Energy-norm a posteriori error estimates for singularly perturbed reaction-diffusion problems on anisotropic meshes. Neumann boundary conditions

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1 Introduction

This paper addresses finite element approximations to singularly perturbed semilinear reaction-diffusion equations of the form

$$-\varepsilon^2 \triangle u + f(x,y;u) = 0 \;\text{ for } (x,y)\in\Omega, \quad \partial_\nu u = \psi \text{ on } \Gamma_N, \quad u = 0 \text{ on } \Gamma_D, \tag{1}$$

posed in a, possibly non-Lipschitz, polygonal domain $\Omega \subset \mathbb{R}^2$. Here $0 < \varepsilon \leqslant 1$. The boundary segments Γ_D and Γ_N are disjoint with $\bar{\Gamma}_D \cup \bar{\Gamma}_N = \partial\Omega$, and ∂_ν denotes the outward normal derivative. The function f is continuous on $\Omega \times \mathbb{R}$ and satisfies $f(\cdot;s) \in L_\infty(\Omega)$ for all $s \in \mathbb{R}$, and the one-sided Lipschitz condition $f(x,y;v) - f(x,y;w) \geqslant C_f[v-w]$ whenever $v \geqslant w$, with some constant $C_f \geqslant 0$ such that $C_f + \varepsilon^2 \geqslant 1$.

Our goal is to give residual-type a posteriori error estimates on reasonably general anisotropic meshes (such as on Fig. 1, left, and Fig. 2) in the energy norm $|||\cdot|||_{\varepsilon;\Omega}$. The latter is an appropriately scaled $W_2^1(\Omega)$ norm naturally associated with our problem. For any $\mathscr{D} \subseteq \Omega$, it is defined by

$$|||v|||_{\varepsilon;\mathscr{D}} := \left\{\varepsilon^2 \|\nabla v\|^2_{2;\mathscr{D}} + \|v\|^2_{2;\mathscr{D}}\right\}^{1/2}.$$

The case of Dirichlet boundary conditions was considered in the recent article [9]. Now we extend this analysis to also allow boundary conditions of Neumann type. In this preliminary contribution, we shape our treatment of Neumann boundary conditions on anisotropic mesh elements in a simpler setting of partially structured meshes (as on Fig. 1, right). The presented approach will be applied to more general anisotropic meshes, such as addressed in [8, 9], in a forthcoming journal article.

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It is worth noting that our estimators in this paper (e.g., (3)), as well as in [8, 9], do not involve the so-called matching functions. (The latter appear in the estimator error constants in [11, 12, 13]; they depend on the unknown error and take moderate values only when the grid is either isotropic, or, being anisotropic, is aligned correctly to the solution, while, in general, may be as large as mesh aspect ratios.)

We discretize (1) using linear finite elements. Let $S_h \subset \{v \in H^1(\Omega)\cap C(\bar\Omega) : v=0 \text{ on } \Gamma_D\}$ be a piecewise-linear finite element space relative to a triangulation $\mathscr{T}$, and let the computed solution $u_h \in S_h$ satisfy

$$\varepsilon^2\langle \nabla u_h, \nabla v_h\rangle + \langle f_h^I, v_h\rangle = \int_{\Gamma_N} \psi\, v_h \quad \forall\, v_h \in S_h, \qquad f_h(\cdot) := f(\cdot\,; u_h). \tag{2}$$

Here $\langle\cdot,\cdot\rangle$ is the $L_2(\Omega)$ inner product, and f_h^I is the standard piecewise-linear Lagrange interpolant of f_h.

To give a flavour of our results, our first estimator reduces to

$$\begin{aligned}\|\!|u_h - u|\!\|_{\varepsilon\,;\Omega} \leqslant C\,\Big\{ &\sum_{z\in\mathscr{N}} \min\{h_z H_z,\, \varepsilon H_z^2 h_z^{-1}\}\,\big\|\varepsilon J\big\|^2_{\infty\,;\gamma_z} + \sum_{z\in\mathscr{N}} |I_z^{\psi}|^2 \\ &+ \sum_{z\in\mathscr{N}} \big\|\min\{1,\, H_z\varepsilon^{-1}\}\, f_h^I\big\|^2_{2\,;\omega_z} + \big\|f_h - f_h^I\big\|^2_{2\,;\Omega}\Big\}^{1/2},\end{aligned} \tag{3}$$

where C is independent of the diameters and the aspect ratios of elements in $\mathscr{T}$, and of ε. Here $\mathscr{N}$ is the set of nodes in $\mathscr{T}$, J is the standard jump in the normal derivative of u_h across an interior element edge, while $J := \partial_\nu u_h - \psi$ on Γ_N, ω_z is the patch of elements surrounding any $z\in\mathscr{N}$, γ_z is the set of edges originating at z and lying in $\omega_z \cup \Gamma_N$, $H_z = \operatorname{diam}(\omega_z)$, and $h_z \simeq H_z^{-1}|\omega_z|$. We also obtain a sharper version of (3), in which the interior-residual factors $\min\{1, H_z\varepsilon^{-1}\}$ are replaced by $\min\{1, h_z\varepsilon^{-1}\}$ and a few other terms are included (see (23) and Corollary 1).

