Chapter 4

# Convergence Theory of Moving Grid Methods

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## Contents

## 4.1 Introduction

The topic of the present chapter is convergence analysis of moving mesh methods applied to *singularly perturbed* differential equations. Such equations arise in semiconductor device modelling, ion transport across biological membranes, pollution dispersion in aqueous media, financial modelling, population dynamics, and many other applications. Solutions to these equations frequently exhibit sharp boundary and interior layers, which are narrow regions where solutions change rapidly [30, 31, 35, 39]. To solve such problems numerically in a reliable and efficient way, one has to use locally refined meshes that are fine in layer regions and standard outside. Hence singularly perturbed problems present an important challenge for adaptive mesh techniques, which are aimed to detect accurate locations and widths of layers. Furthermore, it is crucial that adaptive methods applied to singularly perturbed problems are *robust* with respect to changes in the singular perturbation parameter(s).

The convergence theory that we present here is neither comprehensive nor general. No attempt is made to give an overview of all current activities in the area. Instead we focus on a simple 1d convection-dominated convection-diffusion problem, for which we present maximum norm a posteriori error estimates and then a full analysis of one robust adaptive method. We mainly follow [20,25] trying to make the presentation self-contained and accessible to young researchers and students. Some new results, such as Theorem 4.12, are also included.

These notes present an extended version of lectures given at Peking University in summer 2005. I wish to express my gratitude to Professor Jinchao Xu for his kind hospitality.

## 4.2 Maximum norm a posteriori error estimates for a 1d singularly perturbed convection-diffusion equation

In this section we introduce and discuss our model convection-diffusion problem. Then for various discretizations of this problem we present *maximum norm* error bounds of two types:

- *A priori error estimates*, in which the error is estimated in terms of certain derivatives of the exact solution and the mesh. Such estimates are useful if a priori information about the exact solution and its derivatives is available a priori. Then combining this information with a sharp a priori error estimate, one can construct a suitable layer-adapted mesh. Unfortunately this is rarely the case in applications.

- *A posteriori error estimates*, in which the error is estimated in terms of the mesh and the computed solution, which will be available in the computational process. These are the main subject of the present section. Such estimates can be used by mesh movement/mesh refinement algorithms to construct appropriate layer-adapted meshes starting from an initial unsophisticated mesh; see, e.g., §4.3.

The present §4.2 is heavily based on [20]. But we sharpen some results from [20] and also include completely new results.

### 4.2.1 Model convection-diffusion problem

Consider the simplest singularly perturbed convection-diffusion problem with constant coefficients

$$Tu := -\varepsilon u'' - u' = f(x) \quad \text{for } \ x \in (0,1), \quad u(0) = u(1) = 0, \tag{4.2.1}$$

where $\varepsilon \in (0,1]$ is a small parameter, and $f(x)$ is assumed to be sufficiently smooth.

**Lemma 4.2.1.** *There exists a unique solution $u(x)$ of* (4.2.1)*. It can be decomposed as a sum of smooth and layer functions:*

$$u(x) = y(x) + z(x) \quad \textit{for } \ x \in [0,1], \tag{4.2.2}$$

*where $Tz(x) = 0$ on $[0,1]$, and these functions and their derivatives can be bounded as follows:*

$$\Big|\frac{d^j y}{dx^j}(x)\Big| \le C \qquad \textit{and} \qquad \Big|\frac{d^j z}{dx^j}(x)\Big| \le C\varepsilon^{-j} e^{-x/\varepsilon} \tag{4.2.3}$$

*for $j = 0,1,2,3$, and $x \in [0,1]$.*

*Proof.* Existence of a unique solution follows from the maximum principle. The decomposition (4.2.2) can be obtained from an explicit formula of the solution $u(x)$ of (4.2.1); see [38, Lemma 3.2] for a proof of this result for a more general equation. ☐

Instead of obtaining (4.2.2), we shall derive a similar approximate decomposition $u(x) \approx \tilde{y}(x) + \tilde{z}(x)$, where

$$-\tilde{y}' = f(x) \quad \text{for } \ x \in (0,1), \qquad \tilde{y}(1) = 0,$$

and

$$-\varepsilon\tilde{z}'' - \tilde{z}' = 0 \quad \text{for } \ x \in (0,1), \qquad \tilde{z}(0) = -\tilde{y}(0), \quad \tilde{z}(1) = 0.$$

One can easily check that

$$\tilde{z}(x) = \tilde{z}(0)\,\frac{e^{-x/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}} \approx \tilde{z}(0)\,e^{-x/\varepsilon},$$

and $T[\tilde{u}+\tilde{z}] = -\varepsilon\tilde{y}''+f = \varepsilon f'+f = Tu+\mathcal{O}(\varepsilon)$. Combining this with $\tilde{y}(0)+\tilde{z}(0) = \tilde{y}(1)+\tilde{z}(1) = 0$ and using the maximum/comparison principle, we see that

$$u(x) = \tilde{u}(x) + \tilde{y}(x) + \mathcal{O}(\varepsilon),$$

while the derivatives of $\tilde{y}$ and $\tilde{z}$ satisfy the bounds for $y$ and $z$, respectively, in (4.2.3).

The representation $u \approx \tilde{y} + \tilde{z}$ that we constructed is called a zero-order asymptotic expansion in the asymptotic analysis literature. This particular expansion is very simple since our problem is very simple. In general, one can construct asymptotic expansions for much more complicated nonlinear problems whose solutions cannot be found explicitly; see, e.g., [31, 39]. The functions $y$ and $z$ (and similarly $\tilde{y}$ and $\tilde{z}$) are often referred to as smooth and layer components. The layer component here is a decaying exponential function $\approx Ce^{-x/\varepsilon}$ that changes rapidly at the boundary $x = 0$. Therefore the solution $u(x)$ is said to have an *exponential boundary layer* at $x = 0$ of width $\mathcal{O}(\varepsilon)$.

Since derivatives of the solutions of singularly perturbed problems can be quite large, as in (4.2.3) when $\varepsilon \ll 1$, the accuracy of classical numerical methods on standard unsophisticated meshes depends not only on the number of the mesh nodes $N$, but also on the small perturbation parameter [30, 35]. E.g., the errors for (4.2.1) are usually bounded by $C(\varepsilon N)^{-p}$ (where $p$ is the order of the method) and hence can be quite large and even $\mathcal{O}(1)$ if $\varepsilon \le CN^{-1}$. In contrast, if a suitable layer-adapted mesh is used, such as a Shishkin mesh or a Bakhvalov mesh, then most numerical methods become robust with respect to the small parameter [5, 27, 30, 35]. Ultimately we would like to design adaptive techniques that, without using any a priori information about the exact solution, will produce such appropriate layer-adapted meshes.

### 4.2.2 Numerical methods and notation

Let

$$\left\{ x_i \,\middle|\, 0 = x_0 < x_1 < \ldots < x_{N-1} < x_N = 1 \right\} \tag{4.2.4}$$

be an arbitrary nonuniform mesh on $[0, 1]$. On $\{x_i\}$ we discretize (4.2.1) as follows (most of the notation is collected at the end of the current §4.2.2):

$$T^N u_i^N = -\frac{A^N u_{i+1}^N - A^N u_i^N}{h_{i+1}} = f_i \quad \text{for} \quad i = 1, \ldots, N-1, \quad u_0^N = u_N^N = 0, \tag{4.2.5}$$

where $A^N$ is defined by

$$A^N v_i = \varepsilon D^- v_i + \sigma_i v_i + (1 - \sigma_i) v_{i-1} \quad \text{for } \ i = 1, \ldots, N, \qquad \sigma_i \in [0, 1]. \tag{4.2.6}$$

Here $A^N$ approximates the differential operator $A$ defined by

$$Av(x) = \varepsilon v' + v, \tag{4.2.7}$$

so that $-(Av(x))' = Tv(x)$. Note that such difference operators $A^N$ are at least first-order approximations of $A$. Setting $\sigma_i = 1$ or $\sigma_i = 0.5$ for $i = 1, \ldots, N$, turns (4.2.5), (4.2.6) into a first-order upwind scheme and a central difference scheme respectively.

In §4.2.4 for discretizations (4.2.5), (4.2.6) we give a *first-order* a priori error estimate (Theorem 4.2) and then its *a posteriori* analogue (Theorem 4.5):

$$\left\| u^N(x) - u(x) \right\|_\infty \le 2\Big[ \max_i \{ h_i | D^- u_i^N | \} + C \max_i h_i \Big].$$

Furthermore, these results also hold true for schemes (4.2.39) and (4.2.40); see Remark 4.8. Note that many finite element methods may be represented by (4.2.5), (4.2.39), or (4.2.40) and hence are also covered by our analysis.

