UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

College of Informatics and Electronics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MA4607

SEMESTER: Autumn 2002

MODULE TITLE: Introduction to Fluids

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Prof. S. O'Brien

PERCENTAGE OF TOTAL MARKS: 96%

EXTERNAL EXAMINER: Prof. J.D. Gibbon

INSTRUCTIONS TO CANDIDATES: **Full marks for 4 questions. Number each question carefully** *in the margin provided on your script* **.**

Vectors are written in bold print and are expressed in cartesian coordinates unless otherwise noted. The subscript notation is used intermittently to denote a partial derivative (e.g. $u_x \equiv \frac{\partial u}{\partial x}$**).**
N.B.There are some useful results at the end of the paper.

1 (a) Determine whether the following flows are incompressible:

$$\mathbf{u} = (x, y, z),\ \mathbf{u} = (x, -y, 0).$$

5 %

(b) Define the vorticity $\boldsymbol{\omega}$ of a flow and show that the vorticity of a two dimensional flow $\mathbf{u} = (u(x,y), v(x,y), 0)$ is $\boldsymbol{\omega} = (0, 0, v_x - u_y)$. Find the vorticity of the following flows: $\mathbf{u} = (y, 0, 0),\ \mathbf{u} = (x, y, z)$. 5 %

(c) The velocity vector for a particular flow $\mathbf{u}(x, y, z, t) = (u, v, w)$ has been calculated for all values of x, y, z, t. Explain what is meant by the streamlines of the flow and show that at any particular time they are given by: $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$.
Sketch the streamlines (with direction arrows) for the flow: $\mathbf{u} = (y, -x, 0)$.
Sketch the velocity profile for the shear flow $\mathbf{u} = (y, 0, 0), 0 \leq y \leq 1$.

7%

(d) Let $C(x, y, z, t) = x^2 y e^{-t}$ be the concentration of pollutant in a flow with velocity field $\mathbf{u} = (x, -y, 0)$. Explain what is meant by the convective rate of change $\frac{DC}{Dt}$ of $C(x, y, z, t)$ in a moving fluid with velocity vector $\mathbf{u}(x, y, z, t)$. **Prove** that

$$\frac{DC}{Dt} = \frac{\partial C}{\partial t} + \mathbf{u}.(\nabla C).$$

Calculate the rate of change of pollutant

(i) at a fixed point (a, b, c) in the flow,

(ii) as seen by an observer drifting in the flow at velocity $\mathbf{v}(x, y, z, t) = (xy, 0, 0)$?

(iii) following a fluid particle.

7%

2 (a) Suppose that an inviscid incompressible fluid of constant density is in motion in the presence of a conservative body force $\mathbf{g} = -\nabla\chi$ per unit mass. Let $C(t)$ denote a closed circuit that consists of the same fluid particles as time proceeds. Prove the following lemma:

$$\frac{d}{dt}\int_{C(t)} \mathbf{u}.d\mathbf{r} = \int_{C(t)} \frac{D\mathbf{u}}{Dt}.d\mathbf{r}.$$

Hence prove Kelvin's circulation theorem that the circulation $\Gamma = \int_{C(t)} \mathbf{u}.d\mathbf{r}$ is independent of the time. 12 %

(b) Define what is meant by the terms vortex line, vortex surface. Prove Helmholtz's second theorem that $\Gamma = \int_S \boldsymbol{\omega}.\mathbf{n}dS$ is the same for all sections of a vortex tube. 12 %

3 (a) Consider steady two dimensional viscous flow past a circular cylinder of radius a. The mathematical problem involves finding a velocity field $\mathbf{u} = (u(x,y), v(x,y), 0)$ which satisfies the two dimensional Navier Stokes equations (given on the last page in component form):

$$(\mathbf{u}.\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u},\ \nabla.\mathbf{u} = \mathbf{0}$$

together with the boundary conditions

$$\mathbf{u} = \mathbf{0} \text{ on } x^2 + y^2 = a^2,\ \mathbf{u} \to (U, 0, 0) \text{ as } x^2 + y^2 \to \infty$$

where U is a known constant. Non-dimensionalise this problem (including the boundary conditions) using $a, U, \rho U^2$ as length, velocity and pressure scales. Explain what is meant by the Reynolds number $R = Ua/\nu$ and, without solving the problem, show that the solutions depend on the Reynolds number. What approximations might you make if

