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Energy-norm a posteriori error estimates for singularly perturbed reaction-diffusion problems on anisotropic meshes

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Abstract Residual-type a posteriori error estimates in the energy norm are given for singularly perturbed semilinear reaction-diffusion equations posed in polygonal domains. Linear finite elements are considered on anisotropic triangulations. The error constants are independent of the diameters and the aspect ratios of mesh elements and of the small perturbation parameter. To deal with anisotropic triangulations, a special quasi-interpolation operator is employed that may be of independent interest.

Keywords Anisotropic triangulation · A posteriori error estimate · Energy norm · Singular perturbation · Reaction-diffusion · Quasi-interpolation

Mathematics Subject Classification (2000) 65N15 · 65N30

1 Introduction

This paper addresses finite element approximations to singularly perturbed semilinear reaction-diffusion equations of the form

$$Lu := -\varepsilon^2 \triangle u + f(x, y; u) = 0 \quad \text{for} \;\; (x, y) \in \Omega, \qquad u = 0 \quad \text{on} \;\; \partial\Omega, \qquad (1.1)$$

posed in a, possibly non-Lipschitz, polygonal domain $\Omega \subset \mathbb{R}^2$. Here $0 < \varepsilon \leqslant 1$. We also assume that f is continuous on $\Omega \times \mathbb{R}$ and satisfies $f(\cdot; s) \in L_\infty(\Omega)$ for all $s \in \mathbb{R}$, and the one-sided Lipschitz condition $f(x, y; v) - f(x, y; w) \geqslant C_f[v-w]$ whenever $v \geqslant w$, with some constant $C_f \geqslant 0$. Then there is a unique solution $u \in W^2_\ell(\Omega) \subseteq W^1_q \subset C(\bar{\Omega})$ for some $\ell > 1$ and $q > 2$ [5, Lemma 1]. We additionally assume that $C_f + \varepsilon^2 \geqslant 1$ (as (1.1) can always be reduced to this case by a division by $C_f + \varepsilon^2$).

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Our goal is to give residual-type a posteriori error estimates on reasonably general anisotropic meshes (such as on Fig. 1 and Fig. 2) that one can expect to generate using anisotropic mesh adaptation procedures. The error is estimated in the energy norm $\|\!|\cdot|\!\|_{\varepsilon\,;\Omega}$, which is an appropriately scaled $W_2^1(\Omega)$ norm naturally associated with our problem, defined, for any $\mathcal{D}\subseteq\Omega$, by

$$|\!|\!| v |\!|\!|_{\varepsilon\,;\mathcal{D}} := \Big\{\varepsilon^2\|\nabla v\|^2_{2\,;\mathcal{D}} + \|v\|^2_{2\,;\mathcal{D}}\Big\}^{1/2}.$$

We discretize (1.1) using linear finite elements. Let $S_h\subset H_0^1(\Omega)\cap C(\bar\Omega)$ be a piecewise-linear finite element space relative to a triangulation $\mathcal{T}$, and let the computed solution $u_h\in S_h$ satisfy

$$\varepsilon^2\langle\nabla u_h,\nabla v_h\rangle+\langle f_h^I,v_h\rangle=0\quad\forall\, v_h\in S_h,\qquad\qquad f_h(\cdot):=f(\cdot;u_h).\qquad(1.2)$$

Here $\langle\cdot,\cdot\rangle$ is the $L_2(\Omega)$ inner product, and f_h^I is the standard piecewise-linear Lagrange interpolant of f_h.

To give a flavour of our results, assuming that all mesh elements are anisotropic, our first estimator reduces to

$$|\!|\!| u_h-u |\!|\!|_{\varepsilon\,;\Omega}\leqslant C\,\Big\{\sum_{z\in\mathcal{N}}\min\{h_zH_z,\,\varepsilon H_z^2h_z^{-1}\}\,\big\|\varepsilon J_z\big\|^2_{\infty\,;\gamma_z} + \sum_{z\in\mathcal{N}}\big\|\min\{1,\,H_z\varepsilon^{-1}\}\,f_h^I\big\|^2_{2\,;\omega_z}+\big\|f_h-f_h^I\big\|^2_{2\,;\Omega}\Big\}^{1/2},\quad(1.3)$$

where C is independent of the diameters and the aspect ratios of elements in $\mathcal{T}$, and of ε (combine (4.4), (4.5), (6.1), (6.2) with Theorem 7.4). Here $\mathcal{N}$ is the set of nodes in $\mathcal{T}$, J_z is the standard jump in the normal derivative of u_h across an element edge, ω_z is the patch of elements surrounding any $z\in\mathcal{N}$, γ_z is the set of edges in the interior of ω_z, $H_z=\operatorname{diam}(\omega_z)$, and $h_z\simeq H_z^{-1}|\omega_z|$.

To relate (1.3) to interpolation error bounds, as well as to possible adaptive-mesh construction strategies, note that $|J_z|$ may be interpreted as approximating the diameter of ω_z under the metric induced by the squared Hessian matrix of the exact solution (while f_h^I approximates $\varepsilon^2\triangle u$).

We also obtain a sharper estimator under some additional mesh assumptions: a version of (1.3) will be given, in which, roughly speaking $\min\{1,\,H_z\varepsilon^{-1}\}$ is replaced by $\min\{1,\,h_z\varepsilon^{-1}\}$ with a few other terms included.

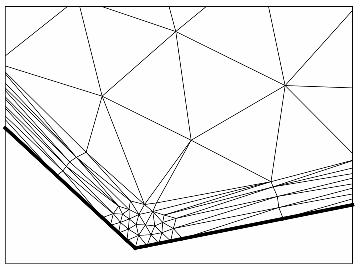


Fig. 1 Example of a mesh considered in §6, that satisfies conditions $\mathcal{A}1$–$\mathcal{A}5$.

Locally anisotropic meshes offer an efficient way of computing reliable numerical approximations of solutions that exhibit sharp boundary and interior layers. (In the context of equations of type (1.1) with $\varepsilon \ll 1$, see, e.g., [7,12,17] and references therein.) But such anisotropic meshes are frequently constructed a priori or by heuristic methods, while the majority of available a posteriori error estimators assume shape regularity of the mesh. [1]

In the case of shape-regular triangulations, residual-type a posteriori error estimates for equations of type (1.1) and its version in $\mathbb{R}^3$ were proved in [19] in the energy norm, and more recently in [5] in the maximum norm. (Note that in this case, (1.3) becomes almost identical with the estimator of [19].) The case of anisotropic meshes having a tensor-product structure was addressed in [18] for the Laplace equation and in [8,4] for problems of type (1.1), with the error estimators given, respectively, in the H^1 norm and the maximum norm.

In this paper, we are interested in a more challenging case of general anisotropic meshes. In this case, a posteriori error estimates can be found in [13,15] for the Laplace equation in the H^1 norm, and in [14,15] for a linear constant-coefficient version of (1.1) in the energy norm. Note that the error constants in the estimators of [13–15] involve the so-called matching functions; the latter depend on the unknown error and take moderate values only when the grid is either isotropic, or, being anisotropic, is aligned correctly to the solution, while, in general, they may be as large as mesh aspect ratios.

The presence of such matching functions in the estimator is clearly undesirable. It is entirely avoided in a more recent paper [10], where the error estimators for problem (1.1) are given in the maximum norm on anisotropic meshes. The application of a scaled trace theorem when estimating the jump residual terms (which typically causes the mesh aspect ratios to appear in the estimator) is combined in [10] with a certain technical trick; see Remark 5.3. In the present paper, we extend the analysis of [10] to estimate the error in the energy norm and thus also avoid any matching functions in the estimators.

Our general approach follows [10], but a number of nontrivial changes are introduced as we estimate the error in a different norm. In particular, the scaled trace theorem is considerably modified. More importantly, while a version of the Lagrange interpolant sufficed in [10], now we have to use quasi-interpolation on anisotropic meshes; the latter may be of independent interest. Compared to [10], we also simplify some mesh assumptions.

The paper is organized as follows. In §2, we make basic triangulation assumptions. §3 generalizes the scaled trace theorem from [10]. In §4, the error is represented in terms of the residual, which is followed, in §5, by a simplified version of our analysis for partially structured meshes. The main results are given in §6, while §7 is devoted to quasi-interpolation operators on anisotropic meshes. We conclude the paper by presenting some numerical results in §8.

Notation. We write $a \simeq b$ when $a \lesssim b$ and $a \gtrsim b$, and $a \lesssim b$ when $a \leqslant Cb$ with a generic constant C depending on Ω and f, but not on either ε or the diameters and the aspect ratios of elements in $\mathcal{T}$. Also, for $\mathcal{D} \subset \bar{\Omega}$, $1 \leqslant p \leqslant \infty$, and $k \geqslant 0$, let $\|\cdot\|_{p\,;\mathcal{D}} = \|\cdot\|_{L_p(\mathcal{D})}$ and $|\cdot|_{k,p\,;\mathcal{D}} = |\cdot|_{W_p^k(\mathcal{D})}$, where $|\cdot|_{W_p^k(\mathcal{D})}$ is the standard Sobolev seminorm, and $\operatorname{osc}(v;\mathcal{D}) = \sup_{\mathcal{D}} v - \inf_{\mathcal{D}} v$ for $v \in L_\infty(\mathcal{D})$.

2 Basic triangulation assumptions

We shall use $z = (x_z, y_z)$, S and T to respectively denote particular mesh nodes, edges and elements, while $\mathcal{N}$, $\mathcal{S}$ and $\mathcal{T}$ will respectively denote their sets. For each $T \in \mathcal{T}$, let H_T be the maximum edge length and $h_T := 2H_T^{-1}|T|$ be the minimum height in T. For each $z \in \mathcal{N}$, let ω_z be the patch of elements surrounding any $z \in \mathcal{N}$, $\mathcal{S}_z$ the set of edges originating at z, and

$$H_z := \operatorname{diam}(\omega_z), \quad h_z := \max_{T\subset\omega_z} h_T, \quad \gamma_z := \mathcal{S}_z \backslash \partial\Omega, \quad \mathring{\gamma}_z := \{S \subset \gamma_z : |S| \lesssim h_z\}. \tag{2.1}$$

Throughout the paper we make the following **Triangulation Assumptions**.

- *Maximum Angle condition.* Let the maximum interior angle in any triangle $T \in \mathcal{T}$ be uniformly bounded by some positive $\alpha_0 < \pi$.
- *Local Element Orientation condition.* For any $z \in \mathcal{N}$, there is a rectangle $R_z \supset \omega_z$ such that $|R_z| \simeq |\omega_z|$.
- Also, let the number of triangles containing any node be uniformly bounded.

Note that the above conditions are automatically satisfied by shape-regular triangulations.

Additionally, we restrict our analysis to the following **Node Types** ([10]; see also Fig. 2), defined using a fixed small constant c_0 (to distinguish between anisotropic and isotropic elements), with the notation $a \ll b$ for $a < c_0 b$.

(1) *Anisotropic Nodes*, whose set is denoted by $\mathcal{N}_{\rm ani}$, are such that

$$h_z \ll H_z, \qquad h_T \simeq h_z \;\text{ and }\; H_T \simeq H_z \quad \forall\, T \subset \omega_z. \tag{2.2}$$

Note that the above implies that $\mathcal{S}_z$ contains at most two edges of length $\lesssim h_z$.

(2) *Semi-Anisotropic Nodes*, whose set is $\mathcal{N}_{\rm s.ani}$, are such that $z \notin \mathcal{N}_{\rm ani}$ and

$$h_z \ll H_z, \qquad H_T \simeq H_z \text{ and } h_T \simeq h_z \quad\text{or}\quad H_T \simeq h_T \simeq h_z \quad \forall\, T \subset \omega_z. \tag{2.3}$$

Also, all edges $S \in \mathcal{S}_z$ of length $|S| \simeq H_z$ lie inside a sector of angle $\simeq \frac{h_z}{H_z}$ centered at z (see Fig. 2, centre).

(3) *Isotropic Nodes*, whose set is denoted by $\mathcal{N}_{\rm iso}$, are such that

$$\begin{gathered} h_z \simeq H_z, \qquad h_T \simeq H_T \;\text{ or }\; H_T \simeq H_z \quad \forall\, T \subset \omega_z, \\ \forall\, S \in \mathcal{S}_z \quad \exists\, \tilde T \subset \omega_z \;:\; S \subset \partial \tilde T, \quad h_{\tilde T} \simeq H_{\tilde T}, \end{gathered} \tag{2.4}$$

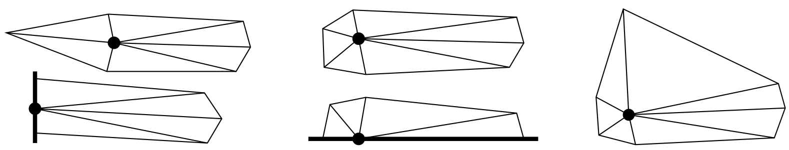


Fig. 2 Examples of anisotropic nodes $z \in \mathcal{N}_{\rm ani}$ (left), semi-anisotropic nodes $z \in \mathcal{N}_{\rm s.ani}$ (centre), an isotropic node $z \in \mathcal{N}_{\rm iso}$ (right), and a node $z \in \mathcal{N}_{\rm ani} \cap \mathcal{N}^*_{\partial\Omega}$ (bottom left).

where the triangle $\tilde{T}$ is not necessarily in $\mathcal{T}$, but can be constructed as above using S as one of its edges. By this definition, at least one T in ω_z is isotropic and satisfies $h_T \simeq H_T \simeq H_z$, while some T in ω_z may be anisotropic, and others, being isotropic, may have $H_T \ll H_z \simeq h_z$ (see Fig. 2, right). Note that if z is surrounded only by shape-regular elements, then it is isotropic.

(1*) One typically expects anisotropic elements near the boundary to be aligned along it. To distinguish the boundary nodes for which this is not the case, we introduce a special set of boundary nodes $\mathcal{N}^*_{\partial\Omega}$ as follows:

$$\mathcal{N}^*_{\partial\Omega} := \left\{ z \in \partial\Omega \cap \mathcal{N}\backslash\mathcal{N}_{\rm iso} \quad : \quad |\mathcal{S}_z \cap \partial\Omega| \lesssim h_z \quad \text{or} \quad z \text{ is a corner of } \Omega \right\}. \tag{2.5}$$

It will be assumed throughout the paper that $\mathcal{N}^*_{\partial\Omega}\cap\mathcal{N}_{\rm s.ani} = \varnothing$ so $\mathcal{N}^*_{\partial\Omega} \subset \mathcal{N}_{\rm ani}$.

3 Scaled trace bounds

In this section, we formulate a version of the scaled trace theorem for permitted (possibly anisotropic) nodes using, with $p = 1, 2$, the scaled $W^1_p(\mathcal{D})$ norm

$$|\!|\!|v|\!|\!|_{p\,;\mathcal{D}} := ({\rm diam}\mathcal{D})^{-1}\|v\|_{p\,;\mathcal{D}} + \|\nabla v\|_{p\,;\mathcal{D}}\,.$$

In particular, in view of ${\rm diam}(\omega_z) = H_z$ and ${\rm diam}(T) \simeq H_T$,

$$|\!|\!|v|\!|\!|_{p\,;\omega_z} = H_z^{-1}\|v\|_{p\,;\omega_z} + \|\nabla v\|_{p\,;\omega_z}\,, \qquad |\!|\!|v|\!|\!|_{p\,;T} \simeq H_T^{-1}\|v\|_{p\,;T} + \|\nabla v\|_{p\,;T}\,. \tag{3.1}$$

Lemma 3.1 ([10]) *For any node $z \in \mathcal{N}$ of type* (2.2), (2.3), *or* (2.4), *and any function $v \in W^1_1(\omega_z)$, one has*

$$\|v\|_{1\,;\mathring{\gamma}_z} + \frac{h_z}{H_z}\|v\|_{1\,;\gamma_z\backslash\mathring{\gamma}_z} + \frac{h_z}{H_z}\|v\|_{1\,;\bar{S}_z} \lesssim |\!|\!|v|\!|\!|_{1\,;\omega_z}\,, \tag{3.2}$$

where γ_z and $\mathring{\gamma}_z$ are from (2.1), while $\bar{S}_z \subset \omega_z$ is any segment that originates at z and satisfies $|\bar{S}_z| \simeq H_z$.

Proof A version of this lemma is given in [10, Lemma 3.1], only an inspection of the proof reveals that whenever an edge $S \in \gamma_z$ is shared by two isotropic triangles, one of them being T, of diameter $\simeq H_T \ll H_z$, we still need to show that $\|v\|_{1\,;S} \lesssim |\!|\!|v|\!|\!|_{1\,;\omega_z}$ (rather than $\|v\|_{1\,;S} \lesssim |\!|\!|v|\!|\!|_{1\,;T}$ used in [10]). Such edges can be found only if $z \in \mathcal{N}_{\rm s.ani}$ or $z \in \mathcal{N}_{\rm iso}$. For each such S, there is $S' \in \gamma_z$ such that S' is an edge of a triangle of diameter $\simeq H_z$, $|S| \simeq |S'|$, and there are only isotropic triangles of diameter $\simeq |S|$ between S and S'. Then $\|v\|_{1\,;S} \lesssim \|v\|_{1\,;S'} + \|\nabla v\|_{1,\,\omega_z}$ (which can be shown, e.g., by an application of Lemma 7.1 given in §7 below). The desired result follows. □

Corollary 3.2 *Under the conditions of Lemma 3.1, one has*

$$\|v\|_{1\,;\mathring{\gamma}_z} + \frac{h_z}{H_z}\|v\|_{1\,;\gamma_z\backslash\mathring{\gamma}_z} + \frac{h_z}{H_z}\|v\|_{1\,;\bar{S}_z} \lesssim \Big\{h_z\, \|v\|_{2\,;\omega_z}\, |\!|\!|v|\!|\!|_{2\,;\omega_z}\Big\}^{1/2}. \tag{3.3}$$

Proof First, $|\mathring{\gamma}_z| \lesssim h_z$ and $|(\gamma_z\backslash\mathring{\gamma}_z) \cup \bar{S}_z| \lesssim H_z$ imply

$$\|v\|_{1\,;\mathring{\gamma}_z} + \frac{h_z}{H_z}\|v\|_{1\,;(\gamma_z\backslash\mathring{\gamma}_z)\cup\bar{S}_z} \lesssim h_z^{1/2}\Big\{\|v\|^2_{2\,;\mathring{\gamma}_z} + \frac{h_z}{H_z}\|v\|^2_{2\,;\gamma_z\backslash\mathring{\gamma}_z\cup\bar{S}_z}\Big\}^{1/2}.$$

Next, replacing v by v^2 in (3.2) immediately yields

$$\|v\|^2_{2\,;\mathring{\gamma}_z} + \frac{h_z}{H_z}\|v\|^2_{2\,;(\gamma_z\backslash\mathring{\gamma}_z)\cup\bar{S}_z} \lesssim /\!\!/ v^2 /\!\!/_{1\,;\omega_z}\,,$$

while, by (3.1),

$$/\!\!/ v^2 /\!\!/_{1\,;\omega_z} \lesssim H_z^{-1}\|v\|^2_{2\,;\omega_z} + \|v\|_{2\,;\omega_z}\,\|\nabla v\|_{2\,;\omega_z} \simeq \|v\|_{2\,;\omega_z}\,/\!\!/ v /\!\!/_{2\,;\omega_z}\,.$$

Combining the above observations, one gets (3.3). □

4 Representation of the error in terms of the residual

Using the monotonicity of f and $C_f + \varepsilon^2 \geqslant 1$, one gets

$$\begin{aligned}|\!|\!| u_h - u |\!|\!|^2_{\varepsilon\,;\Omega} &\lesssim \varepsilon^2\langle\nabla(u_h-u),\nabla(u_h-u)\rangle + \langle f(\cdot;u_h)-f(\cdot;u),u_h-u\rangle\\ &= \varepsilon^2\langle\nabla u_h,\nabla(u_h-u)\rangle + \langle f(\cdot;u_h),u_h-u\rangle\,,\end{aligned}$$

where we also used (1.1). Next, assuming $|\!|\!| u_h - u |\!|\!|_{\varepsilon\,;\Omega} > 0$, let

$$G := \frac{u_h - u}{|\!|\!| u_h - u |\!|\!|_{\varepsilon\,;\Omega}} \qquad \Rightarrow \qquad |\!|\!| G |\!|\!|_{\varepsilon\,;\Omega} = 1. \tag{4.1}$$

Now, $|\!|\!| u_h - u |\!|\!|_{\varepsilon\,;\Omega} \lesssim \varepsilon^2\langle\nabla u_h,\nabla G\rangle + \langle f(\cdot;u_h),G\rangle$. So (1.2) implies, $\forall\ v_h \in S_h$,

$$|\!|\!| u_h - u |\!|\!|_{\varepsilon\,;\Omega} \lesssim \varepsilon^2\langle\nabla u_h,\nabla(G-v_h)\rangle + \langle f_h^I, G-v_h\rangle + \langle f_h - f_h^I, G\rangle\,. \tag{4.2}$$

Here $\langle f_h - f_h^I, G\rangle =: \mathcal{E}_{\rm quad}$ is the quadrature error, for which (4.1) implies

$$|\mathcal{E}_{\rm quad}| \leqslant |\!|\!| f_h - f_h^I |\!|\!|^*_{\varepsilon\,;\Omega}\,, \tag{4.3}$$

where the norm $|\!|\!| \cdot |\!|\!|^*_{\varepsilon\,;\Omega}$ is dual to $|\!|\!| \cdot |\!|\!|_{\varepsilon\,;\Omega}$; see Remark 4.1 below for further discussion.