The presence of Neumann boundary conditions is reflected in J computed on $\gamma_z\cap\Gamma_N$, and in additional (and, perhaps, unexpected) terms I_z^{ψ}:

$$|I_z^{\psi}|^2 \leqslant \min\{H_z,\, \varepsilon\}\, H_z\, \big|\operatorname{osc}(\varepsilon\psi\,; \gamma_z\cap\Gamma_N)\big|^2.$$

In the case of shape-regular triangulations, $\min\{h_z H_z,\, \varepsilon H_z^2 h_z^{-1}\} \simeq \min\{H_z, \varepsilon\}\, H_z$, while $\|J\|_{\infty\,;\gamma_z} \geqslant \frac12 \operatorname{osc}(J\,;\gamma_z\cap\Gamma_N) \geqslant \frac12\operatorname{osc}(\psi\,;\gamma_z\cap\Gamma_N)$. Hence, in this case, $\sum |I_z^{\psi}|^2$ is bounded by the first sum in (3), so may be skipped. For the case $\varepsilon = 1$, this yields a version of the standard estimator [1, §2.2].

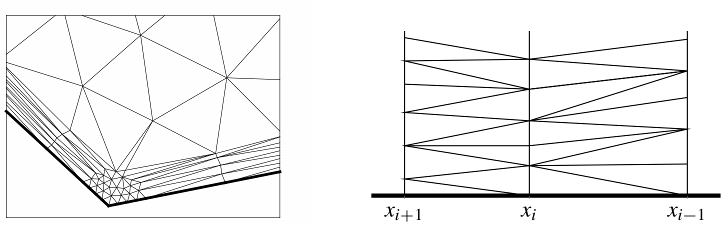

Fig. 1 Example of a mesh considered in [8, 9] (left), partially structured anisotropic mesh (right).

Our interest in this paper is in general anisotropic meshes, since such meshes, when constructed a priori, have been shown to offer an efficient way of computing reliable numerical approximations of solutions that exhibit sharp boundary and interior layers (see, e.g., [2, 5, 10, 15] and references therein). In the case of shape-regular triangulations, residual-type a posteriori estimates for equations of type (1) were proved in [17] in the energy norm, and more recently in [4] in the maximum norm. The case of anisotropic meshes having a tensor-product structure was addressed in [16, 6, 3]. Above, we briefly discussed anisotropic estimators [11, 12, 13].

The paper is organized as follows. In §2, we make basic triangulation assumptions and recall the anisotropic scaled-trace theorem from [8, 9]. The error is represented in terms of the residual in §3. Finally, in §4, we obtain the main results of the paper, a posteriori error estimators on partially structured meshes.

Notation. We write $a \simeq b$ when $a \lesssim b$ and $a \gtrsim b$, and $a \lesssim b$ when $a \leqslant Cb$ with a generic constant C depending on Ω and f, but C does not depend on either ε or the diameters and the aspect ratios of elements in $\mathscr{T}$. Also, for $\mathscr{D} \subset \bar{\Omega}$, $1 \leqslant p \leqslant \infty$, and $k \geqslant 0$, let $\|\cdot\|_{p\,;\mathscr{D}} = \|\cdot\|_{L_p(\mathscr{D})}$ and $|\cdot|_{k,p\,;\mathscr{D}} = |\cdot|_{W_p^k(\mathscr{D})}$, where $|\cdot|_{W_p^k(\mathscr{D})}$ is the standard Sobolev seminorm with integrability index p and smoothness index k.

2 Basic triangulation assumptions. Scaled trace bounds

We shall use $z = (x_z, y_z)$, S and T to respectively denote particular mesh nodes, edges and elements, while $\mathscr{N}$, $\mathscr{S}$ and $\mathscr{T}$ will respectively denote their sets. For each $T \in \mathscr{T}$, let H_T be the maximum edge length and $h_T := 2H_T^{-1}|T|$ be the minimum height in T. For each $z \in \mathscr{N}$, let ω_z be the patch of elements surrounding any $z \in \mathscr{N}$, $\mathscr{S}_z$ the set of edges originating at z, and

$$H_z := \operatorname{diam}(\omega_z), \quad h_z := \max_{T\subset\omega_z} h_T, \quad \gamma_z := \mathscr{S}_z \backslash \Gamma_D, \quad \mathring{\gamma}_z := \{S \subset \gamma_z : |S| \lesssim h_z\}. \quad (4)$$

Throughout the paper we make the following Triangulation Assumptions (that are automatically satisfied by shape-regular triangulations).

- *Maximum Angle condition.* Let the maximum interior angle in any triangle $T \in \mathscr{T}$ be uniformly bounded by some positive $\alpha_0 < \pi$.
- *Local Element Orientation condition.* For any $z \in \mathscr{N}$, a minimal rectangle $R_z \supset \omega_z$ is such that $|R_z| \simeq |\omega_z|$.
- Also, let the number of triangles containing any node be uniformly bounded.

Our analysis in [8, 9] applies to three node types, which we call (i) anisotropic, (ii) semi-anisotropic, and (iii) isotropic nodes (see Fig. 2). As in this preliminary contribution, only type (i) is considered, we skip the definitions (ii) and (iii).

(i) *Anisotropic Nodes*, whose set is denoted by $\mathscr{N}_{\mathrm{ani}}$, are such that

$$h_z < c_0 H_z, \qquad h_T \simeq h_z \text{ and } H_T \simeq H_z \quad \forall\, T \subset \omega_z, \quad (5)$$

where c_0 is a fixed small constant.

