UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

College of Informatics and Electronics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MA4607

SEMESTER: Autumn 2002

MODULE TITLE: Fluids

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Prof. S. O'Brien

PERCENTAGE OF TOTAL MARKS: 90%

EXTERNAL EXAMINER: Prof. J.D. Gibbon

INSTRUCTIONS TO CANDIDATES: **Full marks for 4 questions. Number each question carefully** *in the margin provided on your script* **.**

Vectors are written in bold print and are expressed in cartesian coordinates unless otherwise noted. The subscript notation is used intermittently to denote a partial derivative (e.g. $u_x \equiv \frac{\partial u}{\partial x}$**).**
N.B.There are some useful results at the end of the paper.

1 (a) Define the vorticity vector $\boldsymbol{\omega}$ and comment on its physical significance. Show that the vorticity for a general two dimensional flow $\mathbf{u} = (u(x,y), v(x,y), 0)$ is $(0, 0, v_x - u_y)$.

Calculate the vorticity for the flows with velocity field:

(i) $\mathbf{u} = (x, y, z)$

(ii) (In cylindrical polars, see end of paper); $\mathbf{u} = r\mathbf{e}_r + r\theta\mathbf{e}_\theta + rz\mathbf{e}_z$.

Are these flows rotational or irrotational? 8%

(b) If $C(x, y, z, t)$ is the concentration of pollutant in a flow, what is meant by the convective rate of change $\frac{DC}{Dt}$ of $C(x, y, z, t)$ in a moving fluid with velocity vector $\mathbf{u}(x, y, z, t)$?

Prove that

$$\frac{DC}{Dt} \equiv \frac{\partial C}{\partial t} + \mathbf{u}.(\nabla C) = \frac{\partial C}{\partial t} + (\mathbf{u}.\nabla)C.$$

The concentration of pollutant in a river flowing with velocity field $\mathbf{u} = (x, -y, 0)$ is $C(x, y, t) = x^2 y \exp(-t)$. What is the rate of change of C with respect to time t

(i) at the fixed point $(1, 2, 0)$,

(ii) as viewed from a boat moving with velocity $\mathbf{v}(x, y, z) = (y, 1, 0)$ relative to the river bank,

(iii) following a fluid particle. Comment on this last result.

7%

(c) In a flow the velocity vector $\mathbf{u}(x, y, z, t) = (u, v, w)$ has been calculated for all values of x, y, z, t. Explain what is meant by the streamlines of the flow and prove that at any particular time they are given by: $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$.

Sketch the streamlines (with direction arrows) for the flows: $\mathbf{u} = (y, -x, 0)$ and $\mathbf{u} = (\cos t, \sin t, 0)$.

Sketch the velocity profile for the shear flow $\mathbf{u} = (\sin y, 0, 0), 0 \leq y \leq 2\pi$. 7.5%

2 (a) Starting with Euler's equation in the form:

$$\frac{\partial \mathbf{u}}{\partial t} + (\nabla \times \mathbf{u}) \times \mathbf{u} = -\nabla H(x, y, z, t); \; H \equiv \nabla(\frac{p}{\rho} + \frac{1}{2}\mathbf{u}^2)$$

prove that if an inviscid incompressible fluid is in steady flow, then H is constant along a streamline. Deduce that if the flow is also irrotational then H is a constant throughout the flow. 6%

(b) Consider the steady irrotational inviscid flow of air (assumed incompressible) of density ρ over a thin two dimensional aerofoil inclined at a small angle to the air flow. Assume that the undisturbed free stream velocity vector far from the aerofoil is $\mathbf{u} = (U, 0, 0)$ where U is a given constant.

(i) Using the Bernoulli equation explain why a negative circulation (Γ) is necessary to generate lift on the aerofoil. Further show that the lift is given by $L = -\rho U \Gamma$. 6%

(ii) Use Kelvin's circulation theorem and Stokes' theorem to explain (with the aid of a diagram) how the circulation and starting vortex are generated. 4%

(iii) An aircraft of mass 10^4 Kg. has wings of average width 3m. and average length 30m. Estimate the magnitude of the air speeds associated with the circulation around the wings during level flight at a speed of 100 m s^{-1}. 6.5%

3 (a) A two dimensional flow with velocity vector $\mathbf{u} = (u(x, y), v(x, y), 0)$ in $y \geq 0$ occurs in the vicinity of a stationary wall located along the x axis $y = 0$. Write down the boundary conditions for u and v for this flow on $y = 0$ if

(i) the liquid is inviscid.