Next, let ϕ_z be the standard linear hat function corresponding to $z \in \mathcal{N}$, and $v_h := G_h + \sum_{z\in\mathcal{N}} \bar{g}_z\phi_z \in S_h$, where $G_h \in S_h$ is some interpolant of G, while $\bar{g}_z$ is a certain average of $G - G_h$ near z (to be specified later), but $\bar{g}_z = 0$ for $z \in \partial\Omega$ (so that $v_h \in S_h$). Now, using $g := G - G_h$, one gets $G - v_h = g - \sum_{z\in\mathcal{N}} \bar{g}_z\phi_z = \sum_{z\in\mathcal{N}}(g-\bar{g}_z)\phi_z$. Combining this with (4.2) gives a standard error representation

$$\begin{aligned}|\!|\!| u_h - u |\!|\!|_{\varepsilon\,;\Omega} &\lesssim \sum_{z\in\mathcal{N}}\varepsilon^2\int_{\gamma_z}(g-\bar{g}_z)\phi_z[\![\nabla u_h]\!]\cdot\nu + \sum_{z\in\mathcal{N}}\int_{\omega_z} f_h^I\,(g-\bar{g}_z)\phi_z + \mathcal{E}_{\rm quad}\\ &=: I + I\!I + \mathcal{E}_{\rm quad}\,,\end{aligned} \tag{4.4}$$

which holds for any $G_h \in S_h$ and any $\{\bar{g}_z\}_{z\in\mathcal{N}}$ such that $\bar{g}_z = 0$ whenever $z \in \partial\Omega$.

In (4.4), $[\![\nabla u_h]\!]$ is the standard jump in the gradient of u_h across an interior edge. To be more precise, we adapt the notational convention that the unit normal ν to any edge in γ_z takes the clockwise direction about z, while $[\![w]\!]$, for any w, is the jump in w across any edge in γ_z evaluated in the anticlockwise direction about z. We also use $J_z := [\![\nabla u_h]\!] \cdot \nu$, which is a signed version of $|[\![\nabla u_h]\!]|$; in fact, $[\![\nabla u_h]\!] = J_z\nu$. Occasionally, when computing $[\![\nabla u_h]\!]$ across the boundary edges, we will adapt the convention that $u_h = 0$ in $\mathbb{R}^2\backslash\Omega$.

Remark 4.1 (Quadrature error) For practical computations, (4.3) implies, for any $\mathcal{T}_0 \subset \mathcal{T}$ and any fixed small $\beta > 0$, that

$$|\mathcal{E}_{\rm quad}| \lesssim \Big\{\varepsilon^{-2}\|f_h - f_h^I\|^2_{1+\beta\,;\mathcal{T}_0} + \|f_h - f_h^I\|^2_{2\,;\Omega\backslash\mathcal{T}_0}\Big\}^{1/2}. \tag{4.5}$$

Here we also used the Sobolev embedding $W_p^1(\Omega) \hookrightarrow L_{1+\beta^{-1}}(\Omega)$ with $p < 2$ (as $\frac{1}{p} = \frac{1}{2} + (1+\beta^{-1})^{-1} > \frac{1}{2}$).

5 Error analysis for a partially structured anisotropic mesh

To illustrate our approach, we first present a version of the analysis for a simpler, partially structured, anisotropic mesh in a square domain $\Omega = (0,1)^2$. So, throughout this section, we make the following triangulation assumptions.

A1. Let $\{x_i\}_{i=0}^n$ be an arbitrary mesh on the interval $(0,1)$ in the x direction. Then, let each $T \in \mathcal{T}$, for some i,
(i) have the shortest edge on the line $x = x_i$;
(ii) have a vertex on the line $x = x_{i+1}$ or $x = x_{i-1}$ (see Fig. 3, left).

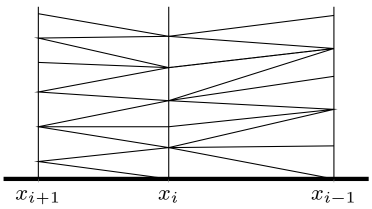
A2. Let $\mathcal{N} = \mathcal{N}_{\rm ani}$, i.e. each mesh node z satisfies (2.2).

A3. *Quasi-non-obtuse anisotropic elements.* Let the maximum angle in any triangle be bounded by $\frac{\pi}{2} + \alpha_1\frac{h_T}{H_T}$ for some positive constant α_1.

These conditions essentially imply that all mesh elements are anisotropic and aligned in the x-direction. They also imply that if $x_z = x_i$, then

$$\omega_z \subseteq \omega_z^* := (x_{i-1}, x_{i+1}) \times (y_z^-, y_z^+), \quad y_z^+ - y_z^- \simeq h_z, \quad \operatorname{diam}\omega_z^* \simeq H_z\,, \tag{5.1}$$

where (y_z^-, y_z^+) is the range of y within ω_z, while $x_{-1} := x_0$ and $x_{n+1} := x_n$.



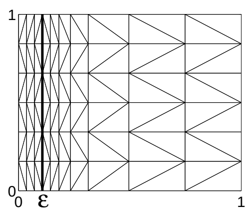


Fig. 3 Partially structured anisotropic mesh (left); triangulation used in §8 (right).

Remark 5.1 The above conditions (in particular A3) imply that there is $J \lesssim 1$ such that $\omega_z^* \subset \omega_z^{(J)}$ for all $z \in \mathcal{N}$, with the notation $\omega_z^{(0)} := \omega_z$ and $\omega_z^{(j+1)}$ for the patch of elements in/touching $\omega_z^{(j)}$. (Note that $J = 1$ for any non-obtuse triangulation, i.e. for the case $\alpha_1 = 0$ in A3.)

Remark 5.2 In comparison with [10, Section 6], our assumption A3 is equivalent to [10, A3] and implies [10, A4], while our A1 and A2 are identical with [10, A1 and A2]. This observation enables us to invoke some auxiliary results from [10].

5.1 Choice of $\bar{g}_z$. Main results

Following [10], the choice of $\bar{g}_z$ in (4.4) is related to the orientation of anisotropic elements, and is crucial in our analysis. Let $\bar{g}_z = 0$ for $z \in \partial\Omega$, and, otherwise, for $x_z = x_i$ with some $1 \leqslant i \leqslant n-1$, let

$$\int_{x_{i-1}}^{x_{i+1}} (g(x, y_z) - \bar{g}_z)\, \varphi_i(x)\, dx = 0. \tag{5.2}$$

Here we use the standard one-dimensional hat function $\varphi_i(x)$ associated with the mesh $\{x_i\}$ (i.e. it has support on (x_{i-1}, x_{i+1}), equals 1 at $x = x_i$, and is linear on (x_{i-1}, x_i) and (x_i, x_{i+1})).

Remark 5.3 For $x_z = x_i$, let $\bar{S}_z \subset \omega_z^*$ be the interval joining (x_{i-1}, y_z) and (x_{i+1}, y_z). Then (5.2) is identical to

$$\int_{\bar{S}_z} (g - \bar{g}_z)\, \varphi_i = 0 \qquad \text{if } \; x_z = x_i, \quad 1 \leqslant i \leqslant n-1. \tag{5.3}$$

Also, for non-obtuse triangulations, it is equivalent to $\int_{\bar{S}_z} (g - \bar{g}_z)\, \phi_z = 0$. The reader may compare this with a more standard choice, denoted here by $\bar{g}_z'$: $\int_{\omega_z} (g - \bar{g}_z')\, \phi_z = 0$ (see, e.g., [16, Lecture 5]).

Remark 5.4 It is sometimes helpful to tweak the definition (5.2) of $\{\bar{g}_z\}_{z\in\mathcal{N}}$ and use instead $\{\bar{g}_z^*\}_{z\in\mathcal{N}}$ defined for $z \in \mathcal{N}\backslash\partial\Omega$ with $x_z = x_i$ by

$$\int_{\omega_z^*} [g(x, y) - \bar{g}_z^*]\, \varphi_i(x) = 0, \tag{5.4}$$

(where ω_z^* is from (5.1)), and $\bar{g}_z^* = 0$ for $z \in \partial\Omega$. Importantly, $\bar{g}_z^*$ is the average of g over a rectangular domain ω_z^* (rather than an irregular patch ω_z) with a one-dimensional weight $\varphi_i(x)$, so (5.4) indeed generalizes (5.2). Note that

$$h_z H_z |\bar{g}_z^*| \lesssim \|g\|_{1\,;\omega_z^*}\,, \quad H_z |\bar{g}_z - \bar{g}_z^*| \lesssim \|\nabla g\|_{1\,;\omega_z^*}\,, \quad |\omega_z^*| \simeq h_z H_z\,. \tag{5.5}$$

Here the first relation is straightforward, while $\int_{\omega_z^*} [g(x, y_z) - g(x, y)]\, \varphi_i(x) \simeq h_z H_z (\bar{g}_z - \bar{g}_z^*)$ implies $H_z |\bar{g}_z - \bar{g}_z^*| \lesssim \|\partial_y g\|_{1\,;\omega_z^*}$ and so the second relation. We also use Remark 5.1.

Theorem 5.1 *Let $g = G - G_h$ with G from (4.1) and any $G_h \in S_h$, and*

$$\Theta := \varepsilon^2 \|\nabla g\|^2_{2\,;\Omega} + \sum_{z\in\mathcal{N}} \bigl(1+\varepsilon^2 H_z^{-2}\bigr)\|g\|^2_{2\,;\omega_z}\,. \tag{5.6}$$

Then $|\!|\!| u_h - u |\!|\!|_{\varepsilon\,;\Omega} \lesssim I + II + \mathcal{E}_{\rm quad}$, where $\mathcal{E}_{\rm quad}$ is bounded by (4.3), and, under conditions A1–A3,

$$|I| \lesssim \Bigl\{\Theta \sum_{z\in\mathcal{N}} \lambda_z \,\bigl\|\varepsilon J_z\bigr\|^2_{\infty\,;\gamma_z}\Bigr\}^{1/2}, \quad \lambda_z := h_z H_z \min\{1,\, \varepsilon H_z h_z^{-2}\}\,, \tag{5.7}$$

$$|II| \lesssim \Bigl\{\Theta \sum_{z\in\mathcal{N}} \bigl\|\lambda'_z\, f_h^I\bigr\|^2_{2\,;\omega_z}\Bigr\}^{1/2}, \qquad \lambda'_z := \min\{1,\, H_z\varepsilon^{-1}\}\,. \tag{5.8}$$

Additionally, one has an alternative bound

$$|II| \lesssim \Bigl\{\Theta \sum_{z\in\mathcal{N}\backslash\mathcal{N}^*_{\partial\Omega}} \bigl\|\min\{1,\, h_z\varepsilon^{-1}\}\, f_h^I\bigr\|^2_{2\,;\omega_z} + \Theta \sum_{z\in\mathcal{N}\backslash\mathcal{N}^*_{\partial\Omega}} \bigl\|\lambda'_z \operatorname{osc}(f_h^I;\omega_z)\bigr\|^2_{2\,;\omega_z} + \Theta \sum_{z\in\mathcal{N}^*_{\partial\Omega}} \bigl\|\lambda'_z\, f_h^I\bigr\|^2_{2\,;\omega_z}\Bigr\}^{1/2}, \tag{5.9}$$

where $\mathcal{N}^*_{\partial\Omega} = \{z\in\mathcal{N} : x_z = 0 \text{ or } x_z = 1\}$ (in agreement with (2.5)).

The proof is given in §5.2 and §5.3 below. We partially follow and invoke some auxiliary results from the proof of [10, Theorem 6.3].

Remark 5.5 It is proved in §7.1 below, under more general conditions than A1–A3, that $\Theta \lesssim |\!|\!| G |\!|\!|_{\varepsilon\,;\Omega} = 1$ (see Theorem 7.4). Combined with this result, Theorem 5.1 offers two a posteriori estimators for $|\!|\!| u_h - u |\!|\!|_{\varepsilon\,;\Omega}$.

5.2 Jump residual. Proof of (5.7)

Proof of (5.7). Split I of (4.4) as $I = \sum_{z\in\mathcal{N}} I_z$, where

$$I_z := \varepsilon^2 \int_{\gamma_z} (g - \bar g_z)\phi_z [\![\nabla u_h]\!]\cdot\nu\,. \tag{5.10}$$

Furthermore, using $[\![\nabla u_h]\!]\cdot\nu = [\![\partial_x u_h]\!]\nu_x + [\![\partial_y u_h]\!]\nu_y$, it can be split as

$$\begin{aligned} I_z = I'_z + I''_z + I'''_z := \varepsilon^2 &\int_{\gamma_z} (g - \bar g_z)\phi_z [\![\partial_x u_h]\!]\,\nu_x \\ + \varepsilon^2 &\int_{\gamma_z} [g - g(x,y_z)]\,\phi_z [\![\partial_y u_h]\!]\,\nu_y \\ + \varepsilon^2 &\int_{\gamma_z} [g(x,y_z) - \bar g_z]\,\phi_z [\![\partial_y u_h]\!]\,\nu_y\,. \end{aligned} \tag{5.11}$$

We now claim that to get the desired assertion (5.7), it suffices to show that

$$|I'_z| + |I''_z| \lesssim \varepsilon ⫽\!\!⫽ g ⫽\!\!⫽_{1\,;\omega^*_z} \, \|\varepsilon J_z\|_{\infty\,;\gamma_z}, \qquad I'''_z = 0, \tag{5.12}$$

and

$$|I_z| \lesssim \varepsilon \frac{H_z}{h_z} \Big\{ h_z \|g\|_{2\,;\omega^*_z} ⫽\!\!⫽ g ⫽\!\!⫽_{2\,;\omega^*_z} \Big\}^{1/2} \|\varepsilon J_z\|_{\infty\,;\gamma_z}, \tag{5.13}$$

and then use Hölder's inequality. Then $|I_z| \lesssim (\theta_z \lambda_z)^{1/2} \|\varepsilon J_z\|_{\infty\,;\gamma_z}$, where

$$\theta_z := \lambda_z^{-1} \varepsilon^2 \min\Big\{ ⫽\!\!⫽ g ⫽\!\!⫽^2_{1\,;\omega^*_z}, \; H_z^2 \, h_z^{-1} \|g\|_{2\,;\omega^*_z} ⫽\!\!⫽ g ⫽\!\!⫽_{2\,;\omega^*_z} \Big\},$$

and (5.7), indeed, follows, in view of $\sum_{z\in\mathcal{N}} \theta_z \lesssim \Theta$. For the latter, note that $\min(a, bc)/\min(1, c) \leqslant a + b$ (for any a, b, $c > 0$) implies $\theta_z \lesssim \varepsilon^2 ⫽\!\!⫽ g ⫽\!\!⫽^2_{2\,;\omega^*_z} + \varepsilon \|g\|_{2\,;\omega^*_z} ⫽\!\!⫽ g ⫽\!\!⫽_{2\,;\omega^*_z}$, and combine this with (3.1), (5.1) and Remark 5.1.

So it remains to prove (5.12) and (5.13). The first desired assertion (5.12) can be found in [10, (6.9)], while (5.13) is obtained from (5.10) using Corollary 3.2 in the spirit of [19, Lemma 3.2], so we only sketch their proofs here. To bound I'_z and I''_z, as well as I_z in (5.13), we use $|\phi_z [\![\nabla u_h]\!]| \leqslant |J_z|$. Note also that $⫽\!\!⫽ g ⫽\!\!⫽_{p\,;\omega_z} \lesssim ⫽\!\!⫽ g ⫽\!\!⫽_{p\,;\omega^*_z}$ as $\omega_z \subseteq \omega^*_z$ and $\operatorname{diam}\omega_z \simeq \operatorname{diam}\omega^*_z$. Now, the bound for I''_z follows from $\int_{\gamma_z} |g(x,y) - g(x, y_z)| \lesssim \|\partial_y g\|_{1\,;\omega^*_z} \lesssim ⫽\!\!⫽ g ⫽\!\!⫽_{1\,;\omega^*_z}$, while to bound I'_z in (5.12) and I_z in (5.13), it respectively remains to show that

$$\int_{\gamma_z} \big(|g\, \nu_x| + |\bar g_z \nu_x| \big) \lesssim ⫽\!\!⫽ g ⫽\!\!⫽_{1\,;\omega^*_z}, \tag{5.14}$$

$$\frac{h_z}{H_z} \int_{\gamma_z} \big(|g| + |\bar g_z| \big) \lesssim \Big\{ h_z \|g\|_{2\,;\omega^*_z} ⫽\!\!⫽ g ⫽\!\!⫽_{2\,;\omega^*_z} \Big\}^{1/2}. \tag{5.15}$$

The above two estimates are respectively obtained by an application of (3.2) and (3.3). When estimating $\bar g_z$, we use (5.3) (where $\bar S_z \subset \omega^*_z$ and $|\bar S_z| \simeq H_z$). For (5.14), we also use $|\nu_x| \lesssim \frac{h_z}{|S|}$ for any $S \in \gamma_z$, in view of A1–A3.

Finally, using A1 and the definition (5.2) of $\bar g_z$, one can represent I'''_z as

$$I'''_z = \varepsilon^2 \Big(\sum_{S\in\gamma_z} [\![\partial_y u_h]\!] \Big) \int_{x_{i-1}}^{x_i} [g(x, y_z) - \bar g_z] \, \varphi_i(x)\, dx, \tag{5.16}$$

where $x_z \in \{x_i\}_{i=1}^n$ (the case of $x_z = x_0$ is similar), $y_z \in (0,1)$. Here we also used $\phi_z = \varphi_i(x)$ on $\gamma_z \backslash \mathring{\gamma}_z$, and $\nu_y |d\nu| = -\operatorname{sgn}(x - x_i)\, dx$. As $\sum_{S\in\gamma_z} [\![\partial_y u_h]\!] = 0$ so $I'''_z = 0$. If $y_z = 0$ or $y_z = 1$, one gets $g(x, y_z) = \bar g_z = 0$ and again $I'''_z = 0$. □

Remark 5.6 An inspection of the above proof shows that it remains valid if $\{\bar g_z\}_{z\in\mathcal{N}}$ defined by (5.2) are replaced by $\{\bar g^*_z\}_{z\in\mathcal{N}}$ from (5.4). Indeed, I_z will include an additional component $I^*_z := \varepsilon^2 \int_{\gamma_z} (\bar g_z - \bar g^*_z) \phi_z [\![\nabla u_h]\!] \cdot \nu$, for which one easily gets $|I^*_z| \leqslant \varepsilon H_z |\bar g_z - \bar g^*_z| \, \|\varepsilon J_z\|_{\infty\,;\gamma_z}$. For this component, a bound of type (5.12) is obtained using the second relation from (5.5), while a bound of type (5.13) follows from the first relation in (5.5) combined with the bound for $\bar g_z$ in (5.15).

5.3 Interior residual. Proof of (5.8) and (5.9)

Now we focus on the interior-residual component $I\!I$ of the error (4.4).