In §4.2.5 we consider the following modification of (4.2.5) from [3]:

$$-\left[ A^N u_{i+1}^N - A^N u_i^N \right] = \int_{\tilde{x}_{i-1/2}}^{\tilde{x}_{i+1/2}} f(x)\, dx \ \text{ for } \ i = 1, \ldots, N-1, \ \ u_0^N = u_N^N = 0, \tag{4.2.8}$$

where

$$\tilde{x}_{i-1/2} := x_{i-1} + h_i\sigma_i, \tag{4.2.9}$$

while $A^N$ is again from (4.2.6) with

$$\sigma_i = \begin{cases} \frac{1}{2}, & \frac{\varepsilon}{h_i} \geq \frac{1}{2}, \\ 1 & \frac{\varepsilon}{h_i} < \frac{1}{2}. \end{cases} \tag{4.2.10a}$$

Alternatively, we can use

$$\sigma_i := \max\left\{\frac{1}{2}, 1 - \frac{\varepsilon}{h_i}\right\} \tag{4.2.10b}$$

or more generally

$$\max\left\{\frac{1}{2}, 1 - \frac{\varepsilon}{h_i}\right\} \leq \sigma_i \leq \frac{1}{2} + C\frac{h_i}{\varepsilon} \tag{4.2.10c}$$

for some constant positive $C$. Here (4.2.10a) and (4.2.10b) are particular cases of (4.2.10c). The lower bound in (4.2.10c) implies that our discrete operator is associated with an $M$-matrix. The upper bound in (4.2.10c) implies that $A^N v_i$ is a second-order approximation of $Av(\tilde{x}_{i-1/2})$. This follows from $A^N v_i - Av(\tilde{x}_{i-1/2}) = -\varepsilon(\sigma_i - 1/2)h_i v''(x_{i-1/2}) + \mathcal{O}(h_i^2)$. First, for (4.2.8),(4.2.6),(4.2.10) we shall recall a *second-order* a priori error estimate from [3] (Theorem 4.9) and then derive its *a posteriori* analogue (Theorem 4.12):

$$\left\|u^N(\cdot) - u(\cdot)\right\|_\infty \leq C\left[\max_i\left(\hbar_i^2\left|DD^- u_i^N\right|\right) + \max_i h_i^2\right].$$

<u>`Notation:`</u> Throughout §4.2 we use the notation

$$\begin{aligned} &D^- v_i = \frac{v_i - v_{i-1}}{h_i}, \quad D^+ v_i = \frac{v_{i+1} - v_i}{h_{i+1}}, \quad Dv_i = \frac{v_{i+1} - v_i}{\hbar_i}, \\ &h_i = x_i - x_{i-1}, \quad \hbar_i = (h_i + h_{i+1})/2, \quad h = \max_i h_i, \\ &I_i = (x_{i-1}, x_i), \quad x_{i-1/2} = x_i - h_i/2, \end{aligned}$$

and $w_i = w(x_i)$ for any continuous function $w(x)$.

By $u^N(x)$ we denote the piecewise linear interpolant of the computed solution $u_i^N$, i.e. the continuous function that is linear on each segment $\bar{I}_i$ and equal to $u_i^N$ at the meshnodes:

$$u^N(x_i) = u_i^N \quad \text{for} \quad i = 0, \ldots N. \tag{4.2.11}$$

In our estimates we use a certain negative norm and the standard maximum and $L_1$ norms given, respectively, by

$$\left\|v(\cdot)\right\|_* = \min_{V:V'=v}\left\|V(\cdot)\right\|_\infty, \quad \left\|v(\cdot)\right\|_\infty = \operatorname*{ess\ sup}_{x\in[0,1]}\left|v(x)\right|, \quad \left\|v(\cdot)\right\|_1 = \int_0^1 \left|v(x)\right| dx. \tag{4.2.12}$$

Note that since

$$\left\|v(\cdot)\right\|_* = \min_{C\in I\!R}\left\|\int_x^1 v(s)\, ds + C\right\|_\infty,$$

this negative norm is well-defined. On the other hand,

$$\|v(\cdot)\|_* = \sup_{u\in W_0^{1,1}(0,1)} \frac{< u, v >}{|u|_{1,1}}$$

(for the notation and relevant theorems see [1]), i.e. this norm is equivalent to the norm in the space $W^{-1,\infty} = \big(W_0^{1,1}\big)'$ and thus is stronger than the $W^{-1,2}$ norm. Note also that $\|f(\cdot)\|_* \le \|f(\cdot)\|_\infty/2$.

For any discrete function $v_i$ we also define the corresponding discrete norms by

$$\|v\|_{\infty,d} = \max_i |v_i|,$$

$$\|v\|_{*,d} = \min_{v_i : D^+V_i = v_i} \|V\|_{\infty,d} = \min_{C\in I\!R} \max_i \Big|\sum_{j=i}^{N} v_j h_{j+1} + C\Big| \le \|v\|_{\infty,d}/2. \tag{4.2.13}$$

Throughout §4.2, $C$, sometimes subscripted, denotes a generic positive constant that is independent of $\varepsilon$ and any mesh used. A subscripted $C$ is fixed in value throughout the section.

### 4.2.3 Stability properties of the differential operator

**Remark 4.1.** The right-hand side of (4.2.1) will often be considered in the form $f(x) = -F'(x)$, where $F(x)$ is at least a bounded piecewise continuous function. This implies that $f(x)$ can have singularities similar to the Dirac delta-function. With such a right-hand side, problem (4.2.1) is solved in the sense of distributions [15]. The explicit formula for the solution is given in the proof of the following lemma, and, to be precise, under our assumptions the weak solution $u(x) \in C^{0,1}\big([0,1]\big) \subset H^1(0,1) \subset C\big([0,1]\big)$, where $C^{0,1}\big([0,1]\big)$ is a standard Hölder space and $H^1(0,1)$ is a Sobolev space.

**Lemma 4.2.2.** *Let $f(x) = -F'(x)$, where $F(x)$ is a bounded piecewise continuous function. Then there exists a unique solution $u(x) \in C\big([0,1]\big)$ of* (4.2.1) *and*

$$\|u(\cdot)\|_\infty \le 2\|Tu(\cdot)\|_*. \tag{4.2.14}$$

*Proof.* Integrating equation (4.2.1) yields the unique continuous solution of (4.2.1):

$$u(x) = W(x) - W(1)\frac{V(x)}{V(1)}, \tag{4.2.15}$$

where

$$W(x) = \int_0^x \frac{F(s)}{\varepsilon} e^{-(x-s)/\varepsilon} ds, \qquad V(x) = \int_0^x \frac{1}{\varepsilon} e^{-(x-s)/\varepsilon} ds. \tag{4.2.16}$$

It is easy to verify that

$$0 \le V(x) \le 1, \qquad V(1) \ge 1 - e^{-1}, \qquad |W(x)| \le V(x)\|F(\cdot)\|_\infty, \tag{4.2.17}$$

which implies $|u(x)| \le 2V(x)\|F(\cdot)\|_\infty$, and furthermore,

$$\|u(x)\|_\infty \le 2\|F(\cdot)\|_\infty.$$

Now it suffices to recall (4.2.12) to complete the proof. □

**Corollary 4.2.1.** *Let $u(x)$ be a solution of* (4.2.1)*; then for any continuous function $v^N(x)$ that is linear on each interval $\bar{I}_i$ and satisfies $v^N(0) = v^N(1) = 0$ we have*

$$\|v^N(x) - u(x)\|_\infty \leq 2\|Tv^N(x) - f(x)\|_*.$$

Obviously, this also holds true for $v^N(x) := u^N(x)$ from (4.2.11), in which case the error is estimated in terms of the *residual in the negative $W^{-1,\infty}$ norm*.

Introduce the Green's function of the operator $T$ that for each $\xi \in (0,1)$ satisfies

$$\begin{aligned} T_x G(x,\xi) = -\varepsilon G_{xx}(x,\xi) - G_x(x,\xi) &= \delta(x-\xi), \qquad x \in (0,1), \\ G(0,\xi) = G(1,\xi) &= 0, \end{aligned} \tag{4.2.18}$$

where $\delta(\cdot)$ is the Dirac $\delta$-distribution. Then the solution $u$ of (4.2.1) is given by

$$u(x) = \int_0^1 G(x,\xi)\, f(\xi)\, d\xi. \tag{4.2.19}$$

Furthermore, $G(x,\xi)$ also solves the following problem for each $x \in (0,1)$:

$$\begin{aligned} T^*_\xi G(x,\xi) = -\varepsilon G_{\xi\xi}(x,\xi) + G_\xi(x,\xi) &= \delta(x-\xi), \qquad x \in (0,1), \\ G(x,0) = G(x,1) &= 0. \end{aligned} \tag{4.2.20}$$

Note that $G_{xx}(\cdot,\xi)$ for each fixed $\xi$ and $G_{\xi\xi}(x,\cdot)$ for each fixed $x$ are distributions.