(1*) One typically expects anisotropic elements near Γ_D to be aligned along it. The boundary nodes for which this is not the case form a special set:

$$\mathscr{N}_D^* := \left\{\, z \in \bar{\Gamma}_D \cap \mathscr{N}_{\rm ani} \;:\; |\mathscr{S}_z \cap \Gamma_D| \lesssim h_z \;\;\text{or}\;\; z \notin \bar{\Gamma}_N \cap \bar{\Gamma}_D \text{ is a corner of } \Omega \,\right\}. \quad (6)$$

Next, we recall a version of the scaled trace theorem for possibly anisotropic nodes using, with $p = 1,2$, the scaled $W_p^1(\mathscr{D})$ norm

$$/\!\!/\!\!/ v /\!\!/\!\!/_{p\,;\mathscr{D}} := (\operatorname{diam}\mathscr{D})^{-1}\|v\|_{p\,;\mathscr{D}} + \|\nabla v\|_{p\,;\mathscr{D}}\,.$$

In particular, in view of $\operatorname{diam}(\omega_z) = H_z$ and $\operatorname{diam}(T) \simeq H_T$,

$$/\!\!/\!\!/ v /\!\!/\!\!/_{p\,;\omega_z} = H_z^{-1}\|v\|_{p\,;\omega_z} + \|\nabla v\|_{p\,;\omega_z}\,, \quad /\!\!/\!\!/ v /\!\!/\!\!/_{p\,;T} \simeq H_T^{-1}\|v\|_{p\,;T} + \|\nabla v\|_{p\,;T}\,. \quad (7)$$

Lemma 2.1 (Anisotropic scaled trace bounds [8, 9]). *For any node $z \in \mathscr{N}$ of type* (5)*, and any function $v \in W_1^1(\omega_z)$, one has*

$$\|v\|_{1\,;\mathring{\gamma}_z} + \frac{h_z}{H_z}\|v\|_{1\,;\gamma_z\backslash\mathring{\gamma}_z} + \frac{h_z}{H_z}\|v\|_{1\,;\bar{S}_z} \lesssim /\!\!/\!\!/ v /\!\!/\!\!/_{1\,;\omega_z}\,, \quad (8)$$

$$\|v\|_{1\,;\mathring{\gamma}_z} + \frac{h_z}{H_z}\|v\|_{1\,;\gamma_z\backslash\mathring{\gamma}_z} + \frac{h_z}{H_z}\|v\|_{1\,;\bar{S}_z} \lesssim \Big\{ h_z\,\|v\|_{2\,;\omega_z} /\!\!/\!\!/ v /\!\!/\!\!/_{2\,;\omega_z} \Big\}^{1/2}, \quad (9)$$

where γ_z and $\mathring{\gamma}_z$ are from (4), while $\bar{S}_z \subset \omega_z$ is any segment that originates at z and satisfies $|\bar{S}_z| \simeq H_z$.

3 Representation of the error in terms of the residual

Using the monotonicity of f and $C_f + \varepsilon^2 \geqslant 1$, one gets

$$\begin{aligned} |\!|\!| u_h - u |\!|\!|^2_{\varepsilon\,;\Omega} &\lesssim \varepsilon^2\langle \nabla(u_h-u), \nabla(u_h-u)\rangle + \langle f(\cdot\,;u_h) - f(\cdot\,;u), u_h - u\rangle \\ &= \varepsilon^2\langle \nabla u_h, \nabla(u_h-u)\rangle + \langle f(\cdot\,;u_h), u_h - u\rangle - \int_{\Gamma_N} \psi\,(u_h-u), \end{aligned}$$

where we also used (1). Next, assuming $|\!|\!| u_h - u |\!|\!|_{\varepsilon\,;\Omega} > 0$, let

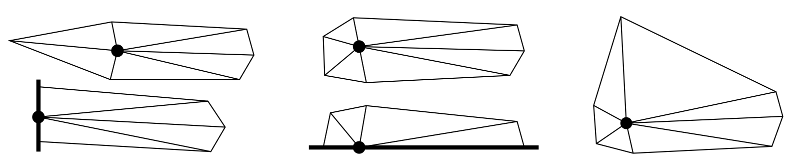

Fig. 2 Examples of anisotropic nodes $z \in \mathscr{N}_{\rm ani}$ (left), semi-anisotropic nodes $z \in \mathscr{N}_{\rm s.ani}$ (centre), an isotropic node $z \in \mathscr{N}_{\rm iso}$ (right), and a node $z \in \mathscr{N}_{\rm ani} \cap \mathscr{N}_D^*$ (bottom left); see [8, 9].

$$G := \frac{u_h - u}{|||u_h - u|||_{\varepsilon\,;\Omega}} \qquad \Rightarrow \qquad |||G|||_{\varepsilon\,;\Omega} = 1. \tag{10}$$

Now, $|||u_h - u|||_{\varepsilon\,;\Omega} \lesssim \varepsilon^2 \langle \nabla u_h, \nabla G\rangle + \langle f(\cdot\,; u_h), G\rangle - \int_{\Gamma_N} \psi G$. So (2) implies, $\forall v_h \in S_h$,

$$|||u_h - u|||_{\varepsilon\,;\Omega} \lesssim \varepsilon^2 \langle \nabla u_h, \nabla(G - v_h)\rangle + \langle f_h^I, G - v_h\rangle - \int_{\Gamma_N} \psi\,(G - v_h) + \langle f_h - f_h^I, G\rangle. \tag{11}$$