Proof of (5.8). In view of Remark 5.6, replace $\{\bar g_z\}$ in (4.4) by $\{\bar g_z^*\}$ of (5.4). Now, (5.5) implies $\|g-\bar g_z^*\|_{2\,;\omega_z} \lesssim \|g\|_{2\,;\omega_z^*}$. So $|I\!I| \lesssim \sum_{z\in\mathcal N}\|g\|_{2\,;\omega_z^*}\big\|f_h^I\big\|_{2\,;\omega_z}$, and an application of Hölder's inequality yields (5.8). We also use $\lambda_z'^{-2}\simeq 1+\varepsilon^2 H_z^{-2}$ and Remark 5.1. □

Proof of (5.9). Here we again use $\{\bar g_z^*\}$ of (5.4) in place of $\{\bar g_z\}$ in (4.4). Let $\mathcal N=\cup_{i=0}^n\mathcal N_i$, where $\mathcal N_i:=\{z: x_z=x_i\}$. Note that $\mathcal N_0\cup\mathcal N_n=\mathcal N^*_{\partial\Omega}$. Now split $I\!I$ of (4.4) as $I\!I=\sum_{i=1}^{n-1}I\!I_i+I\!I_{\rm osc}+I\!I^*_{\partial\Omega}$ as follows:

$$
\begin{aligned}
I\!I_i &:= \sum_{z\in\mathcal N_i}\int_{\omega_z} f_h^I(x_i,y)\,(g-\bar g_z^*)\,\phi_z\,,\\
I\!I_{\rm osc} &:= \sum_{z\in\mathcal N\setminus\mathcal N^*_{\partial\Omega}}\int_{\omega_z}[f_h^I-f_h^I(x_z,y)]\,(g-\bar g_z^*)\,\phi_z\,,\\
I\!I^*_{\partial\Omega} &:= \sum_{z\in\mathcal N^*_{\partial\Omega}}\int_{\omega_z} f_h^I\,(g-\bar g_z^*)\,\phi_z\,.
\end{aligned}
$$

Here, for each $I\!I_{\rm osc}$ and $I\!I^*_{\partial\Omega}$, we immediately get a version of (5.8):

$$
|I\!I_{\rm osc}|\lesssim\Big\{\Theta\sum_{z\in\mathcal N\setminus\mathcal N^*_{\partial\Omega}}\|\lambda_z'\ \mathrm{osc}(f_h^I;\omega_z^*)\|^2_{2\,;\omega_z}\Big\}^{1/2}, \tag{5.17}
$$

$$
|I\!I^*_{\partial\Omega}|\lesssim\Big\{\Theta\sum_{z\in\mathcal N^*_{\partial\Omega}}\|\lambda_z'\,f_h^I\|^2_{2\,;\omega_z}\Big\}^{1/2}, \tag{5.18}
$$

where we also used $\|f_h^I-f_h^I(x_z,y)\|_{\infty\,;\omega_z}\leqslant \mathrm{osc}(f_h^I;\omega_z^*)$.

So it remains to estimate $I\!I_i$ for $1\leqslant i\leqslant n-1$, which can be rewritten as

$$
I\!I_i=\sum_{z\in\mathcal N_i}\int_0^1 f_h^I(x_i,y)\int_{x_{i-1}}^{x_{i+1}}(g-\bar g_z^*)\,\phi_z\,dx\,dy.
$$

Note that $\sum_{z\in\mathcal N_i}\phi_z=\varphi_i(x)$, so

$$
\sum_{z\in\mathcal N_i}\int_{x_{i-1}}^{x_{i+1}} g\,\phi_z\,dx=\int_{x_{i-1}}^{x_{i+1}} g\,\varphi_i\,dx=:\hat g_i(y)\int_{x_{i-1}}^{x_{i+1}}\varphi_i\,dx=\sum_{z\in\mathcal N_i}\int_{x_{i-1}}^{x_{i+1}}\hat g_i(y)\,\phi_z\,dx\,,
$$

where $\hat g_i(y)$ was deliberately defined similarly to $\bar g_z$ in (5.2):

$$
\int_{x_{i-1}}^{x_{i+1}}(g(x,y)-\hat g_i(y))\,\varphi_i(x)\,dx=0.
$$

This observation yields

$$\begin{aligned} II_i &= \sum_{z\in\mathcal{N}_i}\int_0^1 f_h^I(x_i,y)\int_{x_{i-1}}^{x_{i+1}}\bigl(\hat g_i(y)-\bar g_z^*\bigr)\,\phi_z\,dx\,dy\\ &= \sum_{z\in\mathcal{N}_i}\int_{\omega_z} f_h^I(x_i,y)\,\bigl(\hat g_i(y)-\bar g_z^*\bigr)\,\phi_z\,. \end{aligned}$$

Now, using $|f_h^I(x_i,y)-f_h^I(x,y)|\leqslant \operatorname{osc}(f_h^I;\omega_z^*)$, one gets

$$|II_i| \lesssim \sum_{z\in\mathcal{N}_i}\|\hat g_i(y)-\bar g_z^*\|_{2\,;\omega_z}\Bigl\{\|f_h^I\|_{2\,;\omega_z}+\|\operatorname{osc}(f_h^I;\omega_z^*)\|_{2\,;\omega_z}\Bigr\} =: II_i'+II_i^{\rm osc}\,.$$

To complete the proof, it remains to show that

$$\|\hat g_i(y)-\bar g_z^*\|_{2\,;\omega_z} \lesssim \min\Bigl\{\|g\|_{2\,;\omega_z^*},\, h_z\|\nabla g\|_{2\,;\omega_z^*}\Bigr\}. \tag{5.19}$$

Then, combining $\min\{a,h_zb\}\lesssim\min\{\lambda_z',h_z\varepsilon^{-1}\}\,(\lambda_z'^{-1}a+\varepsilon b)$ (for any $a,b>0$) with $\min\{\lambda_z',h_z\varepsilon^{-1}\}=\min\{1,h_z\varepsilon^{-1}\}$, one gets

$$\|\hat g_i(y)-\bar g_z^*\|_{2\,;\omega_z}\lesssim\theta_z'\,\min\{1,\,h_z\varepsilon^{-1}\},\qquad \theta_z':=\lambda_z'^{-1}\|g\|_{2\,;\omega_z^*}+\varepsilon\|\nabla g\|_{2\,;\omega_z^*}.$$

As $\sum_{z\in\mathcal{N}}\theta_z'^2\lesssim\Theta$ (where we again use $\lambda_z'^{-2}\simeq 1+\varepsilon^2H_z^{-2}$ and Remark 5.1), one concludes that, indeed, $\sum_{i=1}^{n-1}II_i^{\rm osc}$ is bounded as $|II_{\rm osc}|$ in (5.17), while $\sum_{i=1}^{n-1}II_i'$ is bounded by the first term in (5.9). Combining these with (5.17) and (5.18), and then replacing $\operatorname{osc}(f_h^I;\omega_z^*)$ by $\operatorname{osc}(f_h^I;\omega_z)$ (again in view of Remark 5.1), yields the desired assertion (5.9).

To prove (5.19), first, $\|\hat g_i(y)-\bar g_z^*\|_{2\,;\omega_z}\lesssim\|g\|_{2\,;\omega_z^*}$ follows from the definition of $\hat g_i$ and (5.4). Also, for any $y\in(y_z^-,y_z^+)$, one has $H_z|\hat g_i(y)-\bar g_z^*|\lesssim\|\partial_y g\|_{1\,;\omega_z^*}\lesssim(h_zH_z)^{1/2}\|\nabla g\|_{2\,;\omega_z^*}$. So one gets (5.19), and thus (5.9). □

6 A posteriori error analysis for general anisotropic meshes

We start this section by listing all additional mesh assumptions that will be used in the present §6, as well as in §7.2. (An example of a mesh that satisfies $\mathcal{A}1$–$\mathcal{A}5$ is pictured on Fig. 1.)

- $\mathcal{A}1$. If $z\in\mathcal{N}$ is a corner of Ω, then $z\in\mathcal{N}_{\rm iso}$ (i.e. all corners are isotropic nodes and none of them is in $\mathcal{N}_{\partial\Omega}^*$).
- $\mathcal{A}2$. Let the maximum triangle angle at any $z\in\mathcal{N}_{\partial\Omega}^*$ be bounded by $\frac{\pi}{2}+\alpha_1\frac{h_z}{H_z}$ for some positive constant α_1 (see Fig. 4 (left)).
- $\mathcal{A}3$. If T is anisotropic (i.e. $h_T\ll H_T$), then $\operatorname{diam}\omega_T'\simeq H_T$, where ω_T and ω_T' respectively denote the patches of elements in or touching T and ω_T.
- $\mathcal{A}4$. If $z\in\mathcal{N}_{\rm iso}$, then the patch $\mathring\omega_z:=\{T\subset\omega_z : h_T\simeq H_T\ll H_z\}$ of small isotropic triangles within ω_z is such that $\mathring\omega_z\cup\partial\mathring\omega_z\backslash\{z\}$ is connected.

Remark 6.1 (Assumptions $\mathcal{A}1$–$\mathcal{A}4$) Assumption $\mathcal{A}1$ is reasonable as typical corner singularities are isotropic, while $\mathcal{A}2$ is made to simplify the presentation. $\mathcal{A}1$–$\mathcal{A}3$ are used to get our first estimator (given by Theorem 6.1), while for the second estimator (of Theorem 6.2), we use $\mathcal{A}1$, $\mathcal{A}3$, as well as $\mathcal{A}5$ given below. $\mathcal{A}4$ is used only in §7.2 to establish desired approximation properties of quasi-interpolants on anisotropic meshes. It is automatically satisfied if all triangles surrounding z are shape-regular. Note also that $\mathring{\omega}_z = \varnothing$ for $z \in \mathcal{N}_{\rm ani}$, while for $z \in \mathcal{N}_{\rm s.ani}$, $\mathcal{A}4$ is also automatically satisfied.

To get a version of the estimator (5.7), (5.9) for a more general mesh, it is essential that we look at sequences of short edges that connect (semi-) anisotropic nodes. This concept is implemented with the help of the following definition from [10].

Definition. A *(Semi-)Anisotropic Path*, or simply a *Path*, is an ordered sequence $\{z_j\}_{j=1}^k$ of nodes in $\mathcal{N}_{\rm ani}\backslash\mathcal{N}^*_{\partial\Omega}$ (or $\mathcal{N}_{\rm s.ani}$), for some $k > 1$ (which may differ for different paths and for which no upper-bound assumption is made), such that each z_j, $j = 1, \ldots, k-1$, is connected to z_{j+1} by an edge of length $\simeq h_{z_j} \simeq h_{z_{j+1}}$, and each of the start and end nodes z_l, $l = 1, k$, is either on $\partial\Omega$, or connected by an edge of length $\simeq h_{z_l}$ to an isotropic node. (E.g., if $z_1 \notin \partial\Omega$, then is it connected to some node $z_0 \in \mathcal{N}_{\rm iso}$ by an edge of length $\simeq h_{z_1}$ so $H_{z_0} \simeq H_{z_1}$.)

Let $\mathcal{N}_i \subset \mathcal{N}_{\rm ani}\backslash\mathcal{N}^*_{\partial\Omega}$ be an anisotropic path for $i = 1, \ldots, n_{\rm ani}$, and $\mathcal{N}_i \subset \mathcal{N}_{\rm s.ani}$, be a semi-anisotropic path for $i = n_{\rm ani} + 1, \ldots, n_{\rm ani} + n_{\rm s.ani}$ (with no upper-bound assumption made on $n_{\rm ani}$ or $n_{\rm s.ani}$). Furthermore, let $\mathcal{N}_{\rm paths} := \cup_{i=1}^{n_{\rm ani}+n_{\rm s.ani}} \mathcal{N}_i$. Note that $\mathcal{N}\backslash\mathcal{N}_{\rm paths}$ may include (semi-)anisotropic nodes that do not belong to any path.

$\mathcal{A}5$. *Path Element Orientation condition.* For each (semi-)anisotropic path $\mathcal{N}_i$, $i = 1, \ldots, n_{\rm ani} + n_{\rm s.ani}$, let there exist a smooth curve $\mathcal{C}_i$, the curvature of which is uniformly bounded by a constant $\varrho \geqslant 0$, such that each $z \in \mathcal{N}_i$ is at a distance $\lesssim H_z$ to $\mathcal{C}_i$, an extension of the longest edge $\hat{S}_z \subset \mathcal{S}_z$ intersects $\mathcal{C}_i$, and the normal $\nu_{\mathcal{C}_i}$ to $\mathcal{C}_i$ at the intercept satisfies $|\sin(\angle(\hat{S}_z, \nu_{\mathcal{C}_i}))| \lesssim \frac{h_z}{H_z}$.

Remark 6.2 ($\mathcal{A}5$) Assumption $\mathcal{A}5$ essentially means that a smooth curve $\mathcal{C}_i$ is almost orthogonal to each ω_z for $z \in \mathcal{N}_i$ (as well as to its superset rectangle R_z from the Local Element Orientation condition). Note that [10, $\mathcal{A}3$] is a stronger assumption as it corresponds to $\varrho = 0$. In the latter case, $\mathcal{A}5$ is equivalent to $|\sin(\angle(\hat{S}_z, \hat{S}_{z_i}))| \lesssim \frac{h_z}{H_z}$, where $z_i \in \mathcal{N}_i$ has the minimal $\frac{h_z}{H_z}$ within $\mathcal{N}_i$.

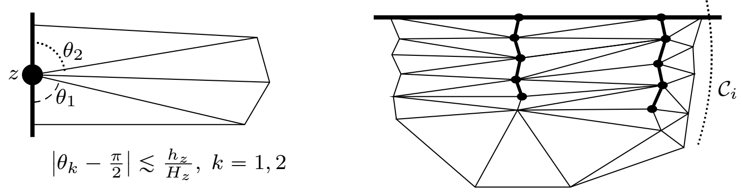


Fig. 4 An anisotropic node $z \in \mathcal{N}^*_{\partial\Omega}$ that satisfies $\mathcal{A}2$ (left); an anisotropic path and a semi-anisotropic path highlighted, with a suitable curve $\mathcal{C}_i$ from $\mathcal{A}5$ (right).

6.1 Main results

Our estimators for more general anisotropic meshes are presented in the form of Theorems 6.1 and 6.2.

Theorem 6.1 *Let $g = G - G_h$ with G from* (4.1) *and any $G_h \in S_h$, and Θ be defined by* (5.6)*. Then $|||u_h - u|||_{\varepsilon\,;\Omega} \lesssim I + II + \mathcal{E}_{\rm quad}$, where $\mathcal{E}_{\rm quad}$ is bounded by* (4.3)*, and, under conditions $\mathcal{A}1$ and $\mathcal{A}2$,*

$$|I| \lesssim \Big\{\Theta \sum_{z\in\mathcal{N}} \lambda_z\, \|\varepsilon J_z\|^2_{\infty\,;\gamma_z}\Big\}^{1/2}, \quad \lambda_z := h_z H_z \min\{1,\, \varepsilon H_z h_z^{-2}\}\,. \tag{6.1}$$

For II, one has

$$|II| \lesssim \Big\{\Theta' \sum_{z\in\mathcal{N}} \|\lambda'_z\, f_h^I\|^2_{2\,;\omega_z}\Big\}^{1/2}, \qquad \lambda'_z := \min\{1,\, H_z\varepsilon^{-1}\}\,, \tag{6.2}$$

where $\Theta' = \Theta$, while, under condition $\mathcal{A}3$, one has a sharper version of this bound with

$$\Theta' = \sum_{T\in\mathcal{T}} \big\{1 + \varepsilon^2(\operatorname{diam}\omega_T)^{-2}\big\}\, \|g\|^2_{2\,;T}\,. \tag{6.3}$$

Remark 6.3 In [10], a similar result is obtained in the maximum norm assuming a version of $\mathcal{A}2$ for all $z \in \mathcal{N}_{\rm ani}$. The improvement in this paper is made by employing bounds of type (6.11) (rather than bounding I'''_z). A similar approach was invoked in a recent paper [11, §6].

Theorem 6.2 *Under the conditions of Theorem 6.1 and $\mathcal{A}1, \mathcal{A}3, \mathcal{A}5$ (but without assuming $\mathcal{A}2$), let $\bar H := \max_{z\in\mathcal{N}_{\rm paths}} H_z$ be sufficiently small. Then*

$$\begin{aligned}|II| \lesssim \Big\{&\widetilde\Theta \sum_{z\in\mathcal{N}_{\rm paths}} \|\min\{1,\, h_z\varepsilon^{-1}\}\, f_h^I\|^2_{2\,;\omega_z} + \widetilde\Theta' \sum_{z\in\mathcal{N}\setminus\mathcal{N}_{\rm paths}} \|\lambda'_z\, f_h^I\|^2_{2\,;\omega_z}\\ &+ \widetilde\Theta' \sum_{z\in\mathcal{N}_{\rm paths}} \|\lambda'_z \operatorname{osc}(f_h^I;\omega_z)\|^2_{2\,;\omega_z} + \widetilde\Theta' \sum_{z\in\mathcal{N}_{\rm paths}} \|\varrho \bar H \lambda'_z\ f_h^I\|^2_{2\,;\omega_z}\Big\}^{1/2},\end{aligned} \tag{6.4}$$

where

$$\widetilde\Theta' = \sum_{T\in\mathcal{T}} \Big\{1 + \frac{\varepsilon^2}{H_T \operatorname{diam}\omega_T}\Big\}\, \|g\|^2_{2\,;T}\,, \qquad \widetilde\Theta = \widetilde\Theta' + \sum_{\substack{T\in\mathcal{T}:\\ |T|\simeq|\omega_T|}} \varepsilon^2 \|\nabla g\|^2_{2\,;T}. \tag{6.5}$$

Remark 6.4 A comparison of (6.5) with (6.3) and (5.6) shows that $\Theta' \lesssim \Theta$ and $\Theta' \lesssim \widetilde\Theta' \lesssim \widetilde\Theta$.

Theorem 6.1 is proved in §6.2, while §§6.3–6.4 below are devoted to the proof of Theorem 6.2. We also give a more intricate version of the jump residual estimator (6.1) in the form of Lemma 6.3, for which we need to introduce more notation, and the proof of which is deferred to Appendix A.

Define $\widehat{\omega}_z \subset \omega'_z$ as follows. If $z \in \mathcal{N}_{\rm s.ani}\backslash\partial\Omega$ and ω_z includes exactly two anisotropic triangles, let $\widehat{\omega}_z$ be the patch of elements in or sharing a common edge of length $\simeq H_z$ with ω_z. If $z \in \mathcal{N}_{\rm ani}\backslash\partial\Omega$ and ω_z is formed by exactly four triangles, let $\widehat{\omega}_z$ be the patch of elements in or sharing a common edge with ω_z. Otherwise, if $z \in \mathcal{N}_{\rm ani}\backslash\partial\Omega$ and there is a subdomain ω_z^+ of ω_z formed by exactly two triangles sharing a common edge, such that $\partial\omega_z^+ \supset \mathring{\gamma}_z$, let $\widehat{\omega}_z$ be the patch of elements in ω_z or sharing a common edge with ω_z^+. In all other cases, let $\widehat{\omega}_z := \omega_z$.

Lemma 6.3 *Under the conditions of Theorem 6.1 and $\mathcal{A}1$–$\mathcal{A}3$, one has*

$$|I| \lesssim \Bigl\{\Theta^* \sum_{z\in\mathcal{N}} \lambda_z^* \,\|\varepsilon J_z\|^2_{\infty\,;\gamma_z}\Bigr\}^{1/2}, \quad \lambda_z^* := \min\Bigl\{|\widehat{\omega}'_z|,\, \varepsilon H_z^2 h_z^{-1} q_z \frac{|\omega'_z|^{1/2}}{|\omega_z|^{1/2}}\Bigr\}.$$

where $q_z := \max_{T\subset\omega_z} \sqrt{h_z/h_T}$, while ω'_z and $\widehat{\omega}'_z$ denote the patches of elements in or surrounding ω_z and $\widehat{\omega}_z$, respectively, and, with $\widetilde{\Theta}'$ from (6.5),

$$\Theta^* := \widetilde{\Theta}' + \varepsilon^2 \sum_{T\in\mathcal{T}} \frac{|T|}{|\omega_T|} \interleave g \interleave_{2\,;T}\,. \tag{6.6}$$

6.2 Proof of Theorem 6.1

Proof of (6.1). For each fixed $z \in \mathcal{N}$, introduce the following local notation. Let the local cartesian coordinates (ξ,η) be such that $z = (0,0)$, and the unit vector i_ξ in the ξ direction lies along the longest edge $\hat{S}_z \in \mathcal{S}_z$ (see Fig. 5 (left)). In view of $\mathcal{A}2$, let i_ξ for $z \in \mathcal{N}^*_{\partial\Omega}$ be orthogonal to $\partial\Omega$ at z.