**Corollary 4.2.2.** *For the Green's function $G(x,\xi)$ from* (4.2.18) *we have*

$$\max_x \|G_\xi(x,\cdot)\|_1 \leq 2, \qquad \max_x \int_0^1 |G_{\xi\xi}(x,\xi)|\, d\xi \leq 3/\varepsilon. \tag{4.2.21}$$

*Proof.* The first relation follows from (4.2.19) with $f = Tu$ combined with (4.2.14). To get the second desired relation, obtain $G_{\xi\xi}$ from (4.2.20) and note that $\int_0^1 \delta(x-\xi) = 1$. □

The following lemma is a slightly sharpened version of [20, Lemma 2.2]. Furthermore, we simplified the proof [20] imitating the argument from [21, Lemma 2.4].

**Lemma 4.2.3.** *Let $\{x_i\}$ be an arbitrary nonuniform mesh* (4.2.4) *and $f(x) = -F'(x)$, where*

$$F(x) = A_i(x - x_{i-1/2}) \quad \textit{for} \;\; x \in I_i, \quad i = 1, 2, \ldots, N.$$

*Then under the conditions of Lemma* 4.2.2 *we have*

$$\big\|u(\cdot)\big\|_\infty \leq 2 \max_{1\leq i\leq N} \Big\{\frac{h_i^2}{\varepsilon + h_i}|A_i|\Big\}.$$

*Proof.* Fix $x$ and denote $v(\xi) := G(x,\xi)$. Now, by (4.2.19) with $f = -F'$, we have

$$u(x) = \int_0^1 f(\xi)v(\xi)\,d\xi = \sum_{i=1}^{N} A_i \int_{x_{i-1}}^{x_i} (\xi - x_{i-1/2})\,v'(\xi)\,d\xi.$$

Note that

$$v'(\xi) = v'(x_{i-1}) + \int_{x_{i-1}}^{\xi} v''(s)\,ds$$

and $\int_{x_{i-1}}^{x_i} (\xi - x_{i-1/2})\,d\xi = 0$. Hence

$$u(x) = \sum_{i:\,h_i\le\varepsilon} A_i \int_{x_{i-1}}^{x_i} (\xi - x_{i-1/2}) \int_{x_{i-1}}^{\xi} v''(s)\,ds\,d\xi + \sum_{i:\,h_i>\varepsilon} A_i \int_{x_{i-1}}^{x_i} (\xi - x_{i-1/2})\,v'(\xi)\,d\xi.$$

Furthermore,

$$\begin{aligned} |u(x)| &\le \sum_{i:\,h_i\le\varepsilon} |A_i| \frac{h_i^2}{4} \int_{x_{i-1}}^{x_i} |v''(s)|\,ds + \sum_{i:\,h_i>\varepsilon} |A_i| \frac{h_i}{2} \int_{x_{i-1}}^{x_i} |v'(s)|\,ds \\ &\le \max_{i:\,h_i\le\varepsilon} \Big\{ \frac{|A_i|\,h_i^2}{4} \Big\} \int_0^1 |v''(\xi)|\,d\xi + \max_{i:\,h_i>\varepsilon} \Big\{ \frac{|A_i|\,h_i}{2} \Big\} \int_0^1 |v'(\xi)|\,d\xi. \end{aligned}$$

Combining this with the estimates $\int_0^1 |v'(\xi)|\,d\xi \le 2$ and $\int_0^1 |v''(\xi)|\,d\xi \le 3/\varepsilon$, which follow from (4.2.21), gives the desired result. □

### 4.2.4 First-order error estimates

In §4.2.4 we investigate the first-order discretizations (4.2.5),(4.2.6) of our convection-diffusion problem (4.2.1) and obtain certain a priori error estimates and then a posteriori error estimates that are robust with respect to the small parameter $\varepsilon$.

#### First-order a priori error estimate

**Theorem 4.2.** *Suppose that* $f(x) \in C^1\big([0,1]\big)$. *Let* $u(x)$ *be the solution of* (4.2.1) *and* $u_i^N$ *be the solution of* (4.2.5), (4.2.6) *with* $\sigma_i$ *satisfying*

$$1 - \frac{\varepsilon}{h_i} \le \sigma_i \le 1, \qquad i = 1,\ldots,N, \tag{4.2.22}$$

*on an arbitrary nonuniform mesh. Then*

$$\big\|u_i^N - u(x_i)\big\|_{\infty,d} \le 2 \max_{1\le i\le N} \Big[ \int_{x_{i-1}}^{x_i} \big|u'(x)\big|dx + C_1 h_i \Big], \tag{4.2.23a}$$

*with* $C_1 = \|f'\|_\infty/2 + \|f\|_\infty$. *Furthermore,*

$$\big\|u_i^N - u(x_i)\big\|_{\infty,d} \le C\Big[ \big\|\min\{\frac{h_i}{\varepsilon}, 1\}\, e^{-x_{i-1}/\varepsilon} \big\|_{\infty,d} + \max_{1\le i\le N} h_i \Big]. \tag{4.2.23b}$$

*Proof.* A similar theorem was derived in [3, Theorem 3]. We sketch this proof now mainly since it inspired the proof of the analogous a posteriori error estimate given by Theorem 4.5.

- We exploit the standard finite-difference theory approach [36] of applying the discrete operator $T^N$ to the error $u_i^N - u(x_i)$ and then eliminating $u^N$ from the resulting relation using our discrete equation (4.2.5).

This yields

$$T^N[u_i^N - u(x_i)] = f_i - T^N u_i = \; f_i \, + \, D^+ A^N u_i \, .$$

Here we also used $T^N = -D^+ A^N$. Introduce

$$F(x) = \int_x^1 f(x)\, dx \, , \tag{4.2.24}$$

so that $f = -F'$, and its discrete analogue

$$F_i^N = \sum_{j=i}^{N} f_j h_{j+1} \, , \tag{4.2.25}$$

so that $f_i = -D^+ F_i^N$. Now we have

$$T^N[u_i^N - u(x_i)] = D^+[A^N u_i - F_i^N].$$

Note that (4.2.1) implies $Au(x) - F(x) = \text{const}$, and in particular $Au(x_i) - F(x_i) = \text{const}$, which yields $D^+[Au(x_i) - F(x_i)] = 0$ and

$$\begin{aligned} &T^N[u_i^N - u(x_i)] = D^+ \eta_i^N \, , \\ &\eta_i^N := [A^N u_i - Au(x_i)] - [F_i^N - F(x_i)] \, . \end{aligned}$$

Now we recall (4.2.14) and note that under condition (4.2.22) the discrete operator $T^N$ enjoys a similar stability property [3, Theorem 3]. Hence

$$\big\|u_i^N - u(x_i)\big\|_{\infty,d} \le 2\big\|T^N[u_i^N - u(x_i)]\big\|_{\infty,*} \le 2\big\|\eta_i^N\big\|_{\infty,d} \, . \tag{4.2.26}$$

Here we also used (4.2.13). Finally, by (4.2.1), we get

$$\begin{aligned} \big|\eta_i^N\big| \;&\le\; \varepsilon\big|D^- u_i^N - u'(x_i)\big| + h\big\|f'\big\|_\infty/2 \le \varepsilon \int_{x_{i-1}}^{x_i} \big|u''(x)\big|\, dx + h\big\|f'\big\|_\infty/2 \\ &\le\; \int_{x_{i-1}}^{x_i} \big|u'(x)\big|\, dx + C_1 h \, , \end{aligned}$$

and hence (4.2.23a). Estimate (4.2.23b) follows, by (4.2.2),(4.2.3). □

**Remark 4.3.** Condition (4.2.22) ensures the discrete maximum/comparison principle for the discrete operator $T^N$ (4.2.5), (4.2.6), and also the stability property (4.2.26) which we used in the above proof. Note that (4.2.22) and hence (4.2.26) do not generally hold true for the central difference scheme with $\sigma_i = 0.5$.

**Remark 4.4.** Given (4.2.23b), one can construct suitable layer-adapted meshes, e.g., of Bakhvalov or Shishkin types, on which the errors in the discreet maximum norm do not exceed $CN^{-1}$ (or $CN^{-1}\ln N$ in the case of the Shishkin mesh) uniformly with respect to $\varepsilon$ [5, 27, 30, 35].