Here $\langle f_h - f_h^I, G\rangle =: \mathscr{E}_{\rm quad}$ is the quadrature error, for which $\|G\|_{2\,;\Omega} \leqslant 1$ implies

$$|\mathscr{E}_{\rm quad}| \lesssim \|f_h - f_h^I\|_{2\,;\Omega}\,. \tag{12}$$

Next, let ϕ_z be the standard linear hat function corresponding to $z \in \mathscr{N}$, and $v_h := G_h + \sum_{z\in\mathscr{N}} \bar g_z \phi_z \in S_h$, where $G_h \in S_h$ is some interpolant of G, while $\bar g_z$ is a certain average of $G - G_h$ near z (to be specified later), but $\bar g_z = 0$ for $z \in \Gamma_D$ (so that $v_h \in S_h$). Now, using $g := G - G_h$, one gets $G - v_h = g - \sum_{z\in\mathscr{N}} \bar g_z \phi_z = \sum_{z\in\mathscr{N}} (g - \bar g_z)\phi_z$. Combining this with (11) gives a standard error representation

$$\begin{aligned} |||u_h - u|||_{\varepsilon\,;\Omega} &\lesssim \sum_{z\in\mathscr{N}} \varepsilon^2 \int_{\gamma_z} J(g - \bar g_z)\phi_z + \sum_{z\in\mathscr{N}} \int_{\omega_z} f_h^I\,(g - \bar g_z)\phi_z + \mathscr{E}_{\rm quad} \\ &=: I + II + \mathscr{E}_{\rm quad}\,, \end{aligned} \tag{13}$$

which holds for any $G_h \in S_h$ and any $\{\bar g_z\}_{z\in\mathscr{N}}$ such that $\bar g_z = 0$ whenever $z \in \Gamma_D$. Here we use a standard definition for J with $J := \partial_\nu u_h\big|_{T'} + \partial_\nu u_h\big|_{T''}$ on an interior edge $\partial T' \cap \partial T'' \neq \varnothing$ (where $T', T'' \in \mathscr{T}$), and $J := \partial_\nu u_h - \psi$ on Γ_N.

4 Error analysis for a partially structured anisotropic mesh

Our ultimate goal is to consider a reasonably general anisotropic mesh such as addressed in [8, 9] (see Fig. 1, left, and Fig. 2). But in this preliminary contribution, to illustrate our approach, we restrict the analysis to a simpler, partially structured, anisotropic mesh in a square domain. To be more precise, let

$$\Omega := (0,1)^2, \qquad \Gamma_N := \{(x,y) \in \partial\Omega :\ x = 1 \text{ or } y = 1\}, \qquad \psi(0,1) = 0. \tag{14}$$

(The condition on ψ is a compatibility condition, as $u(0,y) = 1$ implies $\partial_y u(0,1) = 0$. If it is violated, the mesh node at $(0,1)$ is expected to be isotropic, and a version of our analysis below will apply.) The following triangulation assumptions are made.

A1. Let $\{x_i\}_{i=0}^n$ be an arbitrary mesh on the interval $(0,1)$ in the x direction. Then, let each $T \in \mathscr{T}$, for some i,
(i) have the shortest edge on the line $x = x_i$;
(ii) have a vertex on the line $x = x_{i+1}$ or $x = x_{i-1}$ (see Fig. 1, right).

A2. Let $\mathscr{N} = \mathscr{N}_{\rm ani}$, i.e. each mesh node z satisfies (5).

A3. *Quasi-non-obtuse anisotropic elements.* Let the maximum angle in any triangle be bounded by $\frac{\pi}{2} + \alpha_1 \frac{h_T}{H_T}$ for some positive constant α_1.

These conditions essentially imply that all mesh elements are anisotropic and aligned in the x-direction. They also imply that if $x_z = x_i$, then

$$\omega_z \subseteq \omega_z^* := (x_{i-1}, x_{i+1}) \times (y_z^-, y_z^+), \quad y_z^+ - y_z^- \simeq h_z, \quad \operatorname{diam} \omega_z^* \simeq H_z\,, \tag{15}$$

where (y_z^-, y_z^+) is the range of y within ω_z, and we also use $x_{-1} := x_0$ and $x_{n+1} := x_n$.

Remark 4.1. The above conditions (in particular, A3) imply that there is $J \lesssim 1$ such that $\omega_z^* \subset \omega_z^{(J)}$ for all $z \in \mathscr{N}$, with the notation $\omega_z^{(0)} := \omega_z$ and $\omega_z^{(j+1)}$ for the patch of elements in/touching $\omega_z^{(j)}$. (Note that $J = 1$ for any non-obtuse triangulation.)

4.1 Choice of $\bar g_z$. Main results

Following [8, 9], the choice of $\bar g_z$ in (13) is related to the orientation of anisotropic elements, and is crucial. Let $\bar g_z = 0$ for $z \in \Gamma_D$, and

$$\int_{x_{i-1}}^{x_{i+1}} (g(x, y_z) - \bar g_z)\, \varphi_i(x)\, dx = 0 \qquad \text{for } \; z = (x_i, y_z), \;\; 1 \leqslant i \leqslant n. \tag{16}$$

Here we use the standard one-dimensional hat function $\varphi_i(x)$ associated with the mesh $\{x_i\}$ (i.e. it has support on (x_{i-1}, x_{i+1}), equals 1 at $x = x_i$, and is linear on (x_{i-1}, x_i) and (x_i, x_{i+1})). Note that for $z = (x_i, 0)$, in view of $g = 0$ on Γ_D, the above definition (16) agrees with $\bar g_z = 0$, earlier prescribed on Γ_D.