(ii) the liquid is viscous. 5 %

(b) Viscous incompressible fluid lies at rest in the region $0 \leq y < \infty$. At $t = 0$ the rigid boundary $y = 0$ is jerked into motion in the x direction at constant speed U. Assuming that the pressure gradient is zero, show that the Navier Stokes equations for the velocity field $(u(y, t), v(y, t), 0)$ may be simplified to the diffusion equation:

$$u_t = \nu u_{yy}$$

in the usual notation. Solve this problem using Laplace transforms (see last page) and show that the velocity field takes the form $u(y, t) = U(1 - \text{erf}(\frac{y}{2\sqrt{\nu t}}))$. 9.5%

(c) Viscous incompressible fluid flows between two stationary rigid boundaries $y = \pm h$ under a constant pressure gradient P i.e. $\nabla p = (-P, 0, 0)$. Starting with the Navier Stokes equations show that:

$$u = \frac{P}{2\mu}(h^2 - y^2),\ v = w = 0.$$

What is the mass flux of liquid and the average velocity? 8%

4 (a) Explain what is meant by the stress vector with typical component t_i and the stress tensor with typical component T_{ij}. Explain the relationship $t_i = T_{ij}n_j$.

For the two dimensional flow $\mathbf{u} = (xy, x, 0), p = xy$, calculate the stress tensor at any point in the flow and the stress vector on an infinitesimal element located at $(1, 0, 0)$ with unit normal $(0, 1, 0)$. 6 %

(b) **Derive** Cauchy's equation of motion (see last pages) for a deforming continuous medium. (A statement of the tensor divergence theorem is given on the last page). 6 %

(c) Using Cauchy's equation of motion and the constitutive relation for a Newtonian viscous fluid (see last pages), **derive** the Navier Stokes equations. 6 %

(d) By a consideration of the kinetic energy of a volume V of Newtonian viscous fluid:

$$T = \frac{1}{2}\int_V \rho u_i^2 \, dV$$

show that the rate of energy dissipation in a viscous flow is $2\mu e_{ij}^2$ in the usual notation. 4.5%

5 (a) Consider the problem of steady Stokes (low Reynolds number) flow $\mathbf{0} = -\nabla p + \mu\nabla^2\mathbf{u}$ for an incompressible liquid flowing in a region V with boundary conditions $\mathbf{u} = \mathbf{f}_1(x, y, z)$. **Prove** that there is at most one solution of the problem (uniqueness). 7%

(b) Starting with the steady two dimensional Stokes equations for flow of an incompressible liquid in a **thin** film, derive the thin film equations for the velocity field $\mathbf{u} = (u, v, w)$ in the form:

$$\frac{\partial p}{\partial x} = \mu\frac{\partial^2 u}{\partial z^2}, \ \frac{\partial p}{\partial y} = \mu\frac{\partial^2 v}{\partial z^2}, \ \frac{\partial p}{\partial z} = 0, \ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

You may assume that the film thickness $H \ll L$, the basic length scale in the direction of the flow. 7%

(c) A disc of radius a is separated from a rigid plane by a thin film of incompressible viscous liquid. If the disc is pulled away from the surface in the z direction, the resulting flow has a velocity field of the form $\mathbf{u} = u_r(r, z, t)\mathbf{e}_r + u_z(r, z, t)\mathbf{e}_z$ where the gap between the film thickness is $h(t)$ with $h \ll a$. Using the thin film approximation in the form:

$$p_r = \mu\frac{\partial^2 u_r}{\partial z^2}; \ \ p_z = 0; \ \ \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{\partial u_z}{\partial z} = 0$$

formulate the mathematical problem and show that

$$u_r = \frac{1}{2\mu} p_r z(z-h); \quad p = \frac{3\mu}{h^3}\frac{dh}{dt}(r^2 - a^2).$$

Deduce that the net upward force on the disc is $\frac{3\pi\mu a^4}{2h^3}\frac{dh}{dt}$. 8.5 %

6 Answer any **three** of the following:

(a) Starting with Euler's equation in the form:

$$\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\omega} \times \mathbf{u} = -\nabla H; H = \frac{p}{\rho} + \frac{1}{2}\mathbf{u}^2 - gz$$

and using the vector identities given on the last page, derive the vorticity equation for inviscid flow in the form:

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega}.\nabla)\mathbf{u}.$$

What does this simplify to in the case of two dimensional flow?

(b) Viscous fluid occupies a region $r_1 \leq r \leq r_2$ between two circular cylinders of known radii r_1 and r_2 rotating with angular velocities Ω_1 and Ω_2 respectively. Assuming that the steady flow field $\mathbf{u} = u_\theta(r)\mathbf{e}_\theta$ is governed by the equation:

$$r^2 \frac{d^2 u_\theta}{dr^2} + r\frac{du_\theta}{dr} - u_\theta = 0$$

write down suitable boundary conditions and use these to determine the velocity field. (Hint: the substitution $u = r^\alpha$ may be useful).