- The *Bakhvalov mesh* is defined by $x_i = x(i/N)$ for $i = 0, 1, \ldots, N$, where $x(\xi)$ is the continuous function satisfying

$$x(\xi) = \begin{cases} \begin{cases} \varepsilon\lambda \ln\left[b/(b-\xi)\right] & \text{for} \quad \xi \in [0,\theta] \\ 1 - d(1-\xi) & \text{for} \quad \xi \in [\theta, 1] \end{cases} & \text{if} \quad \varepsilon \le \bar{\varepsilon}_0 \\ \qquad\qquad \xi & \text{otherwise,} \end{cases} \tag{4.2.27}$$
$$b - \varepsilon\bar{C} < \theta < b - \varepsilon C_0, \qquad d = \left(1 - \varepsilon\lambda\ln\left[b/(b-\theta)\right]\right)/(1-\theta),$$

  for some constant $b \in (0,1)$, e.g., $b = 1/2$.

- The Shishkin mesh is a piecewise uniform mesh defined by

$$\{\, x_i \mid x_i = \begin{cases} ih & \text{for} \quad i = 0, \ldots, N/2, \\ x_{N/2} + (i - N/2)H & \text{for} \quad i = N/2+1, \ldots, N, \end{cases} \tag{4.2.28}$$
$$h = 2\tau/N, \quad H = 2(1-\tau)/N, \quad \tau = \min(\varepsilon\lambda \ln N, 1/2)\}.$$

To get first-order convergence (with, in the case of the Shishkin mesh, a logarithmic factor), it is crucial to choose the constant $\lambda \ge 1$ in both (4.2.27) and (4.2.28).

### First-order a posteriori error estimate

Next we shall obtain the first main result of §4.2, an a posteriori analogue of (4.2.23a).

**Theorem 4.5.** *Suppose that* $f(x) \in C^1\big([0,1]\big)$. *Let* $u(x)$ *be a solution of* (4.2.1), $u_i^N$ *be a solution of* (4.2.5), (4.2.6) *with arbitrary* $\sigma_i \in [0,1]$ *on an arbitrary nonuniform mesh, and* $u^N(x)$ *be its piecewise linear interpolant defined in (4.2.11). Then*

$$\begin{aligned} \big\|u^N(x) - u(x)\big\|_\infty &\le 2 \max_{1\le i\le N} \Big[\int_{x_{i-1}}^{x_i} \big|[u^N(x)]'\big| dx + C_2 h_i\Big] \\ &= 2 \max_{1\le i\le N} \Big[\big|u_i^N - u_{i-1}^N\big| + C_2 h_i\Big], \end{aligned} \tag{4.2.29}$$

*where* $C_2 = \|f'\|_\infty/2 + \|f\|_\infty$.

*Proof.* The argument will be similar to the one used in the proof of Theorem 4.2 with the following difference:

- Now we apply the differential operator $T$ to the error $u^N(x) - u(x)$ and then eliminate $u$ from the resulting relation using our original differential equation (4.2.1). Note that $Tu^N(x)$ includes second derivatives of a piecewise linear function, and hence should be understood in the sense of distributions.

By (4.2.1),(4.2.7), we get

$$T[u^N(x) - u(x)] = Tu^N(x) - f(x) = -\big(Au^N(x) - F(x) - a\big)', \tag{4.2.30}$$

where $F(x)$ is defined by (4.2.24), and $a$ is an arbitrary constant. Next note that, by (4.2.5),

$$A^N u_i^N - F_i^N = a \quad \text{for} \quad i = 1, \dots, N \tag{4.2.31}$$

for some constant $a$, which would be the constant that we shall use in (4.2.30). Hence we get

$$T[u^N(x) - u(x)] = \eta'(x), \tag{4.2.32}$$

where

$$\eta(x) = \eta_A(x) - \eta_F(x), \quad \eta_A(x) = A^N u_i^N - Au^N(x), \quad \eta_F(x) = F_i^N - F(x) \\ \text{for } \ x \in I_i, \quad i = 1, \dots, N. \tag{4.2.33}$$

Hence, by (4.2.14) of Lemma 4.2.2, it suffices to prove that

$$\left\| T[u^N(x) - u(x)] \right\|_* \leq \max_i \int_{x_{i-1}}^{x_i} \left| [u^N(x)]' \right| dx + C_2 h \,, \tag{4.2.34}$$

where, by (4.2.12),

$$\left\| T[u^N(x) - u(x)] \right\|_* \leq \left\| \eta(\cdot) \right\|_\infty \leq \left\| \eta_A(\cdot) \right\|_\infty + C_2 h. \tag{4.2.35}$$

Here we also used $|\eta_F(x)| \leq C_2 h$. Now we focus on $\eta_A(x)$ on a particular subinterval $I_i$. Since $\left[u^N(x)\right]' = D^- u_i^N$ for $x \in I_i$, we get

$$\begin{aligned} \eta_A(x) &= \left[\sigma_i u_i^N + (1-\sigma_i) u_{i-1}^N\right] - u^N(x) \\ &= \sigma_i \int_x^{x_i} [u^N(x)]' dx - (1-\sigma_i) \int_{x_{i-1}}^x [u^N(x)]' dx. \end{aligned} \tag{4.2.36}$$

Finally,

$$\sup_{x \in I_i} \left| \eta_A(x) \right| \leq \int_{x_{i-1}}^{x_i} \left| [u^N(x)]' \right| dx,$$

which, combined with (4.2.35), implies (4.2.34). □

**Corollary 4.2.3.** *Suppose that the conditions of Theorem* 4.5 *are satisfied and*

$$\left| u_i^N - u_{i-1}^N \right| + h_i \leq C N^{-1} \quad \textit{for} \quad i = 1, \dots, N. \tag{4.2.37}$$

*Then*

$$\left\| u^N(\cdot) - u(\cdot) \right\|_\infty \leq C N^{-1}. \tag{4.2.38}$$

**Remark 4.6.** Theorem 4.5 and Corollary 4.2.3 hold true for any discretization (4.2.5), (4.2.6), including the unstable *central difference scheme* (while Theorem 4.2 does not). On the other hand, though there exist layer-adapted meshes providing $\varepsilon$-uniform convergence of the central difference scheme [2, 19, 22], to construct a mesh yielding (4.2.37) a posteriori is much more difficult for this scheme than for any stable scheme. Indeed, on an initial uniform mesh, the central difference scheme would yields an oscillatory computed solution, which is of no use as an intermediate computed solution in any a posteriori algorithm.

**Corollary 4.2.4.** *Under the conditions of both Theorem* 4.2 *and Theorem* 4.5*, we have*

$$\|u^N(x) - u(x)\|_\infty \le 2\Big(\|u_i^N - u_{i-1}^N\|_{\infty,d} + C_2 h\Big)$$
$$\le C\Big(\big\|\min\{\frac{h_i}{\varepsilon}, 1\}\, e^{-\beta x_{i-1}/\varepsilon}\big\|_{\infty,d} + h\Big).$$

*Proof.* The first inequality here repeats Theorem 4.5. The second is obtained from (4.2.23b) by using

$$|u_i^N - u_{i-1}^N| \le |u(x_i) - u(x_{i-1})| + C\Big(\big\|\min\{\frac{h_i}{\varepsilon}, 1\}\, e^{-\beta x_{i-1}/\varepsilon}\big\|_{\infty,d} + h\Big)$$

and recalling (4.2.2) and (4.2.3). □

**Remark 4.7.** Corollary 4.2.4 implies that the a posteriori error estimate (4.2.29) is sharp, or at least as sharp as the a priori error estimate (4.2.23b). Furthermore, since there exist a priori chosen meshes such that

$$\min\{\frac{h_i}{\varepsilon}, 1\}\, e^{-\beta x_{i-1}/\varepsilon} + h_i \le CN^{-1}$$

—see Remark 4.4—hence for discretizations (4.2.5),(4.2.6) satisfying (4.2.22) there exist meshes for which we have (4.2.37), and hence first-order convergence (4.2.38).