(i) $R \gg 1$

(ii) $R \ll 1$ 15 %

(b) Ideal fluid moves irrotationally in a simply connected region V bounded by a closed surface S, so that $\mathbf{u} = \nabla\phi$, where ϕ is the velocity potential. Using the vector identities on the last pages, show that $\nabla^2\phi = 0$ and that the kinetic energy $T = \frac{1}{2}\rho\int_V \mathbf{u}^2 dV$ can be written in the form:

$$T = \frac{1}{2}\rho\int_S \phi\frac{\partial\phi}{\partial n}dS.$$

9 %

4 (a) A viscous incompressible fluid film of uniform constant thickness H everywhere and known fluid properties μ, ρ is flowing in the x direction down an infinite plane inclined at α to the horizontal. Assume that the flow is steady, that there is no-slip on $y = 0$ and that on $y = H$ the boundary conditions are $p = 0,\ \frac{du}{dy} = 0$. Justify a further assumption that the velocity vector takes the form $\mathbf{u} = (u(y), v(y), 0)$ and verify from the incompressibility condition that $v = 0$ everywhere. Show that the Navier Stokes equations reduce to

$$\nu u_{yy} = -g\sin\alpha.$$

Find the velocity field $\mathbf{u}$ and estimate the net flux and average velocity.

19 %

(b) A two dimensional flow with velocity vector $\mathbf{u} = (u(x, y), v(x, y), 0)$ in $y \geq 0$ occurs in the vicinity of a stationary wall located along the x axis $y = 0$. Name the equations of motion suitable for each flow and the boundary conditions for u and v on $y = 0$ if

(i) the liquid is inviscid.

(ii) the liquid is viscous. 5 %

5 (a) From the point of view of energy dissipation, explain the difference between a perfectly elastic solid and a viscous liquid. 3 %

(b) Explain what is meant by the stress vector with typical component v_i and the stress tensor with typical component T_{ij}.

Consider the flow with velocity vector $\mathbf{u} = (xy, x, 0)$ and pressure $p = xy$ and calculate the stress tensor at any point in the flow. Use the relationship $v_i = T_{ij}n_j$ to calculate the stress vector acting on an infinitesimal element located at $(1, 0, 0)$ with normal $(0, 1, 0)$. 7 %

(c) **Derive** Cauchy's equation of motion (see last pages) for a deforming continuous medium. (A statement of the tensor divergence theorem is given on the last page). 7 %

(d) Using Cauchy's equation of motion and the constitutive relation for a Newtonian viscous fluid (see last pages), **derive** the Navier Stokes equations. 7 %

6 Do any **2** of the following:

(a) Starting with the steady two dimensional Stokes equations for flow of an incompressible liquid in a **thin** film, derive the thin film equations for the velocity field $\mathbf{u} = (u, v, w)$ in the form:

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial z^2}, \ \frac{\partial p}{\partial y} = \mu \frac{\partial^2 v}{\partial z^2}, \ \frac{\partial p}{\partial z} = 0, \ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

You may assume that the film thickness $H \ll L$, the basic length scale in the direction of the flow. 12 %

(b) Consider Hele-Shaw (viscous) incompressible thin film flow between two plates (located at $z = 0$, $z = h$) which are both flat and parallel, so that h, the gap between the plates, is constant. The flow is driven by a horizontal pressure gradient with $p = p(x, y)$. Use the thin film equations given above to deduce the two components of the velocity field:

$$u = -\frac{1}{2\mu}\frac{\partial p}{\partial x} z(h - z); \ v = -\frac{1}{2\mu}\frac{\partial p}{\partial y} z(h - z).$$

By considering the third component of the velocity w, show that the pressure field must satisfy Laplace's equation. 12 %

(c) Consider the problem of steady Stokes (low Reynolds number) flow $0 = -\nabla p + \mu \nabla^2 \mathbf{u}$ for an incompressible liquid flowing in a region V with boundary conditions $\mathbf{u} = \mathbf{f}_1(x, y, z)$. **Prove** that:

(i) There is at most one solution of the problem (uniqueness).

