dimensionalising the stress tensor show that it takes the approximate form $T_{ij} \approx -p\delta_{ij}$. What is the significance of this for lubricating films? 9 %

(b) Consider incompressible steady Hele-Shaw flow where a thin film of liquid is driven by a pressure gradient in the constant thickness gap H between two square horizontal plates (located at $z = 0, z = H$) of side-length L. Assuming no-slip at and no flow through the plates, use the thin film approximation:

$$0 = -p_x + \mu u_{zz};\ 0 = -p_y + \mu v_{zz};\ 0 = -p_z$$

to derive expressions for the velocity components (u, v, w) in terms of the pressure gradient and deduce that the pressure field must satisfy the 2D Laplace equation. Show that the streamline pattern is independent of z. 9 %

8 Answer any **three** of the following:

(a) Consider the steady irrotational inviscid flow of air (assumed to be incompressible) of density ρ over a thin two dimensional aerofoil inclined at a small angle to the air flow. Assume that the undisturbed free stream velocity vector far from the aerofoil is $\mathbf{u} = (U, 0, 0)$ where U is a given constant. Show that the lift is given by $L = -\rho U \Gamma$ where Γ is the circulation around the wing. 6%

(b) Verify that the two dimensional flow $\mathbf{u} = (u, v) = (x, -y)$ is incompressible. Find the stream function $\psi(x, y)$ such that $u = \psi_y, v = -\psi_x$. Verify that ψ is constant along a streamline. 6%

(c) Consider a slow flow (small Reynolds number) satisfying:

$$\mathbf{0} = -\nabla p + \mu \nabla^2 \mathbf{u}, \quad \nabla . \mathbf{u} = 0.$$

in a region V with boundary conditions on ∂V:

$$\mathbf{u} = \mathbf{f}_1(\mathbf{r}).$$

Given that solutions are unique prove that they are reversible. What consequence does this have for a fish swimming at small Reynolds number? 6%

(d) Starting with the steady two dimensional Navier Stokes equations for an incompressible flow, derive the Prandtl boundary layer equations in the form:

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2};\ \frac{\partial p}{\partial y}=0;\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

and show that the boundary layer thickness is given by $\delta = O(R^{-1/2}L)$ where L is a length scale in the direction of the flow and $R = UL/\nu$ is the Reynolds number.

Using the parameter values on the last pages, estimate the boundary layer thickness on the wings of a formula one racing car travelling at 200 km per hour. 6%

(e) Incompressible viscous liquid is located between two plates at $y = 0, y = h$. The fluid is initially at rest. At $t = 0$, the lower plate $y = 0$ starts to move at given speed U. Assuming the pressure gradient is zero, show that the velocity takes the form $\mathbf{u} = (u(y,t), 0, 0)$ and use this to simplify the Navier Stokes equations. Formulate the resulting mathematical problem for $u(y,t)$ and solve the resulting **steady state** problem for $u(y), (t \to \infty)$. 6%

(f) Find an approximate solution using matched asymptotic expansions to the singularly perturbed ordinary differential equation problem:

$$\epsilon\frac{d^2u}{dy^2}+\frac{du}{dy}-1=0,\ u(0)=0, u(1)=2, \epsilon << 1.$$

(Hint: Use $y = \epsilon Y$ as an "inner" rescaling.) 6%

(g) Let an inviscid, incompressible fluid of constant density be in motion in the presence of a conservative body force $\mathbf{g} = -\nabla\chi$ per unit mass. Let $C(t)$ denote a closed circuit that consists of the same fluid particles as time proceeds. Prove Kelvin's circulation theorem that:

$$\Gamma = \int_{C(t)} \mathbf{u}.d\mathbf{r}$$

is independent of the time. 6%

(h) In a flow the velocity vector $\mathbf{u}(x,y,z,t) = (u,v,w)$ has been calculated for all values of x, y, z, t. Explain what is meant by the **vortex** lines of the flow and prove that at any particular time they are given by: $\frac{dx}{\omega_x} = \frac{dy}{\omega_y} = \frac{dz}{\omega_z}, \boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$.

Explain the terms vortex surface, vortex tube. What direction do the vortex lines take for the flow: $\mathbf{u} = (y, -x, 0)$. 6%

Useful equations

Navier Stokes equation:

$$\frac{D\mathbf{u}}{Dt}(=\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}.\nabla)\mathbf{u})=-\frac{1}{\rho}\nabla p+\nu\nabla^2\mathbf{u}+\mathbf{g};\ (\nu=\mu/\rho)$$

Navier Stokes in index notation

$$\frac{Du_i}{Dt}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 u_i}{\partial x_j\partial x_j}+g_i \text{ (summation convention)}$$

Navier Stokes in component form:

$$\begin{aligned}
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2})+g_x\\
\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2})+g_y\\
\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2})+g_z\\
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} &= 0 \qquad \text{(incompressibility)}
\end{aligned}$$

Euler equations: put $\nu=0$ in the Navier Stokes equations.

Cauchy's equation of motion:

$$\rho\frac{Du_i}{Dt}=\frac{\partial T_{ij}}{\partial x_j}+\rho g_i$$

Constitutive equation for a Newtonian Fluid:

$$T_{ij}=-p\delta_{ij}+2\mu e_{ij}=-p\delta_{ij}+\mu(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i})(i,j=1,2,3)$$

Useful parameters

Reynolds number $R=\rho UL/\mu\equiv UL/\nu$.
Air: $\nu=15.10^{-6}$m^2 s^{-1}; $\rho=1.205$Kg m^{-3}; $(\nu=\mu/\rho)$

Useful results

$$\nabla\times(\boldsymbol{a}\times\mathbf{b})=(\mathbf{b}.\nabla)\boldsymbol{a}-(\boldsymbol{a}.\nabla)\mathbf{b}+\boldsymbol{a}\nabla.\mathbf{b}-\mathbf{b}\nabla.\boldsymbol{a}$$

$$(\mathbf{u}.\nabla)\mathbf{u}\equiv(\nabla\times\mathbf{u})\times\mathbf{u}+\nabla(\mathbf{u^2}/2),\ \mathbf{u^2}\equiv\mathbf{u}.\mathbf{u}\equiv|\ \mathbf{u}\ |^2$$

$$\nabla.(\phi\nabla\phi)=\phi\nabla^2\phi+\nabla\phi.\nabla\phi,\ \ \phi \text{ a scalar function}$$

$$\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + \mathbf{u}.(\nabla f)$$

Tensor divergence theorem

$$\int\int_S T_{ij} n_j dS = \int\int\int_V \frac{\partial T_{ij}}{\partial x_j} dV.$$