Next, split $\mathcal{S}_z = \mathring{\mathcal{S}}_z \cup \mathcal{S}_z^+ \cup \mathcal{S}_z^-$, where $\mathring{\mathcal{S}}_z = \{S \subset \mathcal{S}_z : |S| \lesssim h_z\}$ (so $\mathring{\gamma}_z = \mathring{\mathcal{S}}_z\backslash\partial\Omega$). Here we also use $\mathcal{S}_z^\pm := \{S \subset \mathcal{S}_z\backslash\mathring{\mathcal{S}}_z : S_\xi \subset \mathbb{R}_\pm\}$, where $S_\xi = \mathrm{proj}_\xi(S)$ denotes the projection of S onto the ξ-axis. Now, let (ξ_z^-, ξ_z^+) be the maximal interval such that $(\xi_z^-, 0) \subset S_\xi$ for all $S \in \mathcal{S}_z^-$ and $(0, \xi_z^+) \subset S_\xi$ for all $S \in \mathcal{S}_z^+$. Also, let $\varphi_z(\xi)$ be the standard piecewise-linear hat-function with support on (ξ_z^-, ξ_z^+) and equal to 1 at $\xi = 0$. Note that if $\mathcal{S}_z^- = \varnothing$ (and $\mathcal{S}_z^+ = \varnothing$), then we set $\xi_z^- = 0$ (and $\xi_z^+ = 0$) and do not use φ_z for $\xi < 0$ (and $\xi > 0$).

We make a few observations on the above definitions for particular node types in the following table (at this stage, see the rows for $\mathcal{S}_z^\pm$ and $\xi_z^\pm$).

	$z \in \mathcal{N}_{\rm ani}\backslash\mathcal{N}^*_{\partial\Omega}$	$z \in \mathcal{N}^*_{\partial\Omega} \subset \mathcal{N}_{\rm ani}$	$z \in \mathcal{N}_{\rm s.ani}$	$z \in \mathcal{N}_{\rm iso}$
$\mathcal{S}_z^\pm$	$\mathcal{S}_z^-, \mathcal{S}_z^+ \neq \varnothing$	$\mathcal{S}_z^- = \varnothing,\ \mathcal{S}_z^+ \neq \varnothing$		$\mathcal{S}_z^- = \mathcal{S}_z^+ = \varnothing$
$\xi_z^\pm$	$\lvert\xi_z^-\rvert \simeq \xi_z^+ \simeq H_z$	$\xi_z^- = 0,\ \xi_z^+ \simeq H_z$		$\xi_z^- = \xi_z^+ = 0$
$\bar{g}_z$	(6.7)	$\bar{g}_z = 0$	(6.7)	$\bar{g}_z = 0$

Next, for $\xi \in [\xi_z^-, \xi_z^+]$ define a continuous function $\bar\eta_z(\xi)$ as follows: (i) $\bar\eta_z(\xi)$ is linear on $[\xi_z^-, 0]$ and $[0, \xi_z^+]$; (ii) $\bar\eta_z(0) = 0$; (iii) $(\xi, \bar\eta_z(\xi)) \in \omega_z$ for all $\xi \in (\xi_z^-, \xi_z^+)$. (For example, one may choose $\bar\eta_z(\xi)$ so that $\{(\xi, \bar\eta_z(\xi)) : \xi \in (\xi_z^-, 0)\}$ lies on any edge in $\mathcal{S}_z^-$, while $\{(\xi, \bar\eta_z(\xi)) : \xi \in (0, \xi_z^+)\}$ lies on any edge in $\mathcal{S}_z^+$; see Fig. 5 (left).)

We are now prepared to specify $\bar g_z$ (see also the row for $\bar g_z$ in the above table). We let $\bar g_z := 0$ if $z \in \partial\Omega$ or $z \in \mathcal{N}_{\rm iso}$ (as for the latter, $\xi_z^- = \xi_z^+ = 0$), and, otherwise, let

$$\int_{\xi_z^-}^{\xi_z^+} \big[g(\xi, \bar\eta_z(\xi)) - \bar g_z\big]\, \varphi_z(\xi)\, d\xi = 0. \tag{6.7}$$

Also, let $\bar S_z^- := \{(\xi, \bar\eta_z(\xi)) : \xi \in (\xi_z^-, 0)\}$ and $\bar S_z^+ := \{(\xi, \bar\eta_z(\xi)) : \xi \in (0, \xi_z^+)\}$, i.e. $\bar S_z^\pm$ is the segment joining $(0,0)$ and $(\xi_z^\pm, \bar\eta_z(\xi_z^\pm))$. So using (6.7) and then (3.2) yields

$$h_z|\bar g_z| \lesssim \frac{h_z}{H_z}\|g\|_{1\,;\bar S_z^- \cup \bar S_z^+} \lesssim |\!|\!| g |\!|\!|_{1\,;\omega_z}\,. \tag{6.8}$$

To ensure that for $z \in (\mathcal{N}_{\rm ani}\backslash\mathcal{N}^*_{\partial\Omega}) \cap \partial\Omega$ and $z \in \mathcal{N}_{\rm s.ani} \cap \partial\Omega$, both (6.7) and (6.8) agree with the definition $\bar g_z = 0$, we choose $\bar\eta_z$ for these nodes such that $\{(\xi, \bar\eta_z(\xi)) : \xi \in (\xi_z^-, \xi_z^+)\} \subset \partial\Omega$ (i.e. $\bar S_z^\pm \subset \mathcal{S}_z \cap \partial\Omega$).

Now we proceed to the estimation of I of (4.4) and split it as $I = \sum_{z\in\mathcal{N}} I_z$, where I_z is given by (5.10). As in the proof of (5.7), we again get (5.15) and hence (5.13), only with ω_z^* replaced by ω_z (as now $\bar S_z^- \cup \bar S_z^+ \subset \omega_z$ in (6.8)).

Next, to get a version of (5.12), split I_z, similarly to (5.11), as

$$\begin{aligned}
I_z = \varepsilon^2 \int_{\gamma_z} (g - \bar g_z)\phi_z [\![\nabla u_h]\!] \cdot \nu &= I_z' + I_z'' + I_z''' + I_z'''' \\
:= \varepsilon^2 \int_{\mathring\gamma_z} (g - \bar g_z)\phi_z [\![\nabla u_h]\!] \cdot \nu &+ \varepsilon^2 \int_{\gamma_z\backslash\mathring\gamma_z} (g - \bar g_z)\phi_z [\![\partial_\xi u_h]\!]\, \nu_\xi \\
+ \varepsilon^2 \int_{\gamma_z\backslash\mathring\gamma_z} &\big[g - g(\xi, \bar\eta_z(\xi))\big]\; \varphi_z [\![\partial_\eta u_h]\!]\, \nu_\eta \\
+ \varepsilon^2 \int_{\gamma_z\backslash\mathring\gamma_z} &\big[g(\xi, \bar\eta_z(\xi)) - \bar g_z\big]\; \varphi_z [\![\partial_\eta u_h]\!]\, \nu_\eta \\
+ \varepsilon^2 \int_{\gamma_z\backslash\mathring\gamma_z} &(g - \bar g_z)\{\phi_z - \varphi_z\} [\![\partial_\eta u_h]\!]\, \nu_\eta\,,
\end{aligned} \tag{6.9}$$

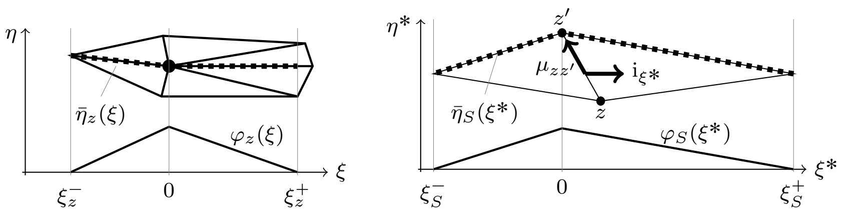


Fig. 5 Local notation associated with a node $z \in \mathcal{N}$ (left), and an edge $S \in \mathcal{S}^*$ with endpoints z and z' (right).

where, with slight abuse of notation, $g = g(\xi, \eta)$.

Note that we have a version of [10, (7.9)]:

$$|I'_z| + |I''_z| + |I''''_z| \lesssim \varepsilon |\!|\!| g |\!|\!|_{1\,;\omega_z} \, \|\varepsilon J_z\|_{\infty\,;\gamma_z} \qquad \text{for } z \in \mathcal{N}. \tag{6.10}$$

Here I''_z is estimated as in the proof of (5.7). To estimate I'_z, we use (3.2), (6.8) and the observation $|\nu_\xi| \lesssim \frac{h_z}{|S|} \simeq \frac{h_z}{H_z}$ for any $S \in \gamma_z \backslash \mathring{\gamma}_z$ (in view of the Local Element Orientation condition). The above bound for I''''_z is obtained in a similar way, only invoking $0 \leqslant \phi_z - \varphi_z \lesssim \frac{h_z}{H_z}$ on $\gamma_z \backslash \mathring{\gamma}_z$ (the latter also follows from the Local Element Orientation condition). So we have obtained (6.10).

To complete the proof, it suffices to show that for some edge subset $\mathcal{S}^* \subset \mathcal{S}$ with some quantities $\mathcal{I}_{S;z}$ associated with any $S \in \mathcal{S}_z \cap \mathcal{S}^*$ (to be specified below), one has

$$\sum_{z \in \mathcal{N}} \sum_{S \in \mathcal{S}_z \cap \mathcal{S}^*} \mathcal{I}_{S;z} = 0, \tag{6.11a}$$

$$|I'''_z + \sum_{S \in \mathcal{S}_z \cap \mathcal{S}^*} \mathcal{I}_{S;z}| \lesssim \varepsilon |\!|\!| g |\!|\!|_{1\,;\omega_z} \, \|\varepsilon J_z\|_{\infty\,;\gamma_z}\,, \tag{6.11b}$$

$$|\mathcal{I}_{S;z}| \lesssim \varepsilon \frac{H_z}{h_z} \Big\{ h_z \|g\|_{2\,;\omega_z} \, |\!|\!| g |\!|\!|_{2\,;\omega_z} \Big\}^{1/2} \|\varepsilon J_z\|_{\infty\,;\gamma_z}\,. \tag{6.11c}$$

Indeed, recalling the alternative bound (5.13) for $|I_z|$ (valid with ω^*_z replaced by ω_z), one then gets $|I_z + \sum_{S \in \mathcal{S}_z \cap \mathcal{S}^*} \mathcal{I}_{S;z}| \lesssim (\theta_z \lambda_z)^{1/2} \|\varepsilon J_z\|_{\infty\,;\gamma_z}$, where

$$\theta_z := \lambda_z^{-1} \varepsilon^2 \, \min\Big\{ |\!|\!| g |\!|\!|^2_{1\,;\omega_z}\,,\; H_z^2\, h_z^{-1} \|g\|_{2\,;\omega_z} |\!|\!| g |\!|\!|_{2\,;\omega_z} \Big\}. \tag{6.12}$$

The desired assertion (6.1) follows in view of $\sum_{z \in \mathcal{N}} \theta_z \lesssim \Theta$; the latter is shown as in the proof of (5.7).

To establish (6.11), we now focus on I'''_z from (6.9). Here $[g(\xi, \bar\eta_z(\xi)) - \bar g_z]\,\varphi_z$ is a function of ξ, while $\nu_\eta |d\nu| = -\mathrm{sgn}(\xi)\, d\xi$. So, using (6.7), one concludes that

$$I'''_z = -\varepsilon^2 \Big(\sum_{S \in [\mathcal{S}_z^- \cup \mathcal{S}_z^+] \backslash \partial\Omega} [\![\partial_\eta u_h]\!] \Big) \int_0^{\xi_z^+} [g(\xi, \bar\eta_z(\xi)) - \bar g_z]\, \varphi_z(\xi)\, d\xi. \tag{6.13}$$

This yields $I'''_z = 0$ for $z \in \mathcal{N}_{\mathrm{iso}} \cup \mathcal{N}_{\mathrm{s.ani}} \cup [(\mathcal{N}_{\mathrm{ani}} \backslash \mathcal{N}^*_{\partial\Omega}) \cap \partial\Omega]$ (see the above table; for $z \in \mathcal{N}_{\mathrm{s.ani}}$, combine (6.7) with $\xi_z^- = 0$).

For the remaining nodes $z \in (\mathcal{N}_{\mathrm{ani}} \backslash \partial\Omega) \cup \mathcal{N}^*_{\partial\Omega}$, note that $(\mathcal{S}_z^- \cup \mathcal{S}_z^+) \cap \partial\Omega = \varnothing$, while $\sum_{S \in \mathcal{S}_z^- \cup \mathcal{S}_z^+} [\![\partial_\eta u_h]\!] = -\sum_{S \in \mathring{\mathcal{S}}_z} [\![\partial_\eta u_h]\!]$ (where $u_h = 0$ in $\mathbb{R}^2 \backslash \Omega$ is used to compute $[\![\partial_\eta u_h]\!]$ for $z \in \mathcal{N}^*_{\partial\Omega}$). Note also that $[\![\partial_\eta u_h]\!] = -\mu \cdot \mathrm{i}_\xi\, J_z$ on any edge $S \in \mathcal{S}_z$, where μ is the unit vector directed from z along S. (The latter follows from $\partial_\eta = \mathrm{i}_\eta \cdot \nabla$ combined with $[\![\nabla u_h]\!] = \nu J_z$ and $\mathrm{i}_\eta \cdot \nu = -\mathrm{i}_\xi \cdot \mu$.) Combining these observations with (6.13), for $z \in (\mathcal{N}_{\mathrm{ani}} \backslash \partial\Omega) \cup \mathcal{N}^*_{\partial\Omega}$, one arrives at

$$I'''_z = -\varepsilon^2 \alpha_z \sum_{S \in \mathring{\mathcal{S}}_z} \mu \cdot \mathrm{i}_\xi\, J_z\,, \qquad \alpha_z := \int_0^{\xi_z^+} [g(\xi, \bar\eta_z(\xi)) - \bar g_z]\, \varphi_z(\xi)\, d\xi\,. \tag{6.14}$$

Recalling that i_ξ for $z \in \mathcal{N}^*_{\partial\Omega}$ is orthogonal to $\partial\Omega$ at z, while $\mathring{\mathcal{S}}_z \subset \partial\Omega$, one gets $\mu \cdot \mathrm{i}_\xi = 0$, and so again $I'''_z = 0$ for any $z \in \mathcal{N}^*_{\partial\Omega}$.

To deal with the remaining case $z \in \mathcal{N}_{\rm ani}\backslash\partial\Omega$, motivated by the observation (6.14), introduce the edge subset $\mathcal{S}^* := \cup_{z\in\mathcal{N}_{\rm ani}\backslash\partial\Omega}\mathring{\mathcal{S}}_z$, and for any $S \in \mathcal{S}^*$ with endpoints z and z', define

$$\mathcal{I}_{S;z} := \varepsilon^2\alpha_S\,\mu_{zz'}\cdot \mathrm{i}_{\xi^*}\,J_z\big|_S\,, \qquad \alpha_S := \int_0^{\xi_S^+}\big[g(\xi^*,\bar\eta_S(\xi^*)) - \bar g_S\big]\,\varphi_S(\xi^*)\,d\xi^*. \tag{6.15}$$

Here $\mu_{zz'}$ is the unit vector directed from z to z', and i_{ξ^*} is the unit vector along the ξ^*-axis. The local cartesian coordinates (ξ^*,η^*) are associated with S and coincide with the local coordinates (ξ,η) associated with either $z\in\mathcal{N}_{\rm ani}$ or $z'\in\mathcal{N}_{\rm ani}$ (at least one of them is always in $\mathcal{N}_{\rm ani}$). The above α_S are defined similarly to α_z, with a version $\int_{\xi_S^-}^{\xi_S^+}[g(\xi^*,\bar\eta_S(\xi^*)) - \bar g_S]\,\varphi_S(\xi^*)\,d\xi^* = 0$ of (6.7) defining $\bar g_S$. The one-dimensional hat function $\varphi_S(\xi^*)$ is associated with the interval (ξ_S^-,ξ_S^+); the latter is the projection of $\omega_z\cap\omega_{z'}$ (which includes at most two triangles) onto the ξ^*-axis. The piecewise-linear function $\bar\eta_S(\xi^*)$ is defined similarly to $\bar\eta_z(\xi)$ under the restriction that any point $(\xi^*,\bar\eta(\xi^*))\in\omega_z\cap\omega_{z'}$ (see Fig. 5(right)).

Note that $\mu_{zz'}+\mu_{z'z}=0$ and $J_z\big|_S = J_{z'}\big|_S$ in (6.15), so $\mathcal{I}_{S;z}+\mathcal{I}_{S;z'}=0$, which implies (6.11a). Next, (6.11c) is obtained similarly to the bound (5.13) for $|I_z|$. To show (6.11b) for $z\in\mathcal{N}_{\rm ani}\backslash\partial\Omega$, note that μ in (6.14) equals $\mu_{zz'}$ in (6.15), while either $\mathrm{i}_\xi=\mathrm{i}_{\xi^*}$ or, by the Local Element Orientation condition, $|\mathrm{i}_\xi-\mathrm{i}_{\xi^*}|\simeq h_zH_z^{-1}$ (assuming, without loss of generality, that $\mathrm{i}_\xi\cdot\mathrm{i}_{\xi^*}>0$). Also, $|\alpha_z-\alpha_S|\lesssim\|\nabla g\|_{1\,;\omega_z}$, while $|\alpha_z|+|\alpha_S|\lesssim H_zh_z^{-1}|\!|\!|g|\!|\!|_{1\,;\omega_z}$ (the latter follows from (6.7), (6.8), and similar observations for g_S and $\bar g_S$). Combining these findings yields (6.11b) for any $z\in\mathcal{N}_{\rm ani}\backslash\partial\Omega$.

Recall that $I_z'''=0$ for $z\notin\mathcal{N}_{\rm ani}\backslash\partial\Omega$, so if $\mathcal{S}_z\cap\mathcal{S}^*=\varnothing$, one immediately gets (6.11b). However, $\mathcal{S}_z\cap\mathcal{S}^*$ may be non-empty for $z\notin\mathcal{N}_{\rm ani}\backslash\partial\Omega$ (as $z\in\mathcal{N}_{\rm iso}$ or $z\in(\mathcal{N}_{\rm ani}\backslash\mathcal{N}^*_{\partial\Omega})\cap\partial\Omega$ may be connected by a short edge to a node in $\mathcal{N}_{\rm ani}\backslash\partial\Omega$). For such $z\in\mathcal{N}_{\rm iso}$, a version of (3.2) (with $h_z\simeq H_z$) implies $|\alpha_S|\lesssim|\!|\!|g|\!|\!|_{1\,;\omega_z}$, while for such $z\in(\mathcal{N}_{\rm ani}\backslash\mathcal{N}^*_{\partial\Omega})\cap\partial\Omega$, one combines $\alpha_z=0$ with $|\alpha_S-\alpha_z|\lesssim\|\nabla u_h\|_{1\,;\omega_z}$. The desired bound (6.11b) for all $z\notin\mathcal{N}_{\rm ani}\backslash\partial\Omega$ follows. This completes the proof of (6.11), and hence of (6.1). □

Remark 6.5 An inspection of the proof of (6.1) shows that it remains valid if the local cartesian coordinates (ξ,η) satisfy $|\angle(\hat S_z,\mathrm{i}_\xi)|\lesssim\frac{h_z}{H_z}$ (rather than $\angle(\hat S_z,\mathrm{i}_\xi)=0$). Also, the requirement that $z=(0,0)$ in the coordinates (ξ,η) is inessential.

Remark 6.6 An inspection of the above proof shows that (6.1) remains valid if we replace $\{\bar g_z\}_{z\in\mathcal{N}}$ by $\{\bar g_z^*\}_{z\in\mathcal{N}}$ such that $\bar g_z^*:=0$ whenever $\bar g_z=0$, and, otherwise,

$$\int_{\widetilde\omega_z}[g-\bar g_z^*]\,\varphi_z(\xi)\,d\xi\,d\eta=0 \tag{6.16}$$

subject to (5.5) with some ω_z^*, for some $J\lesssim 1$, satisfying

$$(\omega_z\cup\widetilde\omega_z)\subset\omega_z^*\subset\omega_z^{(J)},\qquad |\omega_z|\simeq|\widetilde\omega_z|\simeq|\omega_z^*|. \tag{6.17}$$

Furthermore, we require for any T within ω_z^* to have a vertex z' with $H_{z'} \simeq H_z$. Then, indeed, each I_z will additionally include $I_z^* := \varepsilon^2 \int_{\gamma_z} (\bar g_z - \bar g_z^*)\phi_z [\![\nabla u_h]\!]\cdot\nu$, so imitating the argument from Remark 5.6 one again gets (6.1). Note that now θ_z is defined by the following version of (6.12):

$$\theta_z := \lambda_z^{-1}\varepsilon^2 \min\Bigl\{ |\!|\!| g |\!|\!|^2_{1\,;\omega_z^*}\,,\ H_z^2\, h_z^{-1}\|g\|_{2\,;\omega_z} |\!|\!| g |\!|\!|_{2\,;\omega_z} + H_z\, h_z^{-1}\|g\|^2_{2\,;\tilde\omega_z}\Bigr\}. \quad (6.18)$$

The domains $\tilde\omega_z$ to be used in (6.16) are specified in [10, Remark 7.5], using a sufficiently small positive constant c^*:

$$\tilde\omega_z := \{(\xi, \bar\eta_z(\xi) + t) : \xi \in (\xi_z^-, \xi_z^+), |t| < c^* h_z\} \subset \hat\omega_z$$

(where the domains $\hat\omega_z \supset \omega_z$ are defined before Lemma 6.3). Now, setting $\omega_z^* := \omega_z \cup \tilde\omega_z$, one gets (6.17) with $J = 1$ and (5.5).

