UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

College of Informatics and Electronics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MA4607

SEMESTER: Autumn 2003-04

MODULE TITLE: Fluids

DURATION OF EXAMINATION: 2 hrs 30 mins

LECTURER: Prof. S. O'Brien

PERCENTAGE OF TOTAL MARKS: 90%

EXTERNAL EXAMINER: Prof. J.D. Gibbon

INSTRUCTIONS TO CANDIDATES: **Full marks for 4 questions. Number each question carefully** *in the margin provided on your script* **.**

Vectors are written in bold print and are expressed in cartesian coordinates unless otherwise noted. The subscript notation is used intermittently to denote a partial derivative (e.g. $u_x \equiv \frac{\partial u}{\partial x}$**).**
N.B.There are some useful results at the end of the paper.

1 (a) Define the vorticity vector $\boldsymbol{\omega}$ and comment on its physical significance. What is meant by irrotational flow? Show that the vorticity for a general two dimensional flow $\mathbf{u} = (u(x,y), v(x,y), 0)$ is $(0, 0, v_x - u_y)$. Calculate the vorticity for the flow with velocity field $\mathbf{u} = (x, y, z)$. 8.5 %

(b) Starting with Euler's equation in the form:

$$\frac{\partial \mathbf{u}}{\partial t} + (\nabla \times \mathbf{u}) \times \mathbf{u} = -\nabla H(x,y,z,t);\ H \equiv \nabla(\frac{p}{\rho} + \frac{1}{2}\mathbf{u}^2)$$

prove the Bernoulli streamline theorem that if an inviscid incompressible fluid is in steady flow, then H is constant along a streamline. Deduce that if the flow is also irrotational then H is a constant throughout the flow.

Explain briefly the significance of this result in the context of steady flow around a two dimensional aerofoil. 7%

(c) If $C(x,y,z,t)$ is the concentration of pollutant in a flow, what is meant by the convective rate of change $\frac{DC}{Dt}$ of $C(x,y,z,t)$ in a moving fluid with velocity vector $\mathbf{u}(x,y,z,t)$?

Prove that

$$\frac{DC}{Dt} \equiv \frac{\partial C}{\partial t} + \mathbf{u}.(\nabla C) = \frac{\partial C}{\partial t} + (\mathbf{u}.\nabla)C.$$

The concentration of pollutant in a river flowing with velocity field $\mathbf{u} = (y, -x, 0)$ is $C(x,y,t) = x^2 y \exp(-t)$. What is the rate of change of C with respect to time t

(i) at the fixed point $(2,3,0)$,

(ii) as viewed from a boat moving with velocity $\mathbf{v}(x,y,z) = (y, x + y, 0)$ relative to the river bank,

(iii) following a fluid particle. Comment on this last result.

7%

2 (a) Starting with Euler's equation in the form:

$$\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\omega} \times \mathbf{u} = -\nabla H; H = \frac{p}{\rho} + \frac{1}{2}\mathbf{u}^2 - gz$$

and using the vector identities given on the last pages, derive the vorticity equation for inviscid flow in the form:

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega}.\nabla)\mathbf{u}.$$

Show that this reduces to $\frac{D\boldsymbol{\omega}}{Dt} = \mathbf{0}$ in the case of two dimensional flow.

10 %

(b) Consider the steady irrotational inviscid flow of air (assumed to be incompressible) of density ρ over a thin two dimensional aerofoil inclined at a small angle to the air flow. Assume that the undisturbed free stream velocity vector far from the aerofoil is $\mathbf{u} = (U, 0, 0)$ where U is a given constant.

(i) Use the vorticity equation to show why this (steady) flow is irrotational. 3 %

(ii) Show that the lift is given by $L = -\rho U \Gamma$. 7%

(iii) Use Kelvin's circulation theorem and Stokes' theorem to explain (with the aid of a diagram) how the circulation and starting vortex are generated. 2.5 %

3 (a) Explain briefly the significance of the Reynolds number. 2.5 %

(b) A two dimensional flow with velocity vector $\mathbf{u} = (u(x,y), v(x,y), 0)$ in $y \geq 0$ occurs in the vicinity of a stationary wall located along the x axis $y = 0$. Write down the boundary conditions for u and v for this flow on $y = 0$ if

(i) the liquid is inviscid.