Remark 4.2. For $x_z = x_i$, let $\bar S_z \subset \omega_z^*$ be the interval joining (x_{i-1}, y_z) and (x_{i+1}, y_z), $1 \leqslant i \leqslant n$. Then (16) is identical to $\int_{\bar S_z} (g - \bar g_z)\, \varphi_i = 0$. Also, for non-obtuse triangulations, it is equivalent to $\int_{\bar S_z} (g - \bar g_z)\, \phi_z = 0$. The reader may compare this with a more standard choice, denoted here by $\bar g_z'$: $\int_{\omega_z} (g - \bar g_z')\, \phi_z = 0$ (see, e.g., [14, Lecture 5]).

Remark 4.3. It is sometimes helpful to tweak the definition (16) of $\{\bar g_z\}_{z \in \mathscr{N}}$ and use instead $\{\bar g_z^*\}_{z \in \mathscr{N}}$ defined for $z \in \mathscr{N} \backslash \Gamma_D$ with $x_z = x_i$ by

$$\int_{\omega_z^*} [g(x,y) - \bar g_z^*]\, \varphi_i(x) = 0, \tag{17}$$

(where ω_z^* is from (15)), and $\bar g_z^* = 0$ for $z \in \Gamma_D$. Note that

$$h_z H_z |\bar g_z^*| \lesssim \|g\|_{1\,;\omega_z^*}\,, \quad H_z |\bar g_z - \bar g_z^*| \lesssim \|\nabla g\|_{1\,;\omega_z^*}\,, \quad |\omega_z^*| \simeq h_z H_z\,. \tag{18}$$

Here the first relation is obvious, while $\int_{\omega_z^*} [g(x, y_z) - g(x,y)]\, \varphi_i(x) \simeq h_z H_z (\bar g_z - \bar g_z^*)$ implies $H_z |\bar g_z - \bar g_z^*| \lesssim \|\partial_y g\|_{1\,;\omega_z^*}$ and so the second relation.

Theorem 4.1 *For the solution u of* (1), (14)*, and the computed solution u_h of* (2)*, let $g = G - G_h$ with G from* (10) *and any $G_h \in S_h$, and*

$$\Theta := \varepsilon^2 \|\nabla g\|_{2\,;\Omega}^2 + \sum_{z \in \mathscr{N}} \big(1 + \varepsilon^2 H_z^{-2}\big) \|g\|_{2\,;\omega_z}^2\,. \tag{19}$$

Then $|||u_h-u|||_{\varepsilon\,;\Omega} \lesssim I + II + \mathscr{E}_{\rm quad}$, *where* $\mathscr{E}_{\rm quad}$ *is bounded by* (12)*, and, under conditions A1–A3,*

$$|I - I^{\psi}| \lesssim \Big\{\Theta \sum_{z\in\mathscr{N}} \lambda_z \,\|\varepsilon J_z\|^2_{\infty\,;\gamma_z}\Big\}^{1/2}, \qquad \lambda_z := h_z H_z \min\{1,\, \varepsilon H_z h_z^{-2}\}, \tag{20}$$

$$|II| \lesssim \Big\{\Theta \sum_{z\in\mathscr{N}} \|\lambda_z' f_h^I\|^2_{2\,;\omega_z}\Big\}^{1/2}, \qquad \lambda_z' := \min\{1,\, H_z\varepsilon^{-1}\}, \tag{21}$$

$$|I^{\psi}| \lesssim \Big\{\Theta \sum_{\substack{z\in\mathscr{N}:\\ |\gamma_z\cap\Gamma_N|\simeq H_z}} \lambda_z' \,\varepsilon H_z\,\big|\operatorname{osc}(\varepsilon\psi\,;\gamma_z\cap\Gamma_N)\big|^2\Big\}^{1/2}. \tag{22}$$

Additionally, one has an alternative bound

$$\begin{aligned}|II| \lesssim \Big\{\Theta \sum_{z\in\mathscr{N}\backslash\mathscr{N}_D^*} \big\|\min\{1,\, h_z\varepsilon^{-1}\}\, f_h^I\big\|^2_{2\,;\omega_z} + \Theta \sum_{z\in\mathscr{N}\backslash\mathscr{N}_D^*} \big\|\lambda_z' \operatorname{osc}(f_h^I\,;\omega_z)\big\|^2_{2\,;\omega_z}&\\ + \Theta \sum_{z\in\mathscr{N}_D^*} \|\lambda_z' f_h^I\|^2_{2\,;\omega_z}\Big\}^{1/2},&\end{aligned} \tag{23}$$

where $\mathscr{N}_D^* = \{z\in\mathscr{N} : x_z = 0\}$ *(in agreement with* (6)*).*

Corollary 1 (A posteriori error estimator). *Under the conditions of Theorem 4.1,* $|||u_h-u|||_{\varepsilon\,;\Omega} \lesssim I+II+\mathscr{E}_{\rm quad}$, *where* $\mathscr{E}_{\rm quad}$ *is bounded by* (12)*, while for I and II one has bounds* (20)–(23) *with* $\Theta := 1$.