**Remark 4.8.** Theorems 4.2 and 4.5 and all the Corollaries and Remarks given in §4.2.4 also hold true for the more general difference schemes

$$\tilde{T}^N u_i^N = -\frac{A^N u_{i+1}^N - A^N u_i^N}{\chi_i} = f_i \tag{4.2.39}$$

with $\chi_i = k_i h_i + (1 - k_i) h_{i+1}$ and $k_i \in [0, 1]$ (compare with (4.2.5)), where $A^N$ remains defined by (4.2.6). In particular, they hold true for the scheme

$$\bar{T}^N u_i^N = -DA_i^N u_i^N = -\frac{A^N u_{i+1}^N - A^N u_i^N}{\hbar_i} = f_i. \tag{4.2.40}$$

Now Theorem 4.5 is derived using $\tilde{F}_i^N = \sum_{j=i}^{N-1} f_j \chi_j$ instead of (4.2.25), so that $f_i = -(\tilde{F}_{i+1}^N - \tilde{F}_i^N)/\chi_i$. In particular, for (4.2.40) we use $\bar{F}_i^N = \sum_{j=i}^{N-1} f_j \hbar_j$ so that $f_i = -D\bar{F}_i^N$.

### 4.2.5 Second-order error estimates

In §4.2.5 we investigate the second-order discretization (4.2.8),(4.2.6) of our convection-diffusion problem (4.2.1) and obtain certain a priori error estimates and then a posteriori error estimates that are robust with respect to the small parameter $\varepsilon$.

#### Second-order a priori error estimate

Here we present a second-order a priori error estimate, which is a version of [3, Theorem 3].

**Theorem 4.9.** *Suppose that* $f(x) \in C^1\big([0,1]\big)$. *Let* $u(x)$ *be a solution of* (4.2.1), *and* $u_i^N$ *be a solution of* (4.2.8), (4.2.6), (4.2.9) *with* $\sigma_i$ *defined by* (4.2.10) *on an arbitrary nonuniform mesh. Then*

$$\big\|u_i^N - u(x_i)\big\|_{\infty,d} \le C \max_{1\le i\le N}\Big[h_i^2 \max_{x\in \bar I_i}\big|u''(x)\big| + h_i^2\Big]. \tag{4.2.41a}$$

*Furthermore,*

$$\big\|u_i^N - u(x_i)\big\|_{\infty,d} \le C\Big[\big\|\min\big\{\frac{h_i^2}{\varepsilon^2},1\big\}\, e^{-x_{i-1}/\varepsilon}\big\|_{\infty,d} + \max_{1\le i\le N} h_i^2\Big]. \tag{4.2.41b}$$

*Proof.* Estimate (4.2.41b) repeats the statement of [3, Theorem 3], while (4.2.41a) is obtained in the proof of that theorem. Note that the argument is very similar to the proof of Theorem 4.2: it is shown first that

$$\big\|u_i^N - u(x_i)\big\|_{\infty,d} \le 2\big\|A^N u_i - Au(\tilde x_{i-1/2})\big\|_{\infty,d}\,;$$

next we see that

$$\big|A^N u_i - Au(\tilde x_{i-1/2})\big| \le \big[\varepsilon(\sigma_i - 1/2)h_i + \mathcal{O}(h_i^2)\big] \max_{x\in\bar I_i}\big|u''(x)\big| + \big\|f'\big\|_\infty \mathcal{O}(h_i^2)$$

and combine this with $\varepsilon(\sigma_i - 1/2)h_i \le Ch_i^2$, which follows from (4.2.10c). ☐

**Remark 4.10.** Given (4.2.41b), one can construct suitable layer-adapted meshes such that the error $\|u_i^N - u(x_i)\|_{\infty,d}$ of the discretization (4.2.8), (4.2.6) does not exceed $CN^{-2}$, e.g., the Bakhvalov mesh (4.2.27) with $\lambda \ge 2$.

**Remark 4.11.** An estimate similar to (4.2.41a), but for a certain modified Streamline Diffusion FEM, was obtained by Chen and Xu [9, 10] from the error estimate $\|u^N - u\|_\infty \le C\|u - u^I\|_\infty$, where $u^I$ is the linear interpolant of the exact solution $u$. The latter estimate was a particular case of a more general quasi-optimal-approximation result: $\|u^N - u\|_\infty \le C\|u - v^N\|_\infty$ for all $v^N$ in the standard linear finite element space. Furthermore, since, by (4.2.2),(4.2.3), we have $|u''(x)| \sim 1 + \varepsilon^{-2}e^{-x/\varepsilon}$, the analogue of (4.2.41a) in [9, 10] motivated the authors to consider the mesh that equidistributes the monitor function $M(x) = \sqrt{1 + \varepsilon^{-2}e^{-x/\varepsilon}}$ and show $\varepsilon$-unform second-order convergence of the resulting numerical method; see §4.3.2 for a discussion of monitor-function equidistribution.

### Second-order a posteriori error estimate

Now we present an a posteriori analogue of (4.2.41a). Note that this result is new; see [20, Theorem 4.2] for a similar second-order a posteriori error estimate for a different second-order discretization of a more general quasilinear problem.

**Theorem 4.12.** *Let* $u(x)$ *be a solution of* (4.2.1), $u_i^N$ *be a solution of* (4.2.8), (4.2.6), (4.2.9) *with* $\sigma_i$ *defined by* (4.2.10) *on an arbitrary nonuniform mesh, and* $u^N(x)$ *be its piecewise linear interpolant defined in* (4.2.11). *Then*

$$\big\|u^N(\cdot) - u(\cdot)\big\|_\infty \le C\Big[\max_{1\le i\le N-1}\Big(\hbar_i^2\big|DD^- u_i^N\big|\Big) + \max_{1\le i\le N} h_i^2\Big]. \tag{4.2.42}$$

*Proof.* We start by imitating the argument used in the proof of Theorem 4.5, but with (4.2.31) replaced by

$$A^N u_i^N - F(\tilde{x}_{i-1/2}) = a, \quad \text{for} \quad i = 1, \dots, N\,.$$

Hence we get (4.2.32),(4.2.33) with a slightly different $\eta_F$:

$$\eta_F(x) := F(\tilde{x}_{i-1/2}) - F(x) = \int_{\tilde{x}_{i-1/2}}^{x} f(x)\,dx, \quad x \in I_i, \quad i = 1, \dots N\,,$$

for which we obtain

$$\eta_F(x) = \big[x - \tilde{x}_{i-1/2}\big] f(\xi_{i-1}) + \mathcal{O}(h^2)\,, \qquad x \in I_i\,, \tag{4.2.43}$$

where $\xi_i \in [x_{i-1}, x_{i+1}]$ is arbitrary. We choose $\xi_0 := \xi_1$, while $\xi_i$ for $i = 1, \dots, N-1$ is chosen to satisfy

$$f(\xi_i) := [\tilde{x}_{i+1/2} - \tilde{x}_{i-1/2}]^{-1} \int_{\tilde{x}_{i-1/2}}^{\tilde{x}_{i+1/2}} f(x)\,dx\,.$$

Relation (4.2.36) from the proof of Theorem 4.5 still holds true and implies that

$$\eta_A = \big[\sigma_i(x_i - x) - (1-\sigma_i)(x - x_{i-1})\big] D^- u_i^N = -\big[x - \tilde{x}_{i-1/2}\big] D^- u_i^N\,.$$

Here we also used $\big[u^N(x)\big]' = D^- u_i^N$. Combining the above relation with (4.2.43), we get

$$\eta(x) = \eta_A(x) - \eta_F(x) = -\big[x - \tilde{x}_{i-1/2}\big]\big(D^- u_i^N + f(\xi_{i-1})\big) + \mathcal{O}(h^2).$$

Decompose $\eta$ using the decomposition $[x - \tilde{x}_{i-1/2}] = [x - x_{i-1/2}] + h_i[\sigma_i - 1/2]$, in which, by (4.2.10c), we have $h_i[\sigma_i - 1/2] \le h_i \min\{1, Ch_i/\varepsilon\} \le Ch_i^2/(h_i + \varepsilon)$. Next we recall that $T[u^N - u] = \eta'$ and apply Lemma 4.2.3 and Lemma 4.2.2, respectively, to the two components of $\eta$. This yields

$$\big\|u^N(\cdot) - u(\cdot)\big\|_\infty \le C \max_{1 \le i \le N} \frac{h_i^2}{h_i + \varepsilon}\big|D^- u_i^N + f(\xi_{i-1})\big| + \mathcal{O}(h^2)\,. \tag{4.2.44}$$