(ii) Reversed boundary conditions lead to a reversed flow (reversibility).

What consequences does the second (reversibility) theorem have for swimming at low Reynolds number. 12%

7 Answer any **2** of the following:

(a) Prove that if an inviscid incompressible fluid is in steady flow, then H is constant along a streamline (first Bernoulli theorem). Deduce that if the flow is also irrotational then H is a constant throughout the flow (second Bernoulli theorem). (You may use Euler's equation in the form:

$$\frac{\partial \mathbf{u}}{\partial t} + (\nabla \times \mathbf{u}) \times \mathbf{u} = -\nabla H(x, y, z, t); \; H \equiv \frac{p}{\rho} + \frac{1}{2}\mathbf{u}^2).$$

Use the second Bernoulli theorem to give a brief account of why circulation is necessary to obtain lift on an aerofoil 12 %

(b) Find an approximate solution using matched asymptotic expansions to the singularly perturbed ordinary differential equation problem:

$$\epsilon \frac{d^2u}{dy^2} + \frac{du}{dy} - 1 = 0, \; u(0) = 0, u(1) = 2, \epsilon << 1.$$

(Hint: Use $y = \epsilon Y$ as an "inner" rescaling.) 12 %

(c) Starting with the steady two dimensional Navier Stokes equations for an incompressible flow, derive the Prandtl boundary layer equations in the form:

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}; \; \frac{\partial p}{\partial y} = 0; \; \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

and show that the boundary layer thickness is given by $\delta = O(R^{-1/2}L)$ where L is a length scale in the direction of the flow and $R = UL/\nu$ is the Reynolds number.

Using the parameter values on the last pages, estimate the boundary layer thickness on a jet travelling at 500 km per hour. 12 %

Useful equations

Navier Stokes equation:

$$\frac{D\mathbf{u}}{Dt}(=\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}.\nabla)\mathbf{u})=-\frac{1}{\rho}\nabla p+\nu\nabla^2\mathbf{u}+\mathbf{g};\ (\nu=\mu/\rho)$$

Navier Stokes in index notation

$$\frac{Du_i}{Dt}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 u_i}{\partial x_j\partial x_j}+g_i \text{ (summation convention)}$$

Navier Stokes in component form:

$$\begin{aligned}
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2})+g_x\\
\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2})+g_y\\
\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2})+g_z\\
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} &= 0 \qquad \text{(incompressibility)}
\end{aligned}$$

Euler equations: put $\nu=0$ in the Navier Stokes equations.

Cauchy's equation of motion:

$$\rho\frac{Du_i}{Dt}=\frac{\partial T_{ij}}{\partial x_j}+\rho g_i$$

Constitutive equation for a Newtonian Fluid:

$$T_{ij}=-p\delta_{ij}+\mu(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i})(i,j=1,2,3)$$

Useful parameters

Reynolds number $R=\rho UL/\mu\equiv UL/\nu$.
Air: $\nu=15.10^{-6}$m^2 s^{-1}; $\rho=1.205$Kg m^{-3}; $(\nu=\mu/\rho)$

Useful results

$$(\mathbf{u}.\nabla)\mathbf{u}\equiv(\nabla\times\mathbf{u})\times\mathbf{u}+\nabla(\mathbf{u^2}/2),\ \mathbf{u^2}\equiv\mathbf{u}.\mathbf{u}\equiv\mid\mathbf{u}\mid^2$$

$$\nabla.(\phi\nabla\phi)=\phi\nabla^2\phi+\nabla\phi.\nabla\phi,\ \phi \text{ a scalar function}$$

$$\frac{Df}{Dt}\equiv\frac{\partial f}{\partial t}+\mathbf{u}.(\nabla f)$$

Tensor divergence theorem:

$$\int\int_S T_{ij} n_j dS = \int\int\int_V \frac{\partial T_{ij}}{\partial x_j} dV.$$