Proof of (6.2) *with* $\Theta' = \Theta$. In view of Remark 6.6, replace $\{\bar g_z\}$ in (4.4) by $\{\bar g_z^*\}$ such that one has (5.5) and (6.17) with $J = 1$. Furthermore, recall that any T within ω_z^* has a vertex z' with $H_{z'} \simeq H_z$, so $\lambda'_{z'} \simeq \lambda'_z$. Now, imitating the proof of (5.8) (given in §5.3) yields the desired result. □

Proof of (6.2) *with* Θ' *of* (6.3). Note that to bound II from (4.4), it suffices to estimate $\sum_{T\in\mathcal T}\int_T |f_I^h\, g|$ and $\sum_{z\in\mathcal N}\int_{\omega_z}|f_I^h\, \bar g_z^*|$. For the former, in each T, choose a vertex z such that $H_z \simeq \operatorname{diam}\omega_T$. Now $\int_T |f_I^h\, g| \lesssim \|g\|_{2\,;T}\|f_h^I\|_{2\,;\omega_z}$. So using Hölder's inequality and noting that $\lambda_z'^{-2} \lesssim 1 + \varepsilon^2 H_z^{-2} \simeq 1 + \varepsilon^2(\operatorname{diam}\omega_T)^{-2}$ implies $\sum_{T\in\mathcal T}\lambda_z'^{-2}\|g\|^2_{2\,;T} \lesssim \Theta'$, one concludes that $\sum_{T\in\mathcal T}\int_T|f_I^h\, g|$ is bounded by the right-hand side in (6.2).

It remains to estimate $\sum_{z\in\mathcal N'}\int_{\omega_z}|f_I^h\,\bar g_z^*|$, where $\mathcal N' := \{z \in \mathcal N : \bar g_z^* \neq 0\}$. On an application of Hölder's inequality, this task is reduced to showing that $\sum_{z\in\mathcal N'}\lambda_z'^{-2}|\omega_z||\bar g_z^*|^2 \lesssim \Theta'$. The latter follows by combining $\lambda_z'^{-2} \lesssim 1 + \varepsilon^2 H_z^{-2}$ with $|\omega_z||\bar g_z^*|^2 \lesssim \|g\|^2_{2\,;\tilde\omega_z}$ (in view of (6.16)) and the following observation. Note that whenever $T \cap \tilde\omega_z \neq \varnothing$, either T is an anisotropic triangle in ω_z, or T touches such a triangle, so $\operatorname{diam}\omega_T \simeq H_z$ (in view of $\mathcal A3$). □

6.3 Some preliminaries for the proof of Theorem 6.2. Changes in $\bar g_z$ and $\bar g_z^*$

Consider a particular path $\mathcal N_i$, $i = 1, \dots, n_{\rm ani} + n_{\rm s.ani}$, and set $\Omega_i := \cup_{z\in\mathcal N_i}\omega_z$. Assuming that $\bar H$ is sufficiently small, $\mathcal A5$ implies that for each $(x, y) \in \Omega_i$, one can define local curvilinear coordinates (ξ^i, η^i), where ξ^i is the signed distance to $\mathcal C_i$, and η^i is the arc-length parameter of the closest point on $\mathcal C_i$. Note that the transformation Jacobian, as well as the Lamé coefficients are $1 + \mathcal O(\varrho\bar H)$ (see, e.g., [7, Appendix A]). Note also that the ξ^i-coordinate curves are straight lines and $|\sin(\angle(S, \mathrm i_{\xi^i}))| \lesssim \frac{h_z}{|S|}$ for any $S \subset \mathcal S_z$ of any node $z \in \mathcal N_i$.

Let (η_z^-, η_z^+) be the range of η^i within ω_z; if z is a path start/end point, let (η_z^-, η_z^+) be the range of η^i within $\omega'_z \cap \Omega_i$, where $\omega'_z := \omega_z^{(1)}$ is the patch of elements in or surrounding ω_z. Then $\mathcal A5$ implies, for any $z \in \mathcal N_i$, that

$$\omega_z \subset \omega_z^* := \Omega_i \cap \bigl\{\eta^i \in (\eta_z^-, \eta_z^+)\bigr\} \subset \omega_z^{(J)}, \quad \eta_z^+ - \eta_z^- \simeq h_z, \quad \operatorname{diam}\omega_z^* \simeq H_z\,, \quad (6.19)$$

with some $J \lesssim 1$ (and $\omega_z^{(J)}$ defined in §5).

Next, we tweak the definitions $\bar{g}_z$ and $\bar{g}_z^*$ for $z \in \mathcal{N}_{\rm paths}$ (while there are no changes for $z \in \mathcal{N}\backslash\mathcal{N}_{\rm paths}$, i.e. $\{\bar{g}_z^*\}$ is as in the proof of (6.2), with $\bar{g}_z^* = \bar{g}_z = 0$ for $z \in \mathcal{N}_{\rm iso} \cup \partial\Omega$). We also need to tweak the definitions of ω_z^* for $z \in \mathcal{N}_{\rm paths}$ close to the path start and end nodes. Note that with the new definitions, (5.5) and (6.17) will hold true for all $z \in \mathcal{N}$. Hence Theorem 6.1 remains true as well (in view of Remark 6.6).

Consider a particular path $\mathcal{N}_i = \{z_j\}_{j=1}^k$ (where $k = k(i)$). Without loss of generality, let $z_k \in \partial\Omega$, and z_1 be connected to some $z_0 \in \mathcal{N}_{\rm iso}$ by an edge of length $\simeq h_{z_1}$ with $H_{z_0} \simeq H_{z_1}$.

(i) If $z \in \mathcal{N}_i$, in the proof of Theorem 6.1, let the local cartesian coordinates (ξ, η) associated with $z \in \mathcal{N}_i$, which will be also denoted (ξ^z, η^z) (to distinguish them from (ξ^i, η^i)) be chosen so that the ξ^z axis goes through z and is normal to $\mathcal{C}_i$ (i.e. it coincides with the ξ^i-coordinate curve through z). Here we use $\mathcal{A}5$ and Remark 6.5. Note that now the definition (6.7) of $\bar{g}_z$ uses $\xi = \xi^z$.

(ii) Next, set $\mathcal{N}_i^{(l)} := \{z \in \mathcal{N}_i : \omega_z \cap \omega_{z_l}^* \neq \varnothing\}$ for $l = 1, k$, and proceed to redefining ω_z^* and $\bar{g}_z^*$ for $z \in \mathcal{N}_i^{(1)} \cup \mathcal{N}_i^{(k)}$. First define ω_z^* by (6.19). Now, for $z \in \mathcal{N}_i^{(1)} \cup \mathcal{N}_i^{(k)}$, tweak the earlier definitions as follows:

$$\begin{aligned} z \in \mathcal{N}_i^{(1)} :& \quad \text{set} \quad \omega_z^* := \omega_z^* \cup \widetilde{\omega}_z, && \bar{g}_z,\, \bar{g}_z^*,\, \widetilde{\omega}_z \text{ from Remark 6.6,} \\ z \in \mathcal{N}_i^{(k)}\backslash\mathcal{N}_i^{(1)} :& \quad \text{set} \quad \omega_z^* := \cup_{z' \in \mathcal{N}_i^{(k)}} \omega_{z'}^*, && \bar{g}_z^* := 0. \end{aligned}$$

For the former case, unless $\bar{g}_z = 0$, we define $\bar{g}_z$ and $\bar{g}_z^*$ by (6.7), (6.16), in which $\varphi_z(\xi) = \varphi_z(\xi^z)$ and $d\xi = d\xi^z$, $d\eta = d\eta^z$. Note that (6.19) and (6.17) remain valid (with J replaced by $2J$, where we also used $\widetilde{\omega}_z \subset \omega_z^{(1)}$; see Remark 6.6). (Note also that (5.5) for $z \in \mathcal{N}_i^{(1)}$ is discussed in Remark 6.6, while for $z \in \mathcal{N}_i^{(k)}\backslash\mathcal{N}_i^{(1)}$, the first relation in (5.5) is straightforward, while the second can be shown by employing, e.g., Lemma 7.1, in view of $z_k \in \partial\Omega$.)

(iii) Finally, for the remaining $z \in \mathcal{N}_i\backslash[\mathcal{N}_i^{(1)} \cup \mathcal{N}_i^{(k)}]$, define $\bar{g}_z^*$ by (6.16) in which $\varphi_z(\xi) = \varphi_z(\xi^i)$ and $d\xi\, d\eta = d\xi^i d\eta^i$, i.e. the integration is performed in the curvilinear coordinates associated with $\mathcal{N}_i$, and $\widetilde{\omega}_z := \{(\xi^i, \eta^i) \in (\xi_z^-, \xi_z^+) \times (\eta_z^-, \eta_z^+)\}$. Note that $\varphi_z(\xi^i) = \varphi_z(\xi^z)$ along the ξ^i-coordinate curve. (Strictly speaking, our definitions guarantee that the support interval (ξ_z^-, ξ_z^+) of φ_z is within ω_z^* only on the ξ^i-coordinate line through z. On the other hand, $\mathcal{A}5$ implies that to satisfy $\widetilde{\omega}_z \subset \omega_z^*$ (if this is not the case), the support interval width reduction of $\lesssim h_z$ will suffice, while all other arguments will remain unchanged.) Now (5.5) is easily established in the curvilinear coordinates (ξ^i, η^i) as in §5.1. This bound in the original coordinates follows (in view of the the transformation Jacobian, as well as the Lamé coefficients being $1 + \mathcal{O}(\varrho\bar{H})$).

Remark 6.7 The above definitions imply, for any $z \in \mathcal{N}_i \backslash [\mathcal{N}_i^{(1)} \cup \mathcal{N}_i^{(k)}]$, that whenever $T \cap \omega_z^* \neq \varnothing$ and T is an anisotropic element, one has $|T| \simeq |\omega_T|$ (in view of the mesh assumptions in §2).

6.4 Interior Residual. Proof of (6.4)

Proof We start with the case of $n_{\rm ani} = n_{\rm s.ani}$, i.e. $\mathcal{N} = \big(\cup_{i=1}^{n_{\rm ani}} \mathcal{N}_i\big) \cup \big(\mathcal{N} \backslash \mathcal{N}_{\rm paths}\big)$. In (4.4), we replace $\{\bar g_z\}$ by $\{\bar g_z^*\}$ from §6.3. Using $\Omega_i = \cup_{z \in \mathcal{N}_i} \omega_z$, set $\Omega_i^* := \Omega_i \backslash (\omega_{z_1}^* \cup \omega_{z_k}^*)$. Within Ω_i^*, we use the local cartesian coordinates $(\xi, \eta) = (\xi^i, \eta^i)$, as described in §6.3, as well as the original coordinates (x, y) (the superindex i will sometimes be skipped when there is no ambiguity). So split $I\!I$ of (4.4) as $I\!I = I\!I_{\backslash \rm paths} + \sum_i [I\!I_i^{(1)} + I\!I_i^{(k(i))} + I\!I_i + I\!I_i^{\rm osc} + I\!I_i^{\varrho}]$ as follows:

$$
\begin{aligned}
I\!I_{\backslash \rm paths} &:= \sum_{z \in \mathcal{N} \backslash \mathcal{N}_{\rm paths}} \int_{\omega_z} f_h^I \, (g - \bar g_z^*) \, \phi_z \,, \\
I\!I_i^{(l)} &:= \sum_{z \in \mathcal{N}_i} \int_{\omega_z \cap \omega_{z_l}^*} f_h^I \, (g - \bar g_z^*) \, \phi_z \qquad \text{for} \;\; l = 1, k(i), \\
I\!I_i &:= \sum_{z \in \mathcal{N}_i} \int_{(x,y) \in \omega_z \cap \Omega_i^*} \widetilde f_h^I \, (g - \bar g_z^*) \, \phi_z \, d\xi^i d\eta^i, \\
I\!I_i^{\rm osc} &:= \sum_{z \in \mathcal{N}_i} \int_{(x,y) \in \omega_z \cap \Omega_i^*} [f_h^I - \widetilde f_h^I] \, (g - \bar g_z^*) \, \phi_z \, d\xi^i d\eta^i, \\
I\!I_i^{\varrho} &:= \sum_{z \in \mathcal{N}_i} \int_{(x,y) \in \omega_z \cap \Omega_i^*} f_h^I \, (g - \bar g_z^*) \, \phi_z \, [dx\, dy - d\xi^i d\eta^i].
\end{aligned}
$$

Here all integrands are in the original variables (x, y), and the function $\widetilde f_h^I$ (which will be specified below) is such that

$$
|f_h^I - \widetilde f_h^I| \leqslant \operatorname{osc}(f_h^I; \omega_z^* \cap \Omega_i^*) \qquad \text{in} \;\; \omega_z \cap \Omega_i^*. \tag{6.20}
$$

Hence, for $I\!I_{\backslash \rm paths}$, as well as $I\!I_i^{\rm osc}$ and $I\!I_i^{\varrho}$, we immediately get versions of (6.2), (6.3):

$$
\begin{aligned}
|I\!I_{\backslash \rm paths}| &\lesssim \Big\{ \Theta' \sum_{z \in \mathcal{N} \backslash \mathcal{N}_{\rm paths}} \|\lambda_z' \, f_h^I\|_{2\,;\omega_z}^2 \Big\}^{1/2}, \\
\sum_i |I\!I_i^{\rm osc}| &\lesssim \Big\{ \Theta' \sum_{z \in \mathcal{N}_{\rm paths}} \|\lambda_z' \, \operatorname{osc}(f_h^I; \omega_z^* \cap \Omega_i)\|_{2\,;\omega_z}^2 \Big\}^{1/2}, \\
\sum_i |I\!I_i^{\varrho}| &\lesssim \Big\{ \Theta' \sum_{z \in \mathcal{N}_{\rm paths}} \|\varrho \bar H \lambda_z' \, f_h^I\|_{2\,;\omega_z}^2 \Big\}^{1/2}.
\end{aligned}
\tag{6.21}
$$

When estimating $I\!I_i^{\rm osc}$ and $I\!I_i^{\varrho}$, we noted that the transformation Jacobian remains $1 + \mathcal{O}(\varrho \bar H)$ within Ω_i^*. We also used the observation that $H_z \simeq H_T \simeq \operatorname{diam} \omega_T$ for any $T \cap \omega_z^* \neq \varnothing$ so $\lambda_z'^{-2} \lesssim 1 + \varepsilon^2 (\operatorname{diam} \omega_T)^{-2}$. Note that in (6.21), as well as similar bounds below, $\operatorname{osc}(f_h^I;\omega_z^*\cap\Omega_i)$ can be replaced by $\operatorname{osc}(f_h^I;\omega_z)$ (in view of (6.19) and also as $\omega_z^*\cap\Omega_i$ includes only anisotropic elements of similar size).

Next, focus on a particular anisotropic path $\mathcal{N}_i=\{z_j\}_{j=1}^k$ (with $k=k(i)$). Note that, by $\mathcal{A}5$ and the maximum angle condition, the polygonal curve joining consecutive nodes of the path $\mathcal{N}_i$ can be described in the coordinates $(\xi,\eta)=(\xi^i,\eta^i)$ by some function $\xi=\kappa_i(\eta)$, while the two disjoint polygonal curves forming $\partial\Omega_i^*\cap\partial\Omega_i$ can be described by some functions $\xi=\kappa_i^\pm(\eta)$. Furthermore, Ω_i^* is a curvilinear rectangle bounded by the curves $\xi=\kappa^\pm(\eta)$ and the lines $\eta=\eta_{z_1}^+$ and $\eta=\eta_{z_k}^-$ (assuming, without loss of generality, that η increases as we move along the path $\mathcal{N}_i$ from the start node z_1 to the end node z_k). Note also that $|\frac{d}{d\eta}\kappa_i|+|\frac{d}{d\eta}\kappa_i^\pm|\lesssim 1$, so

$$\pm[\kappa_i^\pm(\eta)-\kappa_i(\eta)]\simeq H_z \qquad \text{for } \eta\in(\eta_z^-,\eta_z^+)\cap(\eta_{z_1}^+,\eta_{z_k}^-),$$
$$\operatorname{osc}(\kappa(\eta)\,;\omega_z^*)+\operatorname{osc}(\kappa^\pm(\eta)\,;\omega_z^*)\lesssim h_z,$$

where we also used (6.19).

For each $I\!I_i$, using the notation $\breve f_h^I(\xi,\eta):=f_h^I(x,y)$, set $\widetilde f_h^I:=\breve f_h^I(\kappa(\eta),\eta)$:

$$I\!I_i:=\sum_{z\in\mathcal{N}_i}\int_{(x,y)\in\omega_z\cap\Omega_i^*}\breve f_h^I(\kappa(\eta),\eta)\,(g-\bar g_z^*)\,\phi_z\,d\xi\,d\eta\,,$$

where $g=g(x,y)$ and $\phi_z=\phi_z(x,y)$. Now we can rewrite $I\!I_i$ as

$$I\!I_i=\sum_{z\in\mathcal{N}_i}\int_{\eta_{z_1}^+}^{\eta_{z_k}^-}f_h^I(\kappa_i(\eta),\eta)\int_{\kappa_i^-(\eta)}^{\kappa_i^+(\eta)}(g-\bar g_z^*)\,\phi_z\,d\xi\,d\eta,\quad \text{where } (\xi,\eta)=(\xi^i,\eta^i). \tag{6.22}$$

Set $\varphi_i(\xi,\eta):=\sum_{z\in\mathcal{N}_i}\phi_z(x,y)$ and note that for all $\eta\in(\eta_{z_1}^+,\eta_{z_k}^-)$

$$\sum_{z\in\mathcal{N}_i}\int_{\kappa_i^-(\eta)}^{\kappa_i^+(\eta)}g\,\phi_z\,d\xi=\int_{\kappa_i^-(\eta)}^{\kappa_i^+(\eta)}g\,\varphi_i\,d\xi=:\hat g_i(\eta)\int_{\kappa_i^-(\eta)}^{\kappa_i^+(\eta)}\varphi_i\,d\xi=\sum_{z\in\mathcal{N}_i}\int_{\kappa_i^-(\eta)}^{\kappa_i^+(\eta)}\hat g_i(\eta)\,\phi_z\,d\xi. \tag{6.23}$$

Here the central relation defines $\hat g_i(\eta)$. Now, (6.22) can be rewritten as

$$I\!I_i=\sum_{z\in\mathcal{N}_i}\int_{(x,y)\in\omega_z\cap\Omega_i^*}\widetilde f_h^I\,\big(\hat g_i(\eta)-\bar g_z^*\big)\,\phi_z\,d\xi\,d\eta=:\widetilde{I\!I}_i+\widetilde{I\!I}_i^{(1)}+\widetilde{I\!I}_i^{(k)}\,,$$

where $\widetilde{I\!I}_i^{(l)}$, for $l=1,k$, correspond to $z\in\mathcal{N}_i^{(l)}$. Again using (6.20), one gets

$$|\widetilde{I\!I}_i|\lesssim\sum_{z\in\mathcal{N}_i\backslash[\mathcal{N}_i^{(1)}\cup\mathcal{N}_i^{(k)}]}\|\hat g_i(\eta)-\bar g_z^*\|_{2\,;\omega_z}\Big\{\|f_h^I\|_{2\,;\omega_z}+\|\operatorname{osc}(f_h^I;\omega_z^*\cap\Omega_i)\|_{2\,;\omega_z}\Big\}\,.$$

To complete the estimation of $\widetilde{I\!I}_i$, it remains to show for $z\in\mathcal{N}_i\backslash[\mathcal{N}_i^{(1)}\cup\mathcal{N}_i^{(k)}]$ that

$$\|\hat g_i(\eta)-\bar g_z^*\|_{2\,;\omega_z\cap\Omega_i^*}\lesssim\min\Big\{\|g\|_{2\,;\omega_z^*},\,h_z\,|\!|\!|g|\!|\!|_{2\,;\omega_z^*}\Big\}. \tag{6.24}$$

Then, combining $\min\{a, h_z b\} \lesssim \min\{\lambda'_z, h_z\varepsilon^{-1}\}\,(\lambda'^{-1}_z a + \varepsilon b)$ (for any $a, b > 0$) with $\min\{\lambda'_z, h_z\varepsilon^{-1}\} = \min\{1, h_z\varepsilon^{-1}\}$, one gets

$$\|\hat g_i(\eta) - \bar g^*_z\|_{2\,;\omega_z} \lesssim \theta'_z\,\min\{1,\, h_z\varepsilon^{-1}\}, \qquad \theta'_z := \lambda'^{-1}_z\|g\|_{2\,;\omega^*_z} + \varepsilon |\!|\!| g |\!|\!|_{2\,;\omega^*_z}. \tag{6.25}$$

As $\sum_{z\in\mathcal N}\theta'^2_z \lesssim \widetilde\Theta$ (where we again use $\lambda'^{-2}_z \simeq 1+\varepsilon^2 H_z^{-2} \simeq 1+\varepsilon^2(\operatorname{diam}\omega_T)^{-2}$ for any $T\cap\omega^*_z \neq \varnothing$, in view of Remark 6.7), one concludes that $\sum_i |\widetilde{\mathit\Pi}_i|$ is bounded by the right-hand side of (6.4).