(ii) the liquid is viscous. 5 %

(c) A viscous incompressible liquid film of uniform constant thickness H is flowing under gravity down a plane $y = 0$ of infinite extent inclined at α to the horizontal. Assume that the flow is steady, that there is no slip on the plane $y = 0$ and that the conditions $p = 0, \mu \frac{du}{dy} = 0$ hold on the free surface $y = H$. Simplify the Navier Stokes equations accordingly and find an expression for the velocity field. Sketch the velocity profile and find the net flux down the plane. 15 %

4 (a) Explain what is meant by the stress vector with typical component v_i and the stress tensor with typical component T_{ij}. **Prove** the relationship $v_i = T_{ij} n_j$.

For the two dimensional flow $\mathbf{u} = (u(y), 0, 0), p = f(x, y)$, calculate the stress tensor at any point in the flow and the stress vector on an infinitesimal element located at $(1, 0, 0)$ with unit normal $(0, 1, 0)$. 6 %

(b) **Derive** Cauchy's equation of motion (see last pages) for a deforming continuous medium. (A statement of the tensor divergence theorem is given on the last page). 6 %

(c) Using Cauchy's equation of motion and the constitutive relation for a Newtonian viscous fluid (see last pages), **derive** the Navier Stokes equations. 6 %

(d) By a consideration of the kinetic energy of a volume V of Newtonian viscous fluid:

$$T = \frac{1}{2}\int_V \rho u_i^2 \, dV$$

show that the rate of energy dissipation in a viscous flow is $2\mu e_{ij}^2$ in the usual notation. 4.5%

5 (a) Starting with the **steady** Navier Stokes equations, non-dimensionalise and show that the equations reduce to the Stokes (slow flow) approximation $\mathbf{0} = -\nabla p + \mu \nabla^2 \mathbf{u}$ when the Reynolds number is small. 8.5 %

(b) Consider the problem of steady Stokes (low Reynolds number) flow $\mathbf{0} = -\nabla p + \mu \nabla^2 \mathbf{u}$ for an incompressible liquid flowing in a region V with boundary conditions $\mathbf{u} = \mathbf{f}(x, y, z)$ on the boundary δV.

(i) **Prove** that there is at most one solution of the problem (uniqueness).

(ii) Hence prove that solutions are **reversible** by changing the sign of the boundary condition. 14 %

6 Answer any **three** of the following:

(a) Consider steady incompressible irrotational inviscid flow past a sphere of radius a. Far from the sphere, the velocity field is $\mathbf{u} = U\mathbf{k} \equiv U\cos\theta \mathbf{e}_r - U\sin\theta \mathbf{e}_\theta$. The velocity field can be written in terms of the Stokes stream function $\Psi(r, \theta)$:

$$u_r = \frac{1}{r^2 \sin\theta}\frac{\partial \Psi}{\partial \theta}; \; u_\theta = -\frac{1}{r\sin\theta}\frac{\partial \Psi}{\partial r}. \qquad (1)$$

Show that in terms of the stream function, the correct boundary condition at infinity is:

$$\Psi \sim \frac{1}{2}Ur^2 \sin^2\theta \;\; (r \to \infty). \qquad (2)$$

What is the boundary condition on the surface of the sphere $r = a$ in terms of $\mathbf{u}$ and Ψ ?

(b) Incompressible viscous liquid is located between two plates at $y = 0, y = h$. At $t = 0$, the lower plate $y = 0$ starts to move at given speed U. Show that the velocity takes the form $\mathbf{u} = (u(y, t), 0, 0)$ and use this to simplify the Navier Stokes equations. Formulate the resulting mathematical problem for $u(y, t)$ and solve the resulting **steady state** problem for $u(y), (t \to \infty)$.

(c) Find an approximate solution using matched asymptotic expansions to the singularly perturbed ordinary differential equation problem:

$$\epsilon \frac{d^2u}{dy^2} + \frac{du}{dy} - 1 = 0,\ u(0) = 0, u(1) = 2, \epsilon << 1.$$

(Hint: Use $\xi = x/\epsilon$ as an "inner" rescaling.)

(d) Starting with the steady two dimensional Navier Stokes equations for an incompressible flow, derive the Prandtl boundary layer equations in the form:

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2};\ \frac{\partial p}{\partial y} = 0;\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

and show that the boundary layer thickness is given by $\delta = O(R^{-1/2}L)$ where L is a length scale in the direction of the flow and $R = UL/\nu$ is the Reynolds number.