Our discretization (4.2.8),(4.2.6) can be written as $-[A^N u_{i+1}^N - A^N u_i^N] = f(\xi_i)\,[\tilde{x}_{i+1/2} - \tilde{x}_{i-1/2}]$, where $A^N v_i = [\varepsilon - h_i(1-\sigma_i)]D^- v_i + v_i$. Furthermore, we can rewrite this difference scheme as

$$-\hbar_i\big[\varepsilon - h_i(1-\sigma_i)\big]\frac{D^- u_{i+1}^N - D^- u_i^N}{\hbar_i} = \big[h_i(1-\sigma_i) + h_{i+1}\sigma_{i+1}\big]\big(D^- u_{i+1}^N + f(\xi_i)\big)$$

for $i = 1, \dots, N-1$. Here, by (4.2.9), we used $\tilde{x}_{i+1/2} - \tilde{x}_{i-1/2} = h_i(1-\sigma_i) + h_{i+1}\sigma_{i+1}$. Hence for $i = 1, \dots, N-1$ we obtain

$$\begin{aligned} \frac{h_{i+1}^2}{\varepsilon + h_{i+1}}\big|D^- u_{i+1}^N + f(\xi_i)\big| &= \frac{h_{i+1}^2}{\varepsilon + h_{i+1}}\big|DD^- u_i^N\big|\frac{\hbar_i[\varepsilon - h_i(1-\sigma_i)]}{h_i(1-\sigma_i) + h_{i+1}\sigma_{i+1}} \\ &\le 4\hbar_i^2\big|DD^- v_i\big|\,. \end{aligned} \tag{4.2.45}$$

Here we used $0 \le \varepsilon - h_i(1-\sigma_i) < \varepsilon$ and $\sigma_i \ge 1/2$. To get a similar estimate for $i = 0$, note that

$$D^- u_1^N = D^- u_2^N - \hbar_i DD^- u_1^N, \qquad f(\xi_0) = f(\xi_1),$$

and then use a similar argument to show that

$$\frac{h_1^2}{\varepsilon + h_1}\big|D^- u_1^N + f(\xi_0)\big| \le C\hbar_i^2 \big|DD^- u_1^N\big|.$$

Combining this with (4.2.45) and (4.2.44) we complete the proof. □

**Corollary 4.2.5.** *Suppose that the conditions of Theorem 4.12 are satisfied and*

$$\hbar_i^2\big(\big|DD^- u_i^N\big| + 1\big) \le CN^{-2} \quad \textit{for} \quad i = 1, \ldots, N-1\,. \tag{4.2.46}$$

*Then*

$$\|u^N(\cdot) - u(\cdot)\|_\infty \le CN^{-2}\,.$$

**Remark 4.13.** Theorem 4.12 and Corollary 4.2.5 also hold true for the *central difference schemes* defined by (4.2.5),(4.2.6) or (4.2.40),(4.2.6) with $\sigma_i = 1/2$; see also Remark 4.6.

In §4.2 we obtained certain a posteriori error estimates for our model singularly perturbed convection-diffusion problem. Next we shall implement one of them, estimate (4.2.29), in a moving mesh algorithm and analyze its convergence and robustness with respect to the small parameter.

## 4.3 Full analysis of a robust adaptive method for a 1d convection-diffusion problem

Many numerical techniques have been proposed for the solution of convection-diffusion problems; see [35] for a survey. In recent years attention has focused on the use of meshes that are adapted to the singularly perturbed nature of these problems. These meshes are designed to be very fine where sharp layers appear in the solution, so that they yield accurate numerical solutions within those layers. They can be divided into two main classes: *special meshes* that are chosen a priori (e.g., meshes of Bakhvalov and Shishkin type) and *adaptive meshes* that are constructed by the algorithm, starting from an initial unsophisticated mesh. In the present §4.3 we consider only adaptive meshes. We implement the a posteriori error estimate (4.2.29) from §4.2 in a moving mesh algorithm. The mesh used has a fixed number $(N + 1)$ of nodes and is initially uniform, but its nodes are moved adaptively using a simple algorithm of de Boor based on equidistribution of the arc-length of the current computed piecewise linear solution. We present a full analysis of this algorithm. It is proved that a mesh exists that equidistributes the arc-length along the polygonal solution curve and that the corresponding computed solution is first-order accurate, uniformly in the small diffusion coefficient $\varepsilon$. Furthermore, it is shown that after $\mathcal{O}(\ln(1/\varepsilon)/\ln N)$ iterations of the algorithm, the piecewise linear interpolant of the computed solution achieves first-order accuracy in the maximum norm uniformly in $\varepsilon$. Numerical experiments are presented that support our theoretical results.

Our analysis is heavily based on [25], but here we restrict our presentation to the simple constant-coefficient problem (4.2.1). Furthermore, we discuss desirable properties of monitor functions and possible generalizations of our analysis to other differential equations, other discretizations, and other monitor functions. Hence an attempt is made to point out the key ingredients of the analysis [25] to enable its possible extensions to other problems.

### 4.3.1 Upwind difference scheme

To facilitate our presentation, we shall restrict it to the simplest convection-diffusion problem with constant coefficients (4.2.1). Recall that its solutions typically exhibit an exponential boundary layer at $x = 0$; see §4.2.1.

As in §4.2, we consider *arbitrary meshes*

$$\{\, x_i \;\big|\; 0 = x_0 < x_1 < \cdots < x_N = 1 \,\}.$$

For $i = 1, \ldots, N$, as usual, let $h_i = x_i - x_{i-1}$ be the local mesh size.

**Remark 4.14.** *No assumption* is made that our meshes satisfy $\max\limits_{i=1,\ldots,N} h_i \le CN^{-1}$ for some constant $C$, as it is not immediately obvious that the adaptive algorithm of §4.3.3, or any adaptive algorithm in general, yields meshes enjoying this property (we shall prove in Corollary 4.3.2 that at all stages of our algorithm one does in fact have $\max\limits_i h_i \le CN^{-1}$).

We discretize (4.2.1) by the upwind scheme in the form (4.2.5):

$$T^N u_i^N = -\frac{A^N u_{i+1}^N - A^N u_i^N}{h_{i+1}} = f_i \quad \text{for } i = 1, \ldots, N-1, \quad u_0^N = u_N^N = 0, \tag{4.3.47}$$

where the discrete operator $A^N$ is from (4.2.6) with $\sigma_i = 1$:

$$A^N v_i := \varepsilon D^- v_i + v_i\,, \qquad \text{where} \qquad D^- v_i = \frac{v_i - v_{i-1}}{h_i}. \tag{4.3.48}$$

Here $\{u_i^N\}$ is the solution computed on the mesh $\{x_i\}$. Since (4.3.47) is a linear system with an $M$-matrix, there always exists a unique solution to (4.3.47). Clearly, $T^N u^N = f$ in (4.3.47) can be written as

$$-D^+ A^N u_i^N = f_i, \qquad \text{where} \qquad D^+ v_i = \frac{v_{i+1} - v_i}{h_{i+1}}.$$

Discretization (4.3.47),(4.3.48) is covered by our analysis in §4.2.4 and, in particular, satisfies the a posteriori error estimate (4.2.29). Hence constants $C$ and $\bar{C}$ exist such that

$$\begin{aligned} \|u^N(\cdot) - u(\cdot)\|_\infty &\le C \left[ \max_{i=1,\ldots,N} |u_i^N - u_{i-1}^N| + \max_{i=1,\ldots,N} h_i \right] \\ &\le \bar{C} \max_{i=1,\ldots,N} \sqrt{(u_i^N - u_{i-1}^N)^2 + h_i^2}, \end{aligned} \tag{4.3.49}$$

where $u^N(\cdot) \in C[0,1]$ is the piecewise linear interpolant through the knots $(x_i, u_i^N)_{i=0}^N$.

### 4.3.2 Adaptive mesh movement by equidistribution of the arc-length monitor function

#### Monitor functions

One approach to the adaptive construction of suitable meshes for singularly perturbed problems such as (4.2.1) is the use of *monitor functions* and *equidistribution*. Monitor functions are used in many papers (see, for example, [6, 20, 26, 29, 33, 35]) as the basis for adaptive algorithms designed to yield highly nonuniform meshes that resolve layer phenomena such as our boundary layer at $x = 0$. Although it is not straightforward to extend this approach to convection-diffusion problems in two dimensions, nevertheless related algorithms have been developed for such problems and in some cases yield good numerical results [7, 11, 17, 28].

A monitor function $M(x)$ is an arbitrary nonnegative function defined on our domain $[0, 1]$. A mesh $\{x_i\}$ is said to equidistribute $M(\cdot)$ if

$$\int_{x_{i-1}}^{x_i} M(x)\,dx = \frac{1}{N}\int_0^1 M(x)\,dx \quad \text{for} \quad i = 1, \ldots, N. \tag{4.3.50}$$

If $N$ is given, the efficiency of an equidistributing mesh depends on the choice of $M(\cdot)$.