To prove (6.24), we combine $\|\hat g_i(\eta) - \bar g^*_z\|_{2\,;\omega_z\cap\Omega^*_i} \lesssim \|g\|_{2\,;\omega^*_z}$ (which follows from the definition of $\hat g_i$ and the first relation in (5.5)), with $H_z|\bar g_z - \bar g^*_z| \lesssim \|\nabla g\|_{1\,;\omega^*_z}$ from (5.5), and another crucial bound that can be found in [10, (7.29)]:

$$H_z\|\hat g_i(\eta) - \bar g_z\|_{\infty\,;\omega_z\cap\Omega^*_i} \lesssim |\!|\!| g |\!|\!|_{1\,;\omega^*_z}. \tag{6.26}$$

Note that the latter bound is obtained in [10] for the case of (ξ^i, η^i) being cartesian coordinates. But the changes for our case are minimal. In particular, the estimation should be mainly carried out in the variables $(\xi,\eta) = (\xi^i,\eta^i)$, except the bound $\|\varphi_i - \varphi_z\|_{\infty\,;\omega^*_z\cap\Omega^*_i} \lesssim \frac{h_z}{H_z}$, should be obtained in the cartesian coordinates (ξ^z,η^z) associated with z (as the argument relies on the linearity of $\varphi_i - \varphi_z$ in the variable η^z on each mesh element).

For $\mathit\Pi^{(1)}_i$ and $\widetilde{\mathit\Pi}^{(1)}_i$, we estimate $\|g - \bar g^*_z\|^2_{2\,;\omega_z}$ and $\|\hat g_i - \bar g^*_z\|^2_{2\,;\omega_z}$ using the first relation in (5.5) and the definition of $\hat g_i$. Also combine $\|f^I_h\|_{\infty\,;\omega^*_z\cap\Omega_i} \leqslant |f^I_h(z_0)| + \sum_{z'\in\mathcal N^{(1)}_i}\operatorname{osc}(f^I_h\,;\omega^*_{z'}\cap\Omega_i)$ and $|\omega_{z_0}|^{1/2}|f^I_h(z_0)| \lesssim \|f^I_h\|_{2\,;\omega_{z_0}}$ with the observations that $\lambda'_z \simeq \lambda'_{z_1} \simeq \lambda'_{z_0}$ and $|\omega_z|^{1/2} \simeq |\omega_{z_1}|^{1/2} \lesssim |\omega_{z_0}|^{1/2}$ for $z\in\mathcal N^{(1)}_i$. So

$$|\mathit\Pi^{(1)}_i| + |\widetilde{\mathit\Pi}^{(1)}_i| \lesssim \Big\{\sum_{z\in\mathcal N^{(1)}_i}\lambda'^{-2}_z\|g\|^2_{2\,;\omega^*_z}\Big\}^{1/2} \Big\{\|\lambda'_{z_0}f^I_h\|^2_{2\,;\omega_{z_0}} + \sum_{z\in\mathcal N^{(1)}_i}\|\lambda'_z\operatorname{osc}(f^I_h\,;\omega^*_z\cap\Omega_i)\|^2_{2\,;\omega_z}\Big\}^{1/2}. \tag{6.27}$$

So $\sum_i(|\mathit\Pi^{(1)}_i| + |\widetilde{\mathit\Pi}^{(1)}_i|)$ is also bounded by the right-hand side of (6.4).

For $\mathit\Pi^{(k)}_i$ and $\widetilde{\mathit\Pi}^{(k)}_i$, if at least one node in $\mathcal N^{(k)}_i$ is connected by an edge with some node in $\mathcal N_{\rm iso}$, then imitate the estimation of $\mathit\Pi^{(1)}_i$ and $\widetilde{\mathit\Pi}^{(1)}_i$. Otherwise, note that $|T| \simeq |\omega_T|$ for any $T\cap\omega^*_{z_k} \neq \varnothing$. Recall that $\bar g^*_z = 0$ and note that $h_z \simeq h_{z_k}$ within $\omega^*_{z_k}$. Also using the definition of $\hat g_i$, and also $\operatorname{osc}(f^I_h\,;\omega^*_{z_k}) \lesssim \|f^I_h\|_{\infty\,;\omega^*_{z_k}} \simeq |\omega^*_{z_k}|^{-1/2}\|f^I_h\|_{2\,;\omega^*_{z_k}}$, one gets

$$|\mathit\Pi^{(k)}_i| + |\widetilde{\mathit\Pi}^{(k)}_i| \lesssim \theta''_{z_k}\,\big\|\min\{1,\, h_{z_k}\varepsilon^{-1}\}\,f^I_h\big\|_{2\,;\omega^*_{z_k}}, \quad \theta''_{z_k} := (1+\varepsilon h^{-1}_{z_k})\|g\|_{2\,;\omega^*_{z_k}}. \tag{6.28}$$

To estimate θ''_{z_k}, recall a version of [10, (7.26)] $\|g\|_{2\,;\omega^*_{z_k}} \lesssim h_{z_k}\|\nabla g\|_{2\,;\omega^*_{z_k}}$ (obtained, in view of $|\partial\omega_{z_k}\cap\partial\Omega| \simeq H_{z_k}$, by employing a version of Lemma 7.1).

Now, $\theta''_{z_k} \lesssim \|g\|_{2\,;\omega^*_{z_k}} + \varepsilon\|\nabla g\|_{2\,;\omega^*_{z_k}}$, so $\sum_i \theta''^2_{z_k} \lesssim \widetilde{\Theta}$. So $\sum_i(|I\!I_i^{(k)}| + |\widetilde{I\!I}_i^{(k)}|)$ is bounded by the right-hand side of (6.4).

Combining the above bounds for all the ingredients of $I\!I_i$ with $\Theta' \lesssim \widetilde{\Theta}'$, and also (6.19), we complete the proof of (6.4) for the case of no semi-anisotropic paths.

To prove the desired bound in the general case, we also need to bound $I\!I_i$ for each $i = n_{\rm ani}+1, \ldots, n_{\rm ani}+n_{\rm s.ani}$. We imitate the above estimation with the following modifications. First, we choose $\kappa_i^- = \kappa_i$; then all the bounds obtained for an anisotropic path remain valid within the subdomain $\Omega_i \cap \{\xi^i > \kappa(\eta^i)\}$ $=: \Omega_i^{\rm ani}$, which contains only anisotropic elements. Note that we now replace ω^*_z in (6.25) and (6.28) by $\omega^*_z \cap \Omega_i^{\rm ani}$. (For this change in (6.25), it is crucial that $\widetilde{\omega}_z \subset \Omega_i^{\rm ani}$ for $z \in \mathcal{N}_i \backslash [\mathcal{N}_i^{(1)} \cup \mathcal{N}_i^{(k)}]$. To ensure this, one may need to reduce the width of the interval $(\xi_z^-, \xi_z^+) = (\xi_z, \xi_z^+)$, used in the definition of $\widetilde{\omega}_z$ in §6.3, by $\lesssim h_z$.)

The remaining subdomain $\Omega_i \cap \{\xi^i < \kappa(\eta^i)\} =: \Omega_i^{\rm iso}$ contains only isotropic elements, i.e. $\Omega_i^{\rm iso} = \{T \subset \Omega_i : h_T \simeq H_T\}$. This results in an additional component $I\!I_i^{\rm iso}$ in $I\!I_i$, which we estimate imitating the proof of (6.2) (with Θ' of (6.3)) as follows. We need to bound $\sum_{T\subset\Omega_i^{\rm iso}} \int_T |f_I^h\, g|$ and $\sum_{z\in\mathcal{N}_i} \int_{\omega_z\cap\Omega_i^{\rm iso}} |f_I^h\, \bar{g}^*_z|$. For the former, in each T, choose a vertex $z \in \mathcal{N}_i$ such that $H_z \simeq \operatorname{diam}\omega_T$ (by $\mathcal{A}3$). Now $\int_T |f_I^h\, g| \lesssim \|g\|_{2\,;T} \|f_h^I\|_{2\,;\omega_z\cap\Omega_i^{\rm iso}}$. So using Hölder's inequality with the weights $\lambda''_z := \min\{1,\, h_z H_z \varepsilon^{-2}\}$, one gets

$$|I\!I_i^{\rm iso}| \lesssim \Big\{ \sum_{T\subset\Omega_i^{\rm iso}} \lambda''^{-1}_z \|g\|^2_{2\,;T} + \sum_{z\in\mathcal{N}_i} \lambda''^{-1}_z h_z^2 |\bar{g}^*_z|^2 \Big\}^{1/2} \Big\{ \sum_{z\in\mathcal{N}_i} \lambda''_z \|f_h^I\|^2_{2\,;\omega_z\cap\Omega_i^{\rm iso}} \Big\}^{1/2}.$$

Here, note that $\lambda''^{-1}_z \lesssim 1 + \varepsilon^2 h_z^{-1} H_z^{-1}$, where $h_z \simeq H_T$ and $H_z \simeq \operatorname{diam}\omega_T$, so $\sum_i \sum_{T\subset\Omega_i^{\rm iso}} \lambda''^{-1}_z \|g\|^2_{2\,;T} \lesssim \widetilde{\Theta}'$. Next, the first relation in (5.5) implies $h_z^2|\bar{g}^*_z|^2 \lesssim h_z H_z^{-1} \|g\|^2_{2\,;\omega^*_z}$, while $\lambda''^{-1}_z\, h_z H_z^{-1} \lesssim 1 + \varepsilon^2 H_z^{-2}$, so $\sum_i \sum_{z\in\mathcal{N}_i} \lambda''^{-1}_z h_z^2 |\bar{g}^*_z|^2 \lesssim \widetilde{\Theta}'$. Finally, $\|f_h^I\|_{\infty\,;\omega_z\cap\Omega_i^{\rm iso}} \lesssim |f(z)| + \operatorname{osc}(f_h^I;\omega_z)$ and $|\omega_z|^{1/2}|f_h^I(z)| \lesssim \|f_h^I\|_{2\,;\omega_z}$ so

$$\begin{aligned} \lambda''_z \|f_h^I\|^2_{2\,;\omega_z\cap\Omega_i^{\rm iso}} &\lesssim \lambda''_z\, h_z H_z^{-1} \big(\|f_h^I\|^2_{2\,;\omega_z} + \|\operatorname{osc}(f_h^I;\omega_z)\|^2_{2\,;\omega_z} \big) \\ &\lesssim \|\min\{1,\, h_z\varepsilon^{-1}\}\, f_h^I\|^2_{2\,;\omega_z} + \|\lambda'_z \operatorname{osc}(f_h^I;\omega_z)\|^2_{2\,;\omega_z}, \end{aligned} \tag{6.29}$$

where we used $\sqrt{\lambda''_z\, h_z H_z^{-1}} \lesssim \min\{1,\, h_z\varepsilon^{-1}\} \lesssim \lambda'_z$. Combining the above observations, one concludes that $\sum_i |I\!I_i^{\rm iso}|$ is bounded by the right-hand side of (6.4). This completes the proof of (6.4) in the general case. □

7 Estimation of Θ. Quasi-interpolation operator Π_h

For the a posteriori estimators of Theorems 5.1, 6.1, 6.2, as well as Lemma 6.3, to acquire meaning, we still need to estimate Θ of (5.6), and, possibly, Θ' of (6.3), $\widetilde{\Theta}'$ and $\widetilde{\Theta}$ of (6.5), as well as Θ^* of (6.6); see also Remark 6.4.

Remark 7.1 If $H_T \gtrsim \varepsilon$ for all $T \in \mathcal{T}$, then $\Theta + \widetilde{\Theta} \lesssim 1$. Indeed, in view of (4.1), setting $G_h := 0$ yields the desired assertion. Note that this observation may apply to highly anisotropic meshes (as no restrictions on h_z or $\frac{h_z}{H_z}$ are made).

To deal with $g = G - G_h$ in Θ on more general meshes, it is convenient to employ a quasi-interpolant G_h of Clément/Scott-Zhang type. However, such interpolants were originally introduced on shape-regular meshes, and their error bounds are not readily available for anisotropic meshes. A quasi-interpolation error on anisotropic meshes is estimated in [6, §3], but only in the L_2 norm, while an extension of this approach to estimate the gradient of the quasi-interpolation error does not produce as sharp bounds as we require. (We also refer the reader to [2, Chapt. 3] for a discussion of Scott-Zhang-type interpolation on anisotropic tensor-product meshes.)

We construct an interpolant $\Pi_h v \in S_h$ of a function v onto S_h in the spirit the Clément/Scott-Zhang interpolants as follows. For any $z \in \mathcal{N}$, let σ_z be any edge originating at z. If $z \in \partial\Omega$, then it is additionally required that $\sigma_z \subset \partial\Omega$. Then let $\Pi_h v(z)$ be the average value of v over σ_z (also denoted $\mathcal{A}_{\sigma_z} v$):

$$\Pi_h v(z) := \mathcal{A}_{\sigma_z} v := |\sigma_z|^{-1} \int_{\sigma_z} v\,, \tag{7.1}$$

and $\Pi_h v := \sum_{z\in\mathcal{N}} \phi_z |\sigma_z|^{-1} \int_{\sigma_z} v$.

Now we make further important restrictions on the choice of σ_z (without imposing any additional constraints on the mesh).

- For any $z \in \mathcal{N} \backslash \mathcal{N}^*_{\partial\Omega}$, we let $|\sigma_z| \simeq H_z$.
- Furthermore, for any pair of z and z' in $\mathcal{N}_{\rm ani} \backslash \mathcal{N}^*_{\partial\Omega}$ connected by a short edge (such that $|z - z'| \simeq h_z$), we impose that σ_z and $\sigma_{z'}$ have the same orientation in the sense that their respective endpoints $\hat{z} \neq z$ and $\hat{z}' \neq z'$ satisfy $|\hat{z} - \hat{z}'| \lesssim h_z$; see Fig. 6.

The following auxiliary result will play a central role in our analysis throughout this section.

Lemma 7.1 *For any triangle T with vertices z, z', z'' and their respective opposite edges S, S', S'', one has*

$$|\mathcal{A}_{S'}\, v - \mathcal{A}_{S''}\, v| \lesssim |S||T|^{-1} \|\nabla v\|_{1\,;T}\,.$$

Proof This follows from the observation that, with the ζ-axis having the inward normal direction to S, and $\hbar := 2|T||S|^{-1}$, one gets $\mathcal{A}_{S'}\, v - \mathcal{A}_{S''}\, v = \hbar^{-1} \int_0^{\hbar} \big(v|_{S'} - v|_{S''}\big)\, d\zeta$. □

7.1 Quasi-interpolation operator $\varPi_h$ for anisotropic nodes

We first investigate the approximation properties of $\varPi_h$ in the most interesting case of anisotropic nodes.

Lemma 7.2 *(i) If $z \in \mathcal{N}_{\rm ani}$ is a vertex of a triangle T, then*

$$|T|^{1/2}|\varPi_h v(z)| \lesssim \|v\|_{2\,;T} + H_T\|\nabla v\|_{2\,;\omega_z}\,. \tag{7.2}$$

(ii) If $z, z' \in \mathcal{N}_{\rm ani}$ are vertices of a triangle T, then, for $p = 1, 2$,

$$|T|^{1/p}\,\frac{|\varPi_h v(z) - \varPi_h v(z')|}{|z - z'|} \lesssim \|\nabla v\|_{p\,;\omega_z\cup\omega_{z'}}\,. \tag{7.3}$$

Proof (i) We need to consider only $z \in \mathcal{N}_{\rm ani}\backslash\partial\Omega$, as for $z \in \partial\Omega$ the result is obvious. Let S be any longer edge of T, i.e. $|S| \simeq H_T$. A (possibly repeated) application of Lemma 7.1 yields $|\varPi_h v(z) - \mathcal{A}_S\, v| = |\mathcal{A}_{\sigma_z} v - \mathcal{A}_S\, v| \lesssim h_T^{-1}\|\nabla v\|_{1\,;\omega_z}$. Next, by a version of Lemma 3.1 for T (with ω_z replaced by T, $\gamma_z\backslash\mathring{\gamma}_z$ by S, and also h_z by h_T and H_z by H_T), one gets $|\mathcal{A}_S\, v| \lesssim H_T^{-1}\|v\|_{1\,;S} \lesssim h_T^{-1}|\!|\!|v|\!|\!|_{1\,;T}$. Now,

$$|\varPi_h v(z)| \lesssim (h_T H_T)^{-1}\|v\|_{1\,;T} + h_T^{-1}\|\nabla v\|_{1\,;\omega_z}\,, \tag{7.4}$$

so we get (7.2) using $|T| \simeq h_T H_T \simeq |\omega_z|$ in view of $z \in \mathcal{N}_{\rm ani}$.

(ii) Suppose $z, z' \in \mathcal{N}_{\rm ani}\backslash\mathcal{N}^*_{\partial\Omega}$. If $|z - z'| \simeq H_T$, then using the above argument one gets $|\varPi_h v(z) - \varPi_h v(z')| \lesssim h_T^{-1}\|\nabla v\|_{1\,;\omega_z\cup\omega_{z'}}$, while if $|z - z'| \simeq h_T$, then an application of Lemma 7.1 (crucially, due to the choice of σ_z and $\sigma_{z'}$) yields $|\varPi_h v(z) - \varPi_h v(z')| \lesssim H_T^{-1}\|\nabla v\|_{1\,;\omega_z\cup\omega_{z'}}$. Combining the two cases, we get the desired result (7.3) (also using $|T| \simeq h_T H_T \simeq |\omega_z \cup \omega_{z'}|$).

If both $z, z' \in \partial\Omega$, the result is obvious. The remaining case is $z \in \mathcal{N}^*_{\partial\Omega}$ and $z' \notin \partial\Omega$. Then $|z - z'| \simeq H_T$, and the above argument again yields (7.3). □

Corollary 7.3 *If all vertices z, z', z'' of $T \in \mathcal{T}$ are in $\mathcal{N}_{\rm ani}$, then*

$$H_T^{-1}\|v - \varPi_h v\|_{2\,;T} + \|\nabla(v - \varPi_h v)\|_{2\,;T} \lesssim \|\nabla v\|_{2\,;\omega_T}\,,$$

where $\omega_T = \omega_z \cup \omega_{z'} \cup \omega_{z''}$ is the patch of elements touching T.

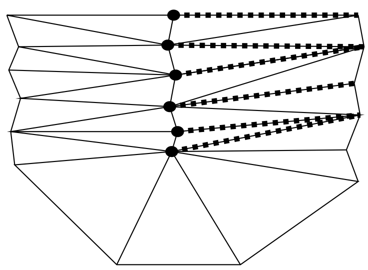


Fig. 6 Choice of σ_z in (7.1): a possible σ_z is highlighted for each highlighted node z.