Proof. Under more general conditions than A1–A3, there exists $G_h \in S_h$ such that $\Theta \lesssim |||G|||_{\varepsilon\,;\Omega} = 1$; see [9, Theorem 7.4]. □

Remark 4.4. An inspection of the proof shows that in the bound (22) for I^{ψ}, one can replace $\gamma_z\cap\Gamma_N$ by $[\gamma_z\backslash\mathring{\gamma}_z]\cap\Gamma_N$, which gives a slightly sharper bound.

Proof of Theorem 4.1. We partially follow and invoke some auxiliary results from the proof of [9, Theorem 5.1]. The proof of (20) and (22) is given in §4.2 below. For the remaining interior-residual bounds (21) and (23), an inspection of [9, Section 5.3] shows that the estimation of the interior-residual component II of the error (13) applies to our case, with the only change in that II involves $\sum_{i=1}^{n} II_i$ (rather than $\sum_{i=1}^{n-1}$), where $II_i := \sum_{z\in\mathscr{N}_i} \int_{\omega_z} f_h^I(x_i,y)\,(g-\bar{g}_z^*)\,\phi_z$, with $\mathscr{N}_i := \{z\in\mathscr{N} : x_z = x_i\}$, while $\mathscr{N}^*_{\partial\Omega}$ of [9] is now denoted $\mathscr{N}_D^* = \mathscr{N}_0$. Note also that Remark 4.5 below is crucial for (21) and (23). □

4.2 Jump Residual. Proof of (20) and (22)

Proof of (20) and (22). Split I of (13) as $I = \sum_{z\in\mathscr{N}} I_z$, where

$$I_z := \varepsilon^2 \int_{\gamma_z} J(g-\bar g_z)\phi_z. \tag{24}$$

When considering J on $\gamma_z\backslash\partial\Omega = \gamma_z\backslash\Gamma_N$, we adapt the notational convention that the unit normal ν to any edge in γ_z takes the clockwise direction about z, while $[\![w]\!]$, for any w, is the jump in w across any edge in γ_z evaluated in the anticlockwise direction about z. Then

$$J\big|_{\gamma_z\backslash\Gamma_N} = [\![\nabla u_h]\!]\cdot\nu = [\![\partial_x u_h]\!]\nu_x + [\![\partial_y u_h]\!]\nu_y\,.$$

So I_z can be split as

$$\begin{aligned} I_z = I_z' + I_z'' + I_z''' + I_z^N := \varepsilon^2 &\int_{\gamma_z\backslash\Gamma^N}(g-\bar g_z)\phi_z[\![\partial_x u_h]\!]\,\nu_x + \varepsilon^2\int_{\mathring{\gamma}_z\cap\Gamma^N} J(g-\bar g_z)\phi_z\\ &+\varepsilon^2\int_{\gamma_z\backslash\Gamma^N}[g-g(x,y_z)]\,\phi_z[\![\partial_y u_h]\!]\,\nu_y\\ &+\varepsilon^2\int_{\gamma_z\backslash\Gamma^N}[g(x,y_z)-\bar g_z]\,\phi_z[\![\partial_y u_h]\!]\,\nu_y\\ &+\varepsilon^2\int_{[\gamma_z\backslash\mathring{\gamma}_z]\cap\Gamma^N}(\partial_\nu u_h-\psi)(g-\bar g_z)\phi_z\,. \end{aligned} \tag{25}$$

For the final term here one has, using $\partial_\nu u_h = \partial_y u_h$ on $[\gamma_z\backslash\mathring{\gamma}_z]\cap\Gamma^N\subset\partial\Omega\cap\{y=1\}$,

$$I_z^N = \varepsilon^2\int_{[\gamma_z\backslash\mathring{\gamma}_z]\cap\Gamma^N}\partial_y u_h(g-\bar g_z)\phi_z - \underbrace{\varepsilon^2\int_{[\gamma_z\backslash\mathring{\gamma}_z]\cap\Gamma^N}\psi(g-\bar g_z)\phi_z}_{=:I_z^\psi}\,. \tag{26}$$

We claim that to get the desired assertions (20) and (22), it suffices to show that

$$|I_z'|+|I_z''| \lesssim \varepsilon /\!\!/\!\!/ g/\!\!/\!\!/_{1;\omega_z^*}\,\|\varepsilon J\|_{\infty;\gamma_z}, \qquad I_z''' + (I_z^N + I_z^\psi) = 0, \tag{27}$$

$$|I_z| \lesssim \varepsilon\frac{H_z}{h_z}\Big\{h_z\|g\|_{2;\omega_z^*}/\!\!/\!\!/ g/\!\!/\!\!/_{2;\omega_z^*}\Big\}^{1/2}\|\varepsilon J\|_{\infty;\gamma_z}\,, \tag{28}$$

and

$$|I_z^\psi| \lesssim \varepsilon\Big\{H_z\|g\|_{2;\omega_z^\psi}/\!\!/\!\!/ g/\!\!/\!\!/_{2;\omega_z^\psi}\Big\}^{1/2}\mathrm{osc}(\varepsilon\psi;[\gamma_z\backslash\mathring{\gamma}_z]\cap\Gamma_N). \tag{29}$$

In (29),

$$\omega_z^\psi := (x_{i-1},x_{i+1})\times(1-H_z,1) \qquad \text{for any } z=(x_i,1),\quad i=1,\dots,n, \tag{30}$$

is an isotropic rectangle with the upper edge $[\gamma_z\backslash\mathring{\gamma}_z]\cap\Gamma_N$ (a similar triangle can be used instead).