**Remark 4.15.** One desirable property of a monitor function is that for some positive constants $C_1' \le C_1$ for all possible solutions $u$ we have

$$C_1' \le M(x), \quad x \in [0, 1]; \qquad \int_0^1 M(x)\,dx \le C_1. \tag{4.3.51}$$

(Here $C_1'$ and $C_1$ are independent of $\varepsilon$ and $N$.) Combined with (4.3.50), this implies

$$h_i \le (C_1/C_1')\,N^{-1};$$

compare this with Remark 4.14. Note also that relations (4.3.51) are used, e.g., in [26, Theorem 1].

Consider a few examples of monitor functions:

1. Consider the monitor function $M(x) = |u'(x)|$. With this choice one can clearly expect *excessively large mesh sizes* in regions where $|u'|$ is small. E.g., if $u'(x) = 0$ on $[0,\ 1/2)$ and $u'(x) = 1$ on $(1/2,\ 0]$, then (4.3.50) yields the mesh with $x_0 = 0$ and $x_i = (1 + i/N)/2$ for $i = 1, \ldots, N$, i.e. $h_1 > 1/2$. Note that this monitor function does not satisfy the lower bound in (4.3.51).

2. To cure this flaw, simply modify the previous monitor function to $M(x) = 1 + |u'(x)|$ or to the so-called *arc-length monitor function*:

$$M_{\rm arc}(x) = \sqrt{1 + (u'(x))^2}, \tag{4.3.52}$$

   whose discrete analogue is the main topic of this whole §4.3. Now (4.3.51) is satisfied. Furthermore, one can show that these monitor functions produce quite satisfactory layer-adapted meshes on which most reasonable discretizations are *first-order accurate*.

3. In an attempt to improve the resulting meshes, some authors consider the monitor function $M(x) = 1 + \alpha|u'(x)|^p$ for some $p \in (0, 1)$ and some constant $\alpha$ that possibly depends on $\varepsilon$. This function clearly satisfies the lower bound in (4.3.51) with $C_1' = 1$. However, there is a problem with the upper bound. Typical possible solutions are, e.g., $u_1 = e^{-x/\varepsilon}$ and $u_2 = x$, whose derivatives are $(-\varepsilon^{-1}e^{-x/\varepsilon})$ and 1, while the values of $\int_0^1 M(x)\,dx$ are $1 + \alpha\mathcal{O}(\varepsilon^{1-p})$ and $1 + \alpha$ respectively. Let $\varepsilon \ll 1$. If $\alpha = \varepsilon^{-(1-p)} \gg 1$, then (4.3.51) is violated. To satisfy (4.3.51)

we need $\alpha = \mathcal{O}(1)$; then this monitor function becomes weaker than (4.3.52), and the resulting mesh in the boundary-layer region is coarser and *less satisfactory* than the one produced by (4.3.52).

4. Note that (4.3.52) produces meshes on which one can get at most first-order accuracy in the layer-region even if higher-order discretaizations of the differential equation are used. To improve this to *second-order accuracy*, one needs more sophisticated monitor functions, e.g.,

$$M(x) = 1 + |u''(x)|^{1/2};$$

see Remark 4.11 regarding a similar monitor function used in [9,10], and also [21,23]. Note that this monitor function enjoys the desirable property (4.3.51).

**Discrete arc-length monitor function**

Since the exact solution $u$ is unknown a priori, the adaptive algorithms use some discrete analogues of certain continuous monitor functions. Hence we shall consider the equidistribution problem (4.3.50) in which the monitor function $M(x)$ is the discrete analogue of (4.3.52):

$$M(x) := M^N(x) = \sqrt{1 + [(u^N)'(x)]^2},$$

where we recall that $u^N(x)$ is the piecewise linear interpolant through the knots $(x_i, u_i^N)$. Note that while most authors analyze $M_{\text{arc}}$ or other monitor functions that use the unknown exact solution, we present the analysis of the discrete monitor function $M^N$ that use the computed solution $u_i^N$. Our target is the following.

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Find $\{(x_i, u_i^N)\}$, with the $\{u_i^N\}$ computed from the $\{x_i\}$ by means of (4.3.47):

$$T^N u_i^N = f_i \quad \text{on mesh } \{x_i\}, \qquad u_0^N = u_N^N = 0$$

such that

$$\underbrace{\sqrt{|u_i^N - u_{i-1}^N|^2 + |x_i - x_{i-1}|^2}}_{\text{arc-length between } x_{i-1} \text{ and } x_i} = \frac{1}{N} \underbrace{\sum_{j=1}^{N} \sqrt{|u_j^N - u_{j-1}^N|^2 + |x_j - x_{j-1}|^2}}_{\text{total arc-length}} \quad \forall i \qquad (4.3.53)$$

Clearly, we can rewrite (4.3.53) as

$$h_i M_i = \frac{1}{N} \sum_{j=1}^{N} h_j M_j \quad \text{for } i = 1, \ldots, N, \quad \text{where } M_i := \sqrt{1 + (D^- u_i^N)^2}. \qquad (4.3.54)$$

Note that here both $\{x_i\}$ and $\{u_i^N\}$ are a priori unknown. Consequently, even though (4.2.1) is linear, the Equidistribution Problem, which requires the simultaneous solution of (4.3.47) and (4.3.54), is nonlinear. We follow [25], which provided answers to the following FOUR FUNDAMENTAL QUESTIONS that had remained unanswered before:

1. Does the equidistribution problem have a solution?
2. If the equidistribution problem has a solution $\{(x_i, u_i^N)\}$, will $u^N(x)$ be an accurate approximation of $u(x)$ on $[0, 1]$?
3. Is there any algorithm that can be proved to yield an accurate solution to the equidistribution problem when it terminates?
4. Can one prove that such an algorithm yields an accurate computed solution after a predetermined number of iterations?

For Question 1, we have the following.

**Theorem 4.16** (existence of a solution to the equidistribution problem)**.** *For each $\varepsilon \in (0, 1]$ and each positive integer $N$, there exists a mesh that provides a solution to the Equidistribution Problem, i.e. that equidistributes the discrete arc-length monitor function in the sense of* (4.3.54)*, for the scheme* (4.3.47) *applied to* (4.2.1).

While the conclusion of this theorem is intuitively reasonable, nevertheless we know of no earlier paper in which existence of an equidistributing mesh had been proved for *any* monitor function.

Question 2 is answered by our next result.

**Theorem 4.17** (accuracy of equidistributed solution)**.** *Let $\{(x_i, u_i^N)\}$ be a solution to the equidistribution problem. Let $u^N(x)$ be the piecewise linear interpolant of $\{(x_i, u_i^N)\}$. Then*

$$\max_{0 \le x \le 1} \left| u^N(x) - u(x) \right| \le CN^{-1} \quad \text{for some constant } C.$$

Question 3 has been considered by several authors. We shall consider a simple algorithm originally due to *de Boor* [12] but since considered by many authors (e.g., [32]). It automatically preserves the mesh ordering without any artificial modifications, and we have experienced no practical difficulties in computing solutions when $\varepsilon$ is small. We shall modify de Boor's original algorithm by using a *less stringent stopping criterion* that, while preserving the accuracy of the final computed solution, requires far fewer iterations. Note that we expect *any other algorithm* based on the same or similar arc-length monitor function to enjoy similar properties and require a similar number of iterations.

For other algorithms no result has been proved regarding the accuracy of any solution to which the algorithm converges. In contrast, we prove such a result for our algorithm, which is stated here in a form that will be made more precise later.

**Theorem 4.18** (accuracy of solution when algorithm terminates)**.** *Suppose that the algorithm reaches its stopping criterion and halts. Let the final mesh generated be $\{x_i^*\}$. Let $\{u_i^*\}$ be the discrete solution computed on this mesh, and let $u^*(x)$ be the piecewise linear interpolant of $\{(x_i^*, u_i^*)\}$. Then for some constant $C$ that is independent of $\varepsilon$ and the mesh, we have*

$$\max_{0 \le x \le 1} \left| u^*(x) - u(x) \right| \le CN^{-1}.$$

We answer Question 4 (under additional mild assumptions) with the following result, which shows that the algorithm is guaranteed to resolve the boundary layer appearing in the solution and estimates how many iterations are needed to achieve this.

**Theorem 4.19** (algorithm yields first-order accurate solution)**.** *Let $N$ be sufficiently large independently of $\varepsilon$ and of the number of iterations taken by the algorithm. Then there exists a positive integer $K$ such that $K \le C' \ln(1/\varepsilon)/\ln N$ and $\|e^{(K)}\|_\infty \le C'' N^{-1}$, where $\|e^{(k)}\|_\infty = \max_{0\le x\le 1} |e^{(k)}(x)|$ is the maximum norm error in the $k$th solution computed by the algorithm, and the constants $C', C''$ are independent of $\varepsilon$ and $N$.*

In [4], Babuška and Rheinboldt consider mesh distributions that are optimal in the sense that the error of the computed solution is minimized in some norm. While we are unable to provide an analogous analysis of the meshes computed by the algorithm, nevertheless we can observe that the final computed mesh satisfies (4.3.101), as does the Bakhvalov mesh (4.2.27), which, by (4.2.23b), implies that the computed solution $u^N$ on either mesh satisfies $\|u - u^N\|_\infty \le CN^{-1}$. See §4.3.7 for a visual comparison of our computed meshes with Bakhvalov and Shishkin meshes.