Proof As $\varPi_h v \in S_h$, from (7.2) and (7.3) one immediately gets $\|\varPi_h v\|_{2\,;T} \lesssim \|v\|_{2\,;T} + H_T\|\nabla v\|_{2\,;\omega_T}$ and $\|\nabla\varPi_h v\|_{2\,;T} \lesssim \|\nabla v\|_{2\,;\omega_T}$. Furthermore, for any constant C, we have $\varPi_h C = C$. As one can find C such that $\|v - C\|_{2\,;T} \lesssim H_T\|\nabla v\|_{2\,;T}$, so

$$\|v - \varPi_h v\|_{2\,;T} = \|(v - C) - \varPi_h(v - C)\|_{2\,;T} \lesssim H_T\|\nabla v\|_{2\,;\omega_T}\,.$$

The desired result follows. □

Remark 7.2 (Quasi-interpolation operator $\varPi_h^*$) Note that $\varPi_h$ is stable in the scaled H^1 norm, but not in the L_2 norm, while the latter property is desirable for the estimation of $\varTheta$ in the case $\varepsilon \lesssim H_T$. To rectify this, we slightly modify $\varPi_h$ for $z \notin \partial\varOmega$ in two steps as follows.

First, we define the interpolation operator $\widetilde{\varPi}_h$ by replacing each edge σ_z that we used by its half $\widetilde{\sigma}_z$ originating at z (see Fig. 7). Then Lemma 7.2 and Corollary 7.3 remain valid. (In particular, in the proof of Lemma 7.2(ii), when dealing with $|z - z'| \simeq H_T$, one also needs the estimate $|\mathcal{A}_{\widetilde{\sigma}_z}\, v - \mathcal{A}_{\sigma_z\setminus\widetilde{\sigma}_z}\, v| \lesssim h_T^{-1}\|\nabla v\|_{1\,;\omega_z}$, which is obtained by a triple application of Lemma 7.1 within a triangle containing σ_z.)

Our final step in constructing $\varPi_h^*$ is to replace each half-edge $\widetilde{\sigma}_z$ by a parallelogram σ_z^* formed by $\widetilde{\sigma}_z$ and the adjacent half of any edge from $\mathring{\gamma}_z$ (see Fig. 7; note that one can always choose σ_z as above and with an adjacent edge from $\mathring{\gamma}_z$). It is crucial that $|\sigma_z^*| \simeq |\omega_z|$. Now $\varPi_h^* v(z) := \mathcal{A}_{\sigma_z^*}\, v$, so

$$|\omega_z|^{1/2}|\varPi_h^* v(z)| \lesssim \|v\|_{2\,;\omega_z}, \qquad |\omega_z||\varPi_h^* v(z) - \widetilde{\varPi}_h v(z)| \lesssim h_z\|\nabla v\|_{1\,;\omega_z}\,. \quad (7.5)$$

Here the first property immediately implies $\|\varPi_h^* v\|_{2\,;T} \lesssim \|v\|_{2\,;\omega_T}$, while the second yields (7.2) and (7.3) for $\varPi_h^*$, i.e. Lemma 7.2 and Corollary 7.3 hold true for $\varPi_h^*$. Combining the above observations, one gets, with $j = 0, 1$,

$$\|v - \varPi_h^* v\|_{2\,;T} \lesssim H_T^j|v|_{j,2\,;\omega_T}, \quad \|\nabla(v - \varPi_h^* v)\|_{2\,;T} \lesssim \|\nabla v\|_{2\,;\omega_T}\,. \qquad (7.6)$$

Theorem 7.4 *If $\mathcal{N} = \mathcal{N}_{\rm ani}$ (i.e. all nodes are anisotropic), then there exists G_h such that $\varTheta$ of* (5.6) *and $\widetilde{\varTheta}$ of* (6.5) *satisfy $\varTheta \lesssim 1$ and $\widetilde{\varTheta}' \lesssim \widetilde{\varTheta} \lesssim 1$.*

Proof Let $G_h := \varPi_h^* G$ and employ (7.6). Then $\varTheta + \widetilde{\varTheta} \lesssim |\!|\!|G|\!|\!|_{\varepsilon\,;\varOmega} \lesssim 1$, where we also used $H_T \simeq \operatorname{diam}\omega_T \simeq H_z$ for any $T \subset \omega_z$. (If $H_T \lesssim \varepsilon$ for all $T \in \mathcal{T}$, one can instead let $G_h := \varPi_h G$ and employ Corollary 7.3.) □

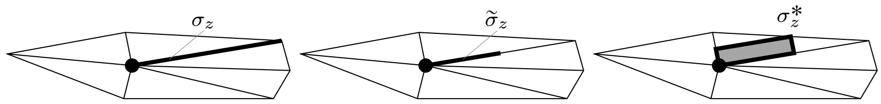


Fig. 7 Choice of a half-edge $\widetilde{\sigma}_z$ (centre) and a parallelogram σ_z^* (right) in Remark 7.2.

7.2 Quasi-interpolation operator Π_h in the general case

Lemma 7.5 *If $z \in \mathcal{N}_{\rm s.ani}$ is a vertex of a triangle T, and an edge S of T is such that $|S| \simeq H_T$, then*

$$|T|^{1/2}|\Pi_h v(z)| \lesssim \|v\|_{2\,;T} + (H_T H_z)^{1/2}\|\nabla v\|_{2\,;\omega_z}\,, \qquad (7.7)$$

$$|T||\Pi_h v(z) - \mathcal{A}_S v| \lesssim h_T\|\nabla v\|_{1\,;\omega_z}\,. \qquad (7.8)$$

Proof Whether T is anisotropic ($H_T \simeq H_z$) or isotropic ($H_T \simeq h_z$), a (possibly repeated) application of Lemma 7.1 yields $|\Pi_h v(z) - \mathcal{A}_S\, v| \lesssim H_T^{-1}\|\nabla v\|_{1\,;\omega_z}$, which immediately implies (7.8). Next, by Lemma 3.1, $|\mathcal{A}_S\, v| \simeq H_T^{-1}\int_S |v| \lesssim h_T^{-1}|\!|\!| v|\!|\!|_{1\,;T}$. So we again get (7.4). Only now the latter implies (7.7), as $|T| \simeq h_T H_T$ and $|\omega_z| \simeq h_T H_z$, in view of $z \in \mathcal{N}_{\rm s.ani}$. □

Lemma 7.6 *(i) Under condition $\mathcal{A}4$, Lemma 7.5 holds true for any $z \in \mathcal{N}_{\rm iso}$. (ii) If $z \in \mathcal{N}_{\rm ani}$ and $z' \in \mathcal{N}_{\rm iso}$ are vertices of a triangle T, and $|z - z'| \simeq h_z$, then one can choose $\sigma_{z'}$ such that (7.3) is valid with $\omega_z \cup \omega_{z'}$ replaced by ω_z.*

Proof (i) First, suppose that $H_T \simeq H_z$. Whether T is isotropic ($h_T \simeq H_z$) or anisotropic ($h_T \ll H_z$), Lemma 7.1 again yields $|\Pi_h v(z) - \mathcal{A}_S\, v| \lesssim H_T^{-1}\|\nabla v\|_{1\,;\omega_z}$ (in view of $\mathcal{A}4$). Combining this with $\mathcal{A}_S\, v \lesssim h_T^{-1}|\!|\!| v|\!|\!|_{1\,;T}$, and also $|T| \simeq h_T H_T$ and $|\omega_z| \simeq H_z^2 \simeq H_T^2$, one gets both (7.7) and (7.8).

Otherwise, $h_T \simeq H_T \ll H_z$ (i.e. $T \subset \mathring{\omega}_z$). Then one can always find a polygon $\widehat{\omega}_z \supset T$ (which is not necessarily a set of mesh elements) such that $\widehat{\omega}_z \supset \sigma_z$ and $|\widehat{\omega}_z| \simeq h_T H_z$, and to which one can apply Lemma 7.5 so that we have (7.7) and (7.8) with ω_z replaced by $\widehat{\omega}_z$. The desired assertions (7.7) and (7.8) for ω_z follow.

(ii) If $z \in \mathcal{N}^*_{\partial\Omega}$, then $z' \in \partial\Omega$ and the assertion is obvious. Otherwise, $z \in \mathcal{N}_{\rm ani}\backslash\mathcal{N}^*_{\partial\Omega}$, the edge $zz' \not\subset \partial\Omega$, and there is at most one such vertex z for each $z' \in \mathcal{N}_{\rm iso}$. Hence choose $\sigma_{z'} \subset (\gamma_{z'} \cap \partial\omega_z)$ with the same orientation as σ_z in the sense that their respective endpoints $\hat{z} \neq z$ and $\hat{z}' \neq z'$ satisfy $|\hat{z} - \hat{z}'| \lesssim h_z$; see Fig. 6. Now the proof of Lemma 7.2(ii) applies with $\omega_z \cup \omega_{z'}$ replaced by ω_z. □

Corollary 7.7 *Under condition $\mathcal{A}4$, for any triangle $T \in \mathcal{T}$,*

$$\|v - \Pi_h v\|_{2\,;T} \lesssim (H_T\,\mathrm{diam}\,\omega_T)^{1/2}\|\nabla v\|_{2\,;\omega_T}\,, \qquad (7.9)$$

$$|T|^{1/2}\|\nabla(v - \Pi_h v)\|_{2\,;T} \lesssim |\omega_T|^{1/2}\|\nabla v\|_{2\,;\omega_T}\,. \qquad (7.10)$$

Proof Combining Lemmas 7.2(i), 7.5 and 7.6(i), one concludes that (7.7) is valid for all $z \in \mathcal{N}$. This immediately implies that

$$\|\Pi_h v\|_{2\,;T} \lesssim \|v\|_{2\,;T} + (H_T\,\mathrm{diam}\,\omega_T)^{1/2}\|\nabla v\|_{2\,;\omega_T}.$$

Now, (7.9) is obtained as in the proof of Corollary 7.3. For the remaining assertion (7.10), it suffices to show that

$$|T|\,\|\nabla\Pi_h v\|_{\infty\,;T} \lesssim \|\nabla v\|_{1\,;\omega_T}\,. \qquad (7.11)$$

For the latter, if T has a pair of vertices $z, z' \in \mathcal{N}_{\rm ani}$, or $z \in \mathcal{N}_{\rm ani}$ and $z' \in \mathcal{N}_{\rm iso}$ with $|z - z'| \simeq h_z \simeq h_T$, then $|z - z'|^{-1}|\varPi_h v(z) - \varPi_h v(z')|$ is estimated using (7.3) or respectively its version described in Lemma 7.6(ii).

To prove (7.11) for other cases, the estimation of $|\varPi_h v(z) - \varPi_h v(z')|$ is reduced to using appropriate bounds for $|\varPi_h v(z) - \mathcal{A}_S v|$ and $|\varPi_h v(z') - \mathcal{A}_S v|$. In particular, if $z, z' \notin \mathcal{N}_{\rm ani}$, both such bounds follow from (7.8), and are combined with $h_T \lesssim |z - z'|$ to get (7.11). The only remaining case is $z \in \mathcal{N}_{\rm ani}$, $z' \notin \mathcal{N}_{\rm ani}$ such that $|z - z'| \simeq H_T$. Then the proof of Lemma 7.2(i) yields $|T||\varPi_h v(z) - \mathcal{A}_S v| \lesssim H_T \|\nabla v\|_{1\,;\omega_z}$, while (7.8) is used again for $|\varPi_h v(z') - \mathcal{A}_S|$, and (7.11) follows. □

Remark 7.3 (Quasi-interpolation operator $\varPi_h^*$) As in Remark 7.2, we slightly modify $\varPi_h$ for $z \notin \partial\Omega$ in two steps. First, we define the interpolation operator $\widetilde{\varPi}_h$ by replacing each edge σ_z that we used by its half $\widetilde{\sigma}_z$ originating at z (see Fig. 7). The above results remain valid, but one needs to be more careful in the proof of (7.11). In particular, if $|z - z'| \simeq H_T$, one may need to combine a version of (7.8) with $|T||\mathcal{A}_{\tilde{S}}\, v - \mathcal{A}_{S\backslash\tilde{S}}\, v| \lesssim H_T \|\nabla v\|_{1\,;T}$, where $\tilde{S}$ denotes any half of S (the latter bound is obtained by a triple application of Lemma 7.1 within T). Otherwise, if $|z - z'| \simeq h_T < c_0 H_T$, and one of z, z', e.g. z, is in $\mathcal{N}_{\rm iso}$, when obtaining a version of (7.8), one may need to use $|\mathcal{A}_{\widetilde{\sigma}_z}\, v - \mathcal{A}_{\sigma_z\backslash\widetilde{\sigma}_z}\, v| \lesssim H_{T'}^{-1}\|\nabla v\|_{1\,;T'}$, where σ_z is an edge of $T' \subset \omega_z$ such that $h_{T'} \simeq H_{T'} \simeq H_T$.

The final step in constructing $\varPi_h^*$ is to replace each half-edge $\widetilde{\sigma}_z$ by a parallelogram σ_z^* formed by $\widetilde{\sigma}_z$ and the adjacent half of an adjacent edge from γ_z (see Fig.7). Note that one can always choose σ_z as above, but so that additionally $|\sigma_z^*| \simeq |\omega_z| \simeq h_z H_z$ and the minimal interior angle in σ_z^* is uniformly bounded. Now $\varPi_h^* v(z) := \mathcal{A}_{\sigma_z^*}\, v$, so we again get (7.5). Now, the first bound in (7.5) immediately implies $\|\varPi_h^* v\|_{2\,;T} \lesssim \|v\|_{2\,;\omega_T}$, while the second yields (7.2), (7.3), (7.7) and (7.8) for $\varPi_h^*$, hence Corollary 7.7 also holds true for $\varPi_h^*$. Combining the above observations, one gets, with $j = 0, 1$,

$$\|v - \varPi_h^* v\|_{2\,;T} \lesssim (H_T \operatorname{diam}\omega_T)^{j/2} |v|_{j,2\,;\omega_T}, \quad \|\nabla(v - \varPi_h^* v)\|_{2\,;T} \lesssim \frac{|\omega_T|^{1/2}}{|T|^{1/2}} \|\nabla v\|_{2\,;\omega_T}. \tag{7.12}$$

Theorem 7.8 *(i) Under condition $\mathcal{A}4$, there exists G_h such that Θ', $\widetilde{\Theta}'$, and $\widetilde{\Theta}$ defined by* (6.3), (6.5) *satisfy $\Theta' + \widetilde{\Theta}' + \widetilde{\Theta} \lesssim 1$. (ii) If, in addition, $|T| \simeq |\omega_T|$ for all $T \in \mathcal{T}$, then $\Theta + \Theta' + \widetilde{\Theta}' + \widetilde{\Theta} \lesssim 1$, where Θ is from* (5.6).

Proof Set $G_h := \varPi_h^* G$ and employ (7.12). (i) Now, $\Theta' \leqslant \Theta' \leqslant \widetilde{\Theta} \lesssim |\!|\!| G |\!|\!|_{\varepsilon\,;\Omega} \lesssim 1$. (If $H_T \operatorname{diam}\omega_T \lesssim \varepsilon^2$ for all $T \in \mathcal{T}$, one can instead let $G_h := \varPi_h G$ and employ Corollary 7.7.) (ii) The additional assumption $|T| \simeq |\omega_T|$ for all $T \in \mathcal{T}$ implies $H_z \gtrsim H_T \simeq \operatorname{diam}\omega_T$ whenever $T \subset \omega_z$ (for the latter relation, see the proof of Theorem 7.8* below). So one additionally gets $\Theta \lesssim 1$. □

Remark 7.4 (Mesh assumption in Theorem 7.8(ii)) The assumption $|T| \simeq |\omega_T|$ for all $T \in \mathcal{T}$ is slightly restrictive (for example, it is not satisfied by the mesh pictured in Fig. 1), but still allows for meaningful anisotropic meshes.

Essentially, this assumption requires the transition between coarse and fine anisotropic parts of the mesh to be slightly more gradual than expected from interpolation error bounds for typical layer solutions of (1.1). Consequently, the number of degrees of freedom may increase to $\simeq H_{\max}^{-2} + \ln^2(1+\varepsilon^{-1}H_{\max})$ compared to the optimal expected value $\simeq H_{\max}^{-2}$, where $H_{\max} := \max_{T\in\mathcal{T}} H_T$.

It remains to estimate Θ of (5.6) (which appears in the jump residual estimator (6.1)) when $|T| \simeq |\omega_T|$ does not hold for all $T \in \mathcal{T}$. Intuitively, one expects that $\Theta \lesssim 1$ for some $G_h \in S_h$. However, it is not a simple task to construct such G_h on a reasonably general anisotropic mesh. Instead, in Lemma 6.3, we give a version of (6.1), which involves Θ^*, and estimate the latter in the following version of Theorem 7.8.

Theorem 7.8* *Under condition $\mathcal{A}4$, there exists G_h such that Θ^*, Θ', $\widetilde{\Theta}'$, and $\widetilde{\Theta}$ defined by* (6.3), (6.5), (6.6) *satisfy* $\Theta^* + \Theta' + \widetilde{\Theta}' + \widetilde{\Theta} \lesssim 1$.

Proof Let $G_h := \Pi_h^* G$ and employ (7.12) to show that $\Theta^* \lesssim \|\!|G|\!\|_{\varepsilon\,;\Omega} \lesssim 1$. We also use

$$\frac{|T|}{|\omega_T|}\,/\!\!/ g /\!\!/_{2\,;T} \lesssim \frac{1}{H_T \operatorname{diam}\omega_T}\|g\|_{2\,;T}^2 + \frac{|T|}{|\omega_T|}\|\nabla g\|_{2\,;T}^2\,.$$

The latter follows from (3.1) combined with $|T|\,|\omega_T|^{-1} \lesssim H_T\,(\operatorname{diam}\omega_T)^{-1}$ (which is obtained from a similar observation $|T|\,|\omega_z|^{-1} \lesssim H_T\,H_z^{-1}$ valid for any vertex z of T). In view of Theorem 7.8, we are done. □

8 Numerical results

We test the estimators of Theorems 6.1 and 6.2, using a simple version of (1.1) with $\Omega = (0,1)^2$ and $f = u - F(x,y)$, where F is such that the unique exact solution $u = 4y\,(1-y)\,[\cos(\pi x/2) - (e^{-x/\varepsilon} - e^{-1/\varepsilon})/(1-e^{-x/\varepsilon})]$ (the latter exhibits a sharp boundary layer at $x = 0$). We consider one a-priori-chosen layer-adapted mesh, as on Fig. 3 (right), which is obtained by drawing diagonals from the tensor product of the Bakhvalov grid $\{\chi(\frac{i}{N})\}_{i=1}^N$ in the x-direction [3] and a uniform grid $\{\frac{j}{M}\}_{j=0}^M$ in the y-direction with $M = \frac{1}{2}N$. The continuous mesh-generating function $\chi(t) = t$ if $\varepsilon > \frac{1}{6}$; otherwise, $\chi(t) = 3\varepsilon\ln\frac{1}{1-2t}$ for $t \in (0, \frac{1}{2} - 3\varepsilon)$ and is linear elsewhere subject to $\chi(1) = 1$.

Theorems 6.1 and 6.2 give the error estimator

$$\|\!|u_h - u|\!\|_{\varepsilon\,;\Omega} \lesssim \mathcal{E} := \Bigl\{\mathcal{E}_{(6.1)}^2 + \mathcal{E}_{(6.4)}^2 + \mathcal{E}_{(4.5)}^2\Bigr\}^{1/2}, \tag{8.1}$$

where $\mathcal{E}_{(\cdot.\cdot)}$ denotes the right-hand side of $(\cdot.\cdot)$ (e.g., $\mathcal{E}_{(4.5)} = \|f_h - f_h^I\|_{2\,;\Omega}$). In $\mathcal{E}_{(4.5)}$ and $\mathcal{E}_{(6.4)}$, we respectively use $\mathcal{T}_0 := \varnothing$ and $\varrho := 0$. Also, all Θ-factors in these estimators are set equal to 1. (The latter is due to all Θ-factors being $\lesssim 1$ on our mesh. If $\varepsilon \lesssim N^{-1}$, one gets $\varepsilon \lesssim H_z$ for all $z \in \mathcal{N}$, so one uses Remark 7.1 combined with Remark 6.4. Otherwise, a calculation shows that $|T| \simeq |\omega_T|$ for all $T \in \mathcal{T}$, so one instead employs Theorem 7.8(ii).) When computing the estimators, we replaced H_z from (2.1) by $\max_{T\subset\omega_z} H_T \simeq H_z$, and quantities

Table 1 Errors $\{\varepsilon^2\|\nabla u_h - (\nabla u)^I\|^2_{2\,;\Omega} + \|u_h - u^I\|^2_{2\,;\Omega}\}^{1/2}$, estimators $\{\mathcal{E}^2_{(6.1)} + \mathcal{E}^2_{(6.4)}\}^{1/2}$, and ratios of $\mathcal{E}_{(6.1)}$ to $\mathcal{E}_{(6.4)}$.