Indeed, combining (25), (26) with (27), (28) yields $|I_z-\widetilde I_z^\psi|\lesssim(\theta_z\lambda_z)^{1/2}\|\varepsilon J\|_{\infty;\gamma_z}$, where $\widetilde I_z^\psi$ is either I_z^ψ or 0, and $|\widetilde I_z^\psi|\leqslant|I_z^\psi|\lesssim(\theta_z^\psi\lambda_z'\varepsilon H_z)^{1/2}\mathrm{osc}(\varepsilon\psi;[\gamma_z\backslash\mathring{\gamma}_z]\cap\Gamma_N)$, in view of (29), where

$$\theta_z := \lambda_z^{-1}\varepsilon^2 \min\Big\{ /\!/\!/ g /\!/\!/^2_{1;\omega_z^*},\, H_z^2\, h_z^{-1} \|g\|_{2;\omega_z^*} /\!/\!/ g /\!/\!/_{2;\omega_z^*} \Big\},$$
$$\theta_z^{\psi} := \lambda_z'^{-1}\varepsilon \|g\|_{2;\omega_z^{\psi}} /\!/\!/ g /\!/\!/_{2;\omega_z^{\psi}}\,.$$

With $I^{\psi} := \sum_{z\in\mathcal{N}} \widetilde{I}_z^{\psi}$, an application of Hölder's inequality yields (20) and (22) in view of $\sum_{z\in\mathcal{N}}(\theta_z + \theta_z^{\psi}) \lesssim \Theta$. For the latter, note that $\min(a,bc)/\min(1,c) \leqslant a+b$ (for any a, b, $c>0$) implies $\theta_z \lesssim \varepsilon^2 /\!/\!/ g /\!/\!/^2_{2;\omega_z^*} + \varepsilon \|g\|_{2;\omega_z^*} /\!/\!/ g /\!/\!/_{2;\omega_z^*}$. Similarly, $\theta_z^{\psi} \lesssim \varepsilon^2 /\!/\!/ g /\!/\!/^2_{2;\omega_z^{\psi}} + (1+\varepsilon^2 H_z^{-2})\|g\|^2_{2;\omega_z^{\psi}}$ (as $\lambda_z'^{-1} \simeq 1+\varepsilon H_z^{-1}$). Combining these two observations with (7), (15), Remark 4.1, and also (30), completes the proof of (20) and (22) from (27), (28), (29).

Hence, it remains to establish (27), (28) and (29). The bounds for I_z' and I_z'' in (27), as well as I_z in (28), can be found in [9, Section 5.2, see (5.12), (5.13)]; they are obtained from (25) and (24) using (8) and (9) respectively. It should be noted that, compared to [9], there is an additional term in I_z', which involves $\int_{\mathring{\gamma}_z\cap\Gamma^N}$ and can be easily estimated again using (8).

The proof of $I_z''' + (I_z^N + I_z^{\psi}) = 0$ in (27) is more delicate. It is convenient to adapt the convention that $u_h = 0$ in $\mathbb{R}^2\backslash\bar{\Omega}$ when computing $[\![\partial_y u_h]\!]$ across the boundary edges. With this convention, one can show, for each $z=(x_i,y_z)$, that

$$I_z''' + (I_z^N + I_z^{\psi}) = \varepsilon^2 \Big(\sum_{S\in\gamma_z\backslash\mathring{\gamma}_z} [\![\partial_y u_h]\!] \Big) \int_{x_{i-1}}^{x_i} [g(x,y_z) - \bar{g}_z]\, \varphi_i(x)\, dx\,, \tag{31}$$

with $\int_{x_{i-1}}^{x_i}$, in the case of $i=0$, replaced by $-\int_{x_i}^{x_{i+1}}$ (see [8, 9] for similar representations of I_z'''). The above representation is obtained noting that $\nu_y = 0$ on $\mathring{\gamma}_z\backslash\Gamma_N$ and $\phi_z = \varphi_i(x)$ on $\gamma_z\backslash\mathring{\gamma}_z$, while $\nu_y ds = dx$ on any edge connecting z to the vertical line $\{x=x_{i-1}\}$ and $\nu_y ds = -dx$ on any edge connecting z to the vertical line $\{x=x_{i+1}\}$. On $[\gamma_z\backslash\mathring{\gamma}_z]\cap\Gamma^N$, one uses $ds=dx$. For $1\leqslant i\leqslant n-1$, we also invoke (16).

Whenever $i=n$ in (31), one immediately gets $I_z''' + (I_z^N + I_z^{\psi}) = 0$ from (16). Otherwise, if $0\leqslant i\leqslant n-1$ and $y_z>0$, noting that $[\![\partial_y u_h]\!] = 0$ on $\mathring{\gamma}_z$, as well as on any element edge lying on $\{x=0\}$, one gets $\sum_{S\in\gamma_z\backslash\mathring{\gamma}_z}[\![\partial_y u_h]\!] = \sum_{S\in\mathscr{S}_z}[\![\partial_y u_h]\!] = 0$, so again $I_z''' + (I_z^N + I_z^{\psi}) = 0$ immediately follows from (16). Finally, if $y_z=0$, one employs $g(x,y_z) = \bar{g}_z = 0$ combined with (16).