Using the parameter values on the last pages, estimate the boundary layer thickness on the wing of a jet travelling at 600 km per hour.

(e) Define what is meant by the terms vortex line, vortex surface. Prove Helmholtz's second theorem that $\Gamma = \int_S \boldsymbol{\omega}.\mathbf{n}dS$ is the same for all sections of a vortex tube.

(f) In a flow the velocity vector $\mathbf{u}(x, y, z, t) = (u, v, w)$ has been calculated for all values of x, y, z, t. Explain what is meant by the **vortex** lines of the flow and prove that at any particular time they are given by: $\frac{dx}{\omega_x} = \frac{dy}{\omega_y} = \frac{dz}{\omega_z}, \boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$.
Explain the terms vortex surface, vortex tube. What direction do the vortex lines take for the flow: $\mathbf{u} = (y, -x, 0)$.

(g) Consider incompressible steady Hele-Shaw flow where a thin film of liquid is driven by a pressure gradient in the constant thickness gap H between two square horizontal plates (located at $z = 0, z = H$) of side-length L. Assuming no-slip at the plates, use the thin film approximation:

$$0 = -p_x + \mu u_{zz};\ 0 = -p_y + \mu v_{zz};\ 0 = -p_z$$

to derive expressions for the velocity components (u, v, w) in terms of the pressure gradient and deduce that the pressure field must satisfy Laplace's equation. Show that the streamline pattern is independent of z.

(h) Starting with the steady two dimensional Stokes equations for flow of an incompressible liquid in a **thin** film, derive the thin film equations for the velocity field $\mathbf{u} = (u, v, w)$ in the form:

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial z^2},\ \frac{\partial p}{\partial y} = \mu \frac{\partial^2 v}{\partial z^2},\ \frac{\partial p}{\partial z} = 0,\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

You may assume that the film thickness $H \ll L$, the basic length scale in the direction of the flow. Deduce that stress tensor takes the approximate form $T_{ij} \approx -p\delta_{ij}$. 22.5 %

Useful equations

Navier Stokes equation:

$$\frac{D\mathbf{u}}{Dt}(=\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}.\nabla)\mathbf{u})=-\frac{1}{\rho}\nabla p+\nu\nabla^2\mathbf{u}+\mathbf{g};\ (\nu=\mu/\rho)$$

Navier Stokes in index notation

$$\frac{Du_i}{Dt}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 u_i}{\partial x_j \partial x_j}+g_i \text{ (summation convention)}$$

Navier Stokes in component form:

$$\begin{aligned}
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2})+g_x\\
\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2})+g_y\\
\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2})+g_z\\
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} &= 0 \qquad \text{(incompressibility)}
\end{aligned}$$

Euler equations: put $\nu=0$ in the Navier Stokes equations.

Cauchy's equation of motion:

$$\rho\frac{Du_i}{Dt}=\frac{\partial T_{ij}}{\partial x_j}+\rho g_i$$

Constitutive equation for a Newtonian Fluid:

$$T_{ij}=-p\delta_{ij}+2\mu e_{ij}=-p\delta_{ij}+\mu(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i})(i,j=1,2,3)$$

Useful parameters

Reynolds number $R=\rho UL/\mu\equiv UL/\nu$.
Air: $\nu=15.10^{-6}$m^2 s^{-1}; $\rho=1.205$Kg m^{-3}; $(\nu=\mu/\rho)$

Useful results

$$\nabla\times(\boldsymbol{a}\times\mathbf{b})=(\mathbf{b}.\nabla)\boldsymbol{a}-(\boldsymbol{a}.\nabla)\mathbf{b}+\boldsymbol{a}\nabla.\mathbf{b}-\mathbf{b}\nabla.\boldsymbol{a}$$

$$(\mathbf{u}.\nabla)\mathbf{u}\equiv(\nabla\times\mathbf{u})\times\mathbf{u}+\nabla(\mathbf{u^2}/2),\ \mathbf{u^2}\equiv\mathbf{u}.\mathbf{u}\equiv|\ \mathbf{u}\ |^2$$

$$\nabla.(\phi\nabla\phi)=\phi\nabla^2\phi+\nabla\phi.\nabla\phi,\ \ \phi \text{ a scalar function}$$

$$\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + \mathbf{u}.(\nabla f)$$

Tensor divergence theorem

$$\int\int_S T_{ij} n_j dS = \int\int\int_V \frac{\partial T_{ij}}{\partial x_j} dV.$$