N	$\varepsilon = 1$	$\varepsilon = 2^{-5}$	$\varepsilon = 2^{-10}$	$\varepsilon = 2^{-15}$	$\varepsilon = 2^{-20}$	$\varepsilon = 2^{-25}$	$\varepsilon = 2^{-30}$
	Errors (odd rows) & Computational Rates (even rows)						
64	3.202e-2	5.081e-3	7.993e-4	1.408e-4	2.489e-5	4.399e-6	7.777e-7
	1.00	0.99	1.00	1.00	1.00	1.00	1.00
128	1.602e-2	2.564e-3	3.991e-4	7.028e-5	1.242e-5	2.196e-6	3.882e-7
	1.00	0.99	1.00	1.00	1.00	1.00	1.00
256	8.011e-3	1.289e-3	1.997e-4	3.511e-5	6.207e-6	1.097e-6	1.940e-7
	1.00	1.00	0.99	1.00	1.00	1.00	1.00
512	4.006e-3	6.461e-4	1.002e-4	1.755e-5	3.102e-6	5.484e-7	9.695e-8
	Estimators (odd rows) & Effectivity Indices (even rows)						
64	1.041e-1	2.102e-2	4.129e-3	7.393e-4	1.308e-4	2.311e-5	4.086e-6
	3.25	4.14	5.17	5.25	5.25	5.25	5.25
128	5.147e-2	1.051e-2	2.050e-3	3.711e-4	6.566e-5	1.161e-5	2.052e-6
	3.21	4.10	5.14	5.28	5.29	5.29	5.29
256	2.559e-2	5.269e-3	1.006e-3	1.858e-4	3.290e-5	5.817e-6	1.028e-6
	3.19	4.09	5.04	5.29	5.30	5.30	5.30
512	1.276e-2	2.645e-3	4.883e-4	9.287e-5	1.647e-5	2.912e-6	5.147e-7
	3.19	4.09	4.87	5.29	5.31	5.31	5.31
	Ratios $\mathcal{E}_{(6.1)}/\mathcal{E}_{(6.4)}$						
64	1.66	1.31	0.82	0.81	0.81	0.81	0.81
128	1.67	1.48	0.83	0.80	0.80	0.80	0.80
256	1.67	1.57	0.86	0.80	0.79	0.79	0.79
512	1.66	1.60	0.93	0.79	0.79	0.79	0.79

Table 2 Error component $\|u-u^I\|_{2\,;\Omega}$ from (8.2), estimator $\mathcal{E}_{(4.5)}$, and its effectivity indices.

N	$\varepsilon = 1$	$\varepsilon = 2^{-5}$	$\varepsilon = 2^{-10}$	$\varepsilon = 2^{-15}$	$\varepsilon = 2^{-20}$	$\varepsilon = 2^{-25}$	$\varepsilon = 2^{-30}$
	Errors $\|u - u^I\|_{2\,;\Omega}$ (odd rows) & Computational Rates (even rows)						
64	2.242e-4	6.120e-4	6.496e-4	6.567e-4	6.571e-4	6.571e-4	6.571e-4
	2.00	2.00	2.01	2.01	2.01	2.01	2.01
128	5.607e-5	1.525e-4	1.617e-4	1.635e-4	1.636e-4	1.636e-4	1.636e-4
	2.00	2.00	2.00	2.00	2.00	2.00	2.00
256	1.402e-5	3.807e-5	4.036e-5	4.077e-5	4.080e-5	4.080e-5	4.080e-5
	2.00	2.00	2.00	2.00	2.00	2.00	2.00
512	3.505e-6	9.510e-6	1.008e-5	1.018e-5	1.019e-5	1.019e-5	1.019e-5
	$\mathcal{E}_{(4.5)}$ (odd rows) & Effectivity Indices $\mathcal{E}_{(4.5)}/\|u - u^I\|_{2\,;\Omega}$ (even rows)						
64	2.661e-3	5.762e-4	6.484e-4	6.567e-4	6.571e-4	6.571e-4	6.571e-4
	11.87	0.94	1.00	1.00	1.00	1.00	1.00
128	6.671e-4	1.435e-4	1.614e-4	1.634e-4	1.636e-4	1.636e-4	1.636e-4
	11.90	0.94	1.00	1.00	1.00	1.00	1.00
256	1.670e-4	3.579e-5	4.029e-5	4.077e-5	4.079e-5	4.080e-5	4.080e-5
	11.91	0.94	1.00	1.00	1.00	1.00	1.00
512	4.178e-5	8.938e-6	1.006e-5	1.018e-5	1.019e-5	1.019e-5	1.019e-5
	11.92	0.94	1.00	1.00	1.00	1.00	1.00

Table 3 Estimators $\mathcal{E}_{(6.2)}$ and effectivity indices (computed as the ratio of $\mathcal{E}_{(6.2)}$ to the right-hand side of (8.2)).

N	$\varepsilon=1$	$\varepsilon=2^{-5}$	$\varepsilon=2^{-10}$	$\varepsilon=2^{-15}$	$\varepsilon=2^{-20}$	$\varepsilon=2^{-25}$	$\varepsilon=2^{-30}$
	Estimators (odd rows)(odd rows) & Effectivity Indices (even rows)						
64	1.268e-1	4.661e-2	1.565e-2	2.850e-3	5.043e-4	8.916e-5	1.576e-5
	3.93	8.19	10.80	3.57	0.74	0.13	0.02
128	6.449e-2	3.111e-2	1.519e-2	2.847e-3	5.043e-4	8.915e-5	1.576e-5
	4.01	11.45	27.08	12.18	2.87	0.54	0.10
256	3.253e-2	1.868e-2	1.435e-2	2.842e-3	5.042e-4	8.915e-5	1.576e-5
	4.05	14.08	59.76	37.45	10.73	2.13	0.38
512	1.634e-2	1.039e-2	1.291e-2	2.831e-3	5.042e-4	8.915e-5	1.576e-5
	4.07	15.84	117.06	102.07	37.94	8.30	1.53

Table 4 Error components (see (8.2)) and computational rates.

N	$\varepsilon=1$	$\varepsilon=2^{-5}$	$\varepsilon=2^{-10}$	$\varepsilon=2^{-15}$	$\varepsilon=2^{-20}$	$\varepsilon=2^{-25}$	$\varepsilon=2^{-30}$
	Errors $\varepsilon^{1/2}\Vert\nabla u_h-(\nabla u)^I\Vert_{2\,;\Omega}$ (odd rows) & Computational Rates (even rows)						
64	3.202e-2	2.872e-2	2.532e-2	2.523e-2	2.522e-2	2.522e-2	2.522e-2
	1.00	0.99	1.00	1.00	1.00	1.00	1.00
128	1.602e-2	1.450e-2	1.265e-2	1.260e-2	1.260e-2	1.260e-2	1.260e-2
	1.00	0.99	1.00	1.00	1.00	1.00	1.00
256	8.011e-3	7.290e-3	6.336e-3	6.297e-3	6.297e-3	6.297e-3	6.297e-3
	1.00	1.00	0.99	1.00	1.00	1.00	1.00
512	4.006e-3	3.655e-3	3.186e-3	3.148e-3	3.148e-3	3.148e-3	3.148e-3
	Errors $\varepsilon^{-1/2}\Vert u_h-u^I\Vert_{2\,;\Omega}$ (odd rows) & Computational Rates (even rows)						
64	1.714e-4	1.173e-3	3.612e-3	3.624e-3	3.624e-3	3.624e-3	3.624e-3
	2.00	2.32	1.06	1.04	1.04	1.04	1.04
128	4.290e-5	2.351e-4	1.736e-3	1.757e-3	1.757e-3	1.757e-3	1.757e-3
	2.00	2.47	1.07	1.02	1.02	1.02	1.02
256	1.073e-5	4.254e-5	8.267e-4	8.643e-4	8.643e-4	8.643e-4	8.643e-4
	2.00	2.44	1.17	1.01	1.01	1.01	1.01
512	2.682e-6	7.826e-6	3.667e-4	4.286e-4	4.287e-4	4.287e-4	4.287e-4

of type $\min\{1, a\,\varepsilon^{-1}\}$ by their smoother analogues $\frac{a}{\varepsilon+a}$ (e.g., λ'_z was replaced by $\frac{H_z}{\varepsilon+H_z}$). We also replaced f_h and u by their quadratic Lagrange interpolants.

For the error in the energy norm, it is reasonable to assume that

$$|||u_h-u|||_{\varepsilon\,;\Omega} \simeq \bigl\{\varepsilon^2\Vert\nabla u_h-(\nabla u)^I\Vert^2_{2\,;\Omega}+\Vert u_h-u^I\Vert^2_{2\,;\Omega}\bigr\}^{1/2}+\Vert u-u^I\Vert_{2\,;\Omega}\,. \quad (8.2)$$

This error decomposition is useful as the two error components in (8.2) exhibit somewhat different behaviour in our experiments, as $\simeq \varepsilon^{1/2}N^{-1}$ and $\simeq N^{-2}$, respectively (compare the upper parts of Tables 1 and 2; see also Table 4 and Remark 8.1). Furthermore, one can identify that $\mathcal{E}_{(6.1)} \simeq \mathcal{E}_{(6.4)}$ essentially estimate the first error component in (8.2) (see Table 1), while $\mathcal{E}_{(4.5)}$ provides an estimator for the remaining error component in (8.2) (see Table 2). For the estimators in Tables 1 and 2, the effectivity indices (computed as the ratio of the estimator to the error) do not exceed 11.92. Table 1 also gives the ratios of $\mathcal{E}_{(6.1)}$ to $\mathcal{E}_{(6.4)}$, which remain between 0.79 and 1.67.

Next, we look at another estimator given by Theorem 6.1, which is a version of (8.1) with $\mathcal{E}_{(6.4)}$ replaced by a less sharp interior-residual estimator $\mathcal{E}_{(6.2)}$.

Table 3 shows that with the inclusion of $\mathcal{E}_{(6.2)}$, the effectivity of the overall estimator considerably deteriorates for certain ε and N.

Remark 8.1 (Energy norm vs the L_2 norm) For $\varepsilon \ll 1$, it is demonstrated by Table 4 that $\|u_h - u^I\|_{2\,;\Omega} \simeq \varepsilon\|\nabla u_h - (\nabla u)^I\|_{2\,;\Omega} \simeq \varepsilon^{1/2}N^{-1}$ (which is consistent with [9, Table 4]), while the upper part of Table 2 shows that $\|u - u^I\|_{2\,;\Omega} \simeq N^{-2}$. These results suggest that for $\varepsilon \ll 1$, one cannot expect $\|u_h - u\|_{2\,;\Omega} \ll |||u_h - u|||_{\varepsilon\,;\Omega}$, and, in fact, both $\|u_h - u\|_{2\,;\Omega}$ and $\|u_h - u^I\|_{2\,;\Omega}$ may be as large as $|||u_h - u|||_{\varepsilon\,;\Omega}$.

For the considered ranges of ε and N, the aspect ratios of the mesh elements take values between 2 and 3.6e+8. Considering these variations, the estimator $\mathcal{E}$ of (8.1) performs quite well and its effectivity indices stabilize as $\varepsilon \to 0$. The less sharp alternative $\mathcal{E}_{(6.2)}$ to $\mathcal{E}_{(6.4)}$ in (8.1) is unsatisfactory for some ε and N. A more comprehensive numerical study of the proposed estimators certainly needs to be conducted, and will be presented elsewhere.

A Proof of Lemma 6.3

Remark A.1 An inspection of the proofs of Lemma 3.1 and Corollary 3.2 reveals that one can get a sharper version of (3.3) (which will help to prove Lemma 6.3):

$$\frac{h_z}{H_z}\|v\|_{1\,;\gamma_z\cup\bar S_z} \lesssim \Big\{h_z q_z \sum_{T\subset\omega_z} \frac{|T|^{1/2}}{|\omega_z|^{1/2}}\,\|v\|_{2\,;T}\, |||v|||_{2\,;T}\Big\}^{1/2}. \tag{A.1}$$

To show (A.1), we first get an auxiliary bound

$$\frac{h_z}{H_z}\|v\|_{1\,;\gamma_z\cup\bar S_z} \lesssim \Big\{h_z\,\frac{|\mathring\omega_z|^{1/2}}{|\omega_z|^{1/2}}\|v\|_{2\,;\mathring\omega_z}\,|||v|||_{2\,;\mathring\omega_z} + h_z\,\|v\|_{2\,;\omega_z\backslash\mathring\omega_z}\,|||v|||_{2\,;\omega_z\backslash\mathring\omega_z}\Big\}^{1/2}, \tag{A.2}$$

where $\mathring\omega_z$ is defined in $\mathcal{A}4$. We only consider $z \notin \mathcal{N}_{\rm ani}$ as $\mathring\omega_z = \varnothing$ otherwise. Let $\widehat\gamma_z := \{S \in \gamma_z : |S| \ll H_z\} = \gamma_z \cap (\mathring\omega_z \cup \partial\mathring\omega_z)$ (note that $\widehat\gamma_z = \mathring\gamma_z$ for $z \in \mathcal{N}_{\rm s.ani}$). Now, employ (3.3) for the patches $\omega_z\backslash\mathring\omega_z$ and $\mathring\omega_z$ separately. Then $\|v\|_{1\,;(\gamma_z\backslash\widehat\gamma_z)\cup\bar S_z}$ is bounded as in (3.3), only with ω_z replaced by $\omega_z\backslash\mathring\omega_z$. Also, $\|v\|_{1\,;\widehat\gamma_z}$ is bounded as in (3.3) but with ω_z replaced by $\mathring\omega_z$, and h_z in the right-hand side replaced by $\operatorname{diam}\mathring\omega_z$. Finally, note that $(h_z/H_z)^2 \operatorname{diam}\mathring\omega_z$ is $\lesssim h_z(h_z/H_z)^{1/2}$ for $z \in \mathcal{N}_{\rm s.ani}$, and $\simeq h_z(\operatorname{diam}\mathring\omega_z/H_z)$ for $z \in \mathcal{N}_{\rm iso}$, so it is $\lesssim h_z(|\mathring\omega_z|/|\omega_z|)^{1/2}$ for any $z \in \mathcal{N}$. Combining these observations yields (A.2). Now we have (A.2), the assertion (A.1) follows as $|T| \simeq |\mathring\omega_z|$ for $T \subset \mathring\omega_z$, while $H_T \simeq H_z$ implies $q_z|T|^{1/2}|\omega_z|^{-1/2} \gtrsim 1$ for $T \subset \omega_z\backslash\mathring\omega_z$.

Proof of Lemma 6.3. We make minor modifications to the proof of (6.1) combined with Remark 6.6 as follows. In (4.4), we use $\{\bar g_z^*\}$ instead of $\{\bar g_z\}$. As $\bar g_z$ and $\bar g_z^*$ are defined slightly differently depending on whether $z \in \mathcal{N}_{\rm paths}$ or not, we consider the two cases separately.

(i) First, consider $z \in \mathcal{N}_{\rm paths}$. Similarly to the proof of (6.1), we get $|I_z| \lesssim (\theta_z \lambda_z^*)^{1/2} \|\varepsilon J_z\|_{\infty\,;\gamma_z}$, only the definition of θ_z is a more complicated version of (6.18), for which one now gets $\sum_{z\in\mathcal{N}_{\rm paths}} \theta_z \lesssim \Theta^*$. If $\lambda_z^* = \lambda_z^{(1)} := |\widehat{\omega}_z'|$, define $\bar g_z$ and $\bar g_z^*$ by (6.7), (6.16) (with $\widetilde{\omega}_z \cup \omega_z = \omega_z^* \subset \widehat{\omega}_z$), and

$$\theta_z := (\lambda_z^*)^{-1} \varepsilon^2 /\!\!/ g /\!\!/^2_{1\,;\omega_z^*} \lesssim \varepsilon^2 \sum_{T\subset\widehat{\omega}_z} |\omega_T|^{-1} /\!\!/ g /\!\!/^2_{1\,;T} \,. \tag{A.3}$$

Here we used (3.1) and also $|\widehat{\omega}_z'| \geqslant |\omega_T|$ (the latter, in view of $\widehat{\omega}_z' \supset \omega_T$ for any $T \subset \widehat{\omega}_z$). Otherwise, i.e. if $\lambda_z^* < \lambda_z^{(1)}$, set

$$\theta_z := (\lambda_z^*)^{-1} \varepsilon^2 H_z^2\, h_z^{-1} q_z \sum_{T\subset\omega_z} \frac{|T|^{1/2}}{|\omega_z|^{1/2}} \|g\|_{2\,;T} /\!\!/ g /\!\!/_{2\,;T} + (\lambda_z^*)^{-1} \varepsilon^2 H_z\, h_z^{-1} \|g\|^2_{2\,;\widetilde{\omega}_z} \,. \tag{A.4}$$

Note that one arrives at this definition of θ_z by replacing the $\{\cdots\}^{1/2}$ term in (5.13) by the $\{\cdots\}^{1/2}$ term from (A.1) (with $v := g$). Consequently, the term $H_z^2\, h_z^{-1} \|g\|_{2\,;\omega_z} /\!\!/ g /\!\!/_{2\,;\omega_z}$ in (6.18) becomes replaced by $H_z h_z^{-1} \{\cdots\}^{1/2}$ squared. Next, to estimate the first term in (A.4), use $|\omega_z'| \geqslant |\omega_T|$ (in view of $\omega_z' \supset \omega_T$). For the second term, note that $q_z \geqslant 1$ and $|\omega_z'|^{1/2} |\omega_z|^{-1/2} \geqslant 1$ implies $(\lambda_z^*)^{-1} \varepsilon^2 H_z\, h_z^{-1} \leqslant \varepsilon H_z^{-1} \lesssim 1 + \varepsilon^2 (H_T \operatorname{diam} \omega_T)^{-1}$ as, by $\mathcal{A}3$, $H_z \simeq H_T \simeq \operatorname{diam} \omega_T$ for any T within $\widetilde{\omega}_z$. Thus, we get

$$\theta_z \lesssim \varepsilon \sum_{T\subset\omega_z} \frac{|T|^{1/2}}{|\omega_T|^{1/2}} \|g\|_{2\,;T} /\!\!/ g /\!\!/_{2\,;T} + \sum_{T\cap\widetilde{\omega}_z\neq\varnothing} \Big\{ 1 + \frac{\varepsilon^2}{H_T \operatorname{diam} \omega_T} \Big\} \|g\|^2_{2\,;T} \,. \tag{A.5}$$

Now, indeed, $\sum_{z\in\mathcal{N}_{\rm paths}} \theta_z \lesssim \Theta^*$ follows from (A.3) and (A.5).

(ii) Now consider $z \notin \mathcal{N}_{\rm paths}$. Using the notation of §6.3, the nodes $z \in \mathcal{N}_i^{(1)} \cup \mathcal{N}_i^{(k)}$ can be treated as in Case (i) (as $\bar g_z$ and $\bar g_z^*$ are defined as in Case (i), or equal 0). So we focus on $z \in \mathcal{N}_i \backslash [\mathcal{N}_i^{(1)} \cup \mathcal{N}_i^{(k)}]$. Note that now $\widetilde{\omega}_z \subset \omega_z^* \cap \Omega_i^{\rm ani}$, where $\Omega_i^{\rm ani} = \{T \subset \Omega_i : h_T \ll H_T\}$ (see the final part of the proof of (6.4) in §6.4). If $\lambda_z^* = \lambda_z^{(1)}$, we again use (A.3), only with ω_z^* replaced by $\omega_z \cup (\omega_z^* \cap \Omega_i^{\rm ani})$. Now for the second relation in (A.3), we again use $|\widehat{\omega}_z'| \geqslant |\omega_T|$ (which follows from $\widehat{\omega}_z' \supset \omega_T$ for any $T \subset \omega_z$, and from $|\widehat{\omega}_z'| \geqslant |\omega_z| \simeq |T| \simeq |\omega_T|$ for any $T \subset \omega_z^* \cap \Omega_i^{\rm ani}$, by Remark 6.7). For $\lambda_z^* < \lambda_z^{(1)}$, we again get (A.4) and then (A.5). So $\sum_{z\notin\mathcal{N}_{\rm paths}} \theta_z \lesssim \Theta^*$. □

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