We now proceed to getting (29). Note that in the definition of I_z^{ψ} in (26) one has $\phi_z = \varphi_i(x)$ and $\int_{[\gamma_z\backslash\mathring{\gamma}_z]\cap\Gamma^N} = \int_{x_{i-1}}^{x_{i+1}} dx$. Now, if $z=(x_i,1)$ for $1\leqslant i\leqslant n$, recalling (16), one can replace ψ in I_z^{ψ} by $\psi - \psi(z)$, so

$$|I_z^{\psi}| \lesssim \varepsilon \Big\{ \int_{[\gamma_z\backslash\mathring{\gamma}_z]\cap\Gamma^N} |g| \Big\}\, \mathrm{osc}(\varepsilon\psi\,; [\gamma_z\backslash\mathring{\gamma}_z]\cap\Gamma_N).$$

This yields (29) by an application of (9), in which ω_z is replaced by the isotropic domain ω_z^{ψ} from (30), and hence h_z replaced by H_z. The remaining case of $z=(0,1)$ is considered similarly, only using $\bar{g}_z = 0$ and $|\psi| \leqslant \mathrm{osc}(\varepsilon\psi\,; [\gamma_z\backslash\mathring{\gamma}_z]\cap\Gamma_N)$ (the latter follows from the final condition in (14)). This completes the proof of (27), (28) and (29), and hence of (20) and (22). □

Remark 4.5. An inspection of the above proof shows that it remains valid if $\{\bar{g}_z\}_{z\in\mathscr{N}}$ defined by (16) are replaced by $\{\bar{g}_z^*\}_{z\in\mathscr{N}}$ from (17). Indeed, I_z will include an additional component $I_z^* := \varepsilon^2\int_{\gamma_z} J(\bar{g}_z - \bar{g}_z^*)$, for which one easily gets $|I_z^*| \leqslant \varepsilon H_z|\bar{g}_z - \bar{g}_z^*|\,\|\varepsilon J\|_{\infty;\gamma_z}$. For this component, a bound of type (27) is obtained using the second relation from (18), while a bound of type (28) follows from the first relation in (18) combined with the bound for $\bar{g}_z$ from [9, (5.15)].

References

1. Ainsworth, M., Oden, J. T.: A posteriori error estimation in finite element analysis. Wiley-Interscience, New York (2000)
2. Bakhvalov, N. S.: On the optimization of methods for solving boundary value problems with boundary layers. Zh. Vychisl. Mat. Mat. Fis. **9**, 841–859 (1969) (in Russian)
3. Chadha, N. M., Kopteva, N.: Maximum norm a posteriori error estimate for a 3d singularly perturbed semilinear reaction-diffusion problem. Adv. Comput. Math. **35**, 33–55 (2011)
4. Demlow, A., Kopteva, N.: Maximum-norm a posteriori error estimates for singularly perturbed elliptic reaction-diffusion problems. Numer. Math. (2015) (published online 14 August 2015)
5. Kopteva, N.: Maximum norm error analysis of a 2d singularly perturbed semilinear reaction-diffusion problem. Math. Comp. **76**, 631–646 (2007)
6. Kopteva, N.: Maximum norm a posteriori error estimate for a 2d singularly perturbed reaction-diffusion problem. SIAM J. Numer. Anal. **46**, 1602–1618 (2008)
7. Kopteva, N.: Linear finite elements may be only first-order pointwise accurate on anisotropic triangulations. Math. Comp. **83**, 2061–2070 (2014)
8. Kopteva, N.: Maximum-norm a posteriori error estimates for singularly perturbed reaction-diffusion problems on anisotropic meshes. SIAM J. Numer. Anal. **53**, 2519–2544 (2015)
9. Kopteva, N.: Energy-norm a posteriori error estimates for singularly perturbed reaction-diffusion problems on anisotropic meshes. `http://www.staff.ul.ie/natalia/pubs.html`
10. Kopteva, N., O'Riordan, E.: Shishkin meshes in the numerical solution of singularly perturbed differential equations. Int. J. Numer. Anal. Model. **7**, 393–415 (2010)
11. Kunert, G.: An a posteriori residual error estimator for the finite element method on anisotropic tetrahedral meshes. Numer. Math. **86**, 471–490 (2000)
12. Kunert, G.: Robust a posteriori error estimation for a singularly perturbed reaction-diffusion equation on anisotropic tetrahedral meshes. Adv. Comput. Math. **15**, 237–259 (2001)
13. Kunert, G., Verfürth, R.: Edge residuals dominate a posteriori error estimates for linear finite element methods on anisotropic triangular and tetrahedral meshes. Numer. Math. **86**, 283–303 (2000)
14. Nochetto, R.H.: Pointwise a posteriori error estimates for monotone semi-linear equations. Lecture Notes at 2006 CNA Summer School Probabilistic and Analytical Perspectives on Contemporary PDEs, `http://www.math.cmu.edu/cna/Summer06/lecturenotes/nochetto/`
15. Roos, H.-G., Stynes, M., Tobiska, T.: Robust Numerical Methods for Singularly Perturbed Differential Equations. Springer, Berlin (2008)
16. Siebert, K. G.: An a posteriori error estimator for anisotropic refinement. Numer. Math. **73**, 373–398 (1996)
17. Verfürth, R.: Robust a posteriori error estimators for a singularly perturbed reaction-diffusion equation. Numer. Math. **78**, 479–493 (1998)