(c) Find an approximate solution using matched asymptotic expansions to the singularly perturbed ordinary differential equation problem:

$$\epsilon \frac{d^2 u}{dy^2} + \frac{du}{dy} - 1 = 0, \ u(0) = 0, u(1) = 2, \epsilon << 1.$$

(Hint: Use $\xi = x/\epsilon$ as an "inner" rescaling.)

(d) Starting with the steady two dimensional Navier Stokes equations for an incompressible flow, derive the Prandtl boundary layer equations in the form:

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}; \ \frac{\partial p}{\partial y} = 0; \ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

and show that the boundary layer thickness is given by $\delta = O(R^{-1/2}L)$ where L is a length scale in the direction of the flow and $R = UL/\nu$ is the Reynolds number.

Using the parameter values on the last pages, estimate the boundary layer thickness on the wing of a jet travelling at 600 km per hour.

(e) Suppose that an inviscid incompressible fluid of constant density is in motion in the presence of a conservative body force $\mathbf{g} = -\nabla\chi$ per unit mass. Let $C(t)$ denote a closed circuit that consists of the same fluid particles as time proceeds. Prove the following lemma:

$$\frac{d}{dt}\int_{C(t)} \mathbf{u}.d\mathbf{r} = \int_{C(t)} \frac{D\mathbf{u}}{Dt}.d\mathbf{r}.$$

Hence prove Kelvin's circulation theorem that the circulation $\Gamma = \int_{C(t)} \mathbf{u}.d\mathbf{r}$ is independent of the time.

(f) Define what is meant by the terms vortex line, vortex surface. Prove Helmholtz's second theorem that $\Gamma = \int_S \boldsymbol{\omega}.\mathbf{n}dS$ is the same for all sections of a vortex tube. 22.5 %

Useful equations

Navier Stokes equation:

$$\frac{D\mathbf{u}}{Dt}\left(=\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}.\nabla)\mathbf{u}\right)=-\frac{1}{\rho}\nabla p+\nu\nabla^2\mathbf{u}+\mathbf{g};\ (\nu=\mu/\rho)$$

Navier Stokes in index notation

$$\frac{Du_i}{Dt}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 u_i}{\partial x_j\partial x_j}+g_i \text{ (summation convention)}$$

Navier Stokes in component form:

$$\begin{aligned}
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+g_x\\
\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+g_y\\
\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+g_z\\
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} &= 0 \qquad \text{(incompressibility)}
\end{aligned}$$

Euler equations: put $\nu = 0$ in the Navier Stokes equations.

Cauchy's equation of motion:

$$\rho\frac{Du_i}{Dt}=\frac{\partial T_{ij}}{\partial x_j}+\rho g_i$$

Constitutive equation for a Newtonian Fluid:

$$T_{ij}=-p\delta_{ij}+2\mu e_{ij}=-p\delta_{ij}+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)(i,j=1,2,3)$$

Useful parameters

Reynolds number $R=\rho UL/\mu\equiv UL/\nu$.
Air: $\nu = 15.10^{-6}$m^2 s^{-1}; $\rho = 1.205$Kg m^{-3}; $(\nu=\mu/\rho)$

Useful results

$$\nabla\times(\boldsymbol{a}\times\mathbf{b})=(\mathbf{b}.\nabla)\boldsymbol{a}-(\boldsymbol{a}.\nabla)\mathbf{b}+\boldsymbol{a}\nabla.\mathbf{b}-\mathbf{b}\nabla.\boldsymbol{a}$$

$$(\mathbf{u}.\nabla)\mathbf{u}\equiv(\nabla\times\mathbf{u})\times\mathbf{u}+\nabla(\mathbf{u}^2/2),\ \mathbf{u}^2\equiv\mathbf{u}.\mathbf{u}\equiv|\ \mathbf{u}\ |^2$$

$$\nabla.(\phi\nabla\phi)=\phi\nabla^2\phi+\nabla\phi.\nabla\phi,\ \phi \text{ a scalar function}$$

$$\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + \mathbf{u}.(\nabla f)$$

Tensor divergence theorem

$$\int\int_S T_{ij} n_j dS = \int\int\int_V \frac{\partial T_{ij}}{\partial x_j} dV.$$

Curl of a vector in (cylindrical) polar coordinates

$$\nabla \times \mathbf{u} = \frac{1}{r}\,\mathrm{det}\begin{bmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ u_r & ru_\theta & u_z \end{bmatrix} \quad (\mathbf{u} = u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta + u_z\mathbf{e}_z).$$

Laplace transform

$$\mathbf{L}[\frac{\partial u(y,t)}{\partial t}] = s\bar{u}(y,s) - u(y,0). \;\; \mathbf{L}[1 - \mathrm{erf}(\frac{a}{2\sqrt{t}})] = \frac{\exp(-a\sqrt{s})}{s}.$$

Error function

$$\mathrm{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x \exp(-s^2) ds.$$