**Outline**

An outline of the present §4.3 is as follows. The algorithm is described in §4.3.3. In §4.3.4 we prove our existence result (Theorem 4.16). In §4.3.5, Theorems 4.17 and 4.18 are proved. §4.3.6 is devoted to a detailed analysis of the algorithm's behavior; this includes an analysis of the intermediate meshes and the intermediate computed solutions for one example and then culminates in Theorem 4.19. Numerical results supporting our theory are presented in §4.3.7. Finally, in §4.3.8 we point out the key ingredients of our analysis and discuss its possible generalizations to other problems, other discretizations, and other monitor functions.

**Notation**

Throughout §4.3 we use the usual maximum norm $\|\cdot\|_\infty$ on $C[0,1]$. As usual, $C$ denotes a generic positive constant that is independent of $\varepsilon$, of the mesh, and of the number of iterations taken by the algorithm of §4.3.3, and can take different values in different places. A subscripted $C$ (e.g., $C_1$) is a constant that is independent of $\varepsilon$, of the mesh, and of the number of iterations taken by the algorithm, but whose value is fixed. As the number of mesh points $(N+1)$ is fixed throughout the algorithm, we generally do not indicate dependence on $N$ in our notation; for example, we write $u^{(k)}$ for the solution $u^N$ computed by the $k$th iteration of the algorithm and $T^{(k)}$ for the difference operator $T^N$ on the $k$th mesh. For any function $f \in C[0,1]$ and any current mesh $\{x_i\}$ we set $f_i = f(x_i)$.

### 4.3.3 The algorithm

Given an arbitrary mesh, our algorithm aims to construct a mesh that solves the Equidistribution Problem. The number $N$ of mesh intervals is fixed throughout.

ALGORITHM

1. ***Initialize mesh***: The initial mesh $\{0,\ 1/N,\ 2/N, \dots, 1\}$ is uniform.

---

2. For $k = 0, 1, \dots$, assuming that the mesh $\{x_i^{(k)}\}$ is given, compute the discrete solution $\{u_i^{(k)}\}$ satisfying

$$T^{(k)}u^{(k)} = f^{(k)} \quad \text{on } \{x_i^{(k)}\}, \qquad \text{with} \quad u_0^{(k)} = u_N^{(k)} = 0, \tag{4.3.55}$$

where $f^{(k)} = \{f_i\}$. Let $h_i^{(k)} = x_i^{(k)} - x_{i-1}^{(k)}$ for each $i$. Let

$$\ell_i^{(k)} = h_i^{(k)}\sqrt{1 + (D^-u_i^{(k)})^2} = \sqrt{(u_i^{(k)} - u_{i-1}^{(k)})^2 + (h_i^{(k)})^2}$$

be the arc-length between the points $(x_{i-1}^{(k)}, u_{i-1}^{(k)})$ and $(x_i^{(k)}, u_i^{(k)})$ in the piecewise linear computed solution $u^{(k)}(x)$. Then the total arc-length of the solution curve $u^{(k)}(x)$ is

$$L^{(k)} := \sum_{i=1}^{N} \ell_i^{(k)} = \sum_{i=1}^{N} h_i^{(k)}\sqrt{1 + (D^-u_i^{(k)})^2} = \sum_{i=1}^{N} \sqrt{(u_i^{(k)} - u_{i-1}^{(k)})^2 + (h_i^{(k)})^2}.$$

---

3. ***Test mesh***: Let $C_0$ be a user-chosen constant with $C_0 > 1$ (see Remark 4.22). If

$$\frac{\max_i \ell_i^{(k)}}{L^{(k)}} \le \frac{C_0}{N}, \tag{4.3.56}$$

then go to step 5. Otherwise, continue to step 4.

---

4. ***Generate new mesh*** *by equidistributing arc-length of current computed solution:* Choose points $0 = x_0^{(k+1)} < x_1^{(k+1)} < \cdots < x_N^{(k+1)} = 1$ such that for each $i$ the distance from $(x_{i-1}^{(k+1)}, u^{(k)}(x_{i-1}^{(k+1)}))$ to $(x_i^{(k+1)}, u^{(k)}(x_i^{(k+1)}))$, *measured along the polygonal solution curve* $u^{(k)}(x)$, equals $L^{(k)}/N$. (This clearly determines the $x_i^{(k+1)}$ uniquely.) Our new mesh is then defined to be $\{x_i^{(k+1)}\}$. Return to step 2.

---

5. Set $\{x_0^*, x_1^*, \dots, x_N^*\} = \{x_i^{(k)}\}$ and $u^* = u^{(k)}$ then *stop*.

**Remark 4.20.** Step 4 of our algorithm is equivalent to

$$\int_{x_{i-1}^{(k+1)}}^{x_i^{(k+1)}} \sqrt{1 + \left[(u^{(k)})'(x)\right]^2}\, dx = \frac{L^{(k)}}{N}, \qquad i = 1, \dots, N, \quad k = 0, 1, \dots, \tag{4.3.57}$$

where $u^{(k)}(x)$ is, as usual, the piecewise linear interpolant of the computed solution $\{u_i^{(k)}\}_{i=0}^N$ on the mesh $\{x_i^{(k)}\}_{i=0}^N$.

**Remark 4.21.** Unlike the mathematical description of Step 4, its numerical implementation is quite straightforward. E.g., the corresponding MatLab command is, roughly speaking:

$$\texttt{x = interp1( cumsum([}0, \ell_1^{(k)}, \ell_i^{(k)}, \ldots, \ell_N^{(k)}\texttt{]), x, } \frac{L^{(k)}}{N}\texttt{[}0, 1, \ldots, N\texttt{] ),}$$

where `x` is the current mesh $[x_0, x_1, \ldots, x_N]$.

**Remark 4.22.** In (4.3.56) we can choose any constant $C_0$ that satisfies $C_0 > 1$. The larger $C_0$ is, the fewer iterations needed by the algorithm, but the final error bound of Theorem 4.29 increases with $C_0$. If we set $C_0 = 1$, then the algorithm is attempting to compute a fixed point of Theorem 4.25, so when $C_0 \approx 1$, we expect that the computed solution lies near such a fixed point. Our numerical experiments—see §4.3.7—show that $C_0 = 2$ produces suitable layer-adapted meshes and requires quite a few iterations. Note also that choosing $C_0 > 1$, we replace our original Equidistribution Problem on p. 164 by the following

QUASI-EQUIDISTRIBUTION PROBLEM

Find $\{(x_i, u_i^N)\}$, with the $\{u_i^N\}$ computed from the $\{x_i\}$ by means of (4.3.47):

$$T^N u_i^N = f_i \quad \text{on mesh } \{x_i\}, \qquad u_0^N = u_N^N = 0$$

such that

$$\underbrace{\sqrt{|u_i^N - u_{i-1}^N|^2 + |x_i - x_{i-1}|^2}}_{\text{arc-length between } x_{i-1} \text{ and } x_i} \le \frac{C_0}{N} \underbrace{\sum_{j=1}^{N} \sqrt{|u_j^N - u_{j-1}^N|^2 + |x_j - x_{j-1}|^2}}_{\text{total arc-length}} \quad \forall i \qquad (4.3.58)$$

**Remark 4.23.** Instead of (4.3.56), one can use the stopping criterion

$$\frac{\max_i \ell_i^{(k)}}{\min_i \ell_i^{(k)}} \le C_0. \qquad (4.3.59)$$

This criterion is stronger than (4.3.56); if (4.3.59) holds true, then

$$L^{(k)} = \sum_{i=1}^{N} \ell_i^{(k)} \ge N \min_i \ell_i^{(k)} \ge (N/C_0) \max_i \ell_i^{(k)},$$

which is equivalent to (4.3.56). The converse implication is false.

**Remark 4.24.** The algorithm is based implicitly on the standard arc-length monitor function of (4.3.52). Some authors [17,26,34] have used a scaled version of this function, defined by $M_{\text{arc},\alpha}(x) = \sqrt{\alpha + (u'(x))^2}$ for some positive $\alpha$ (independent of $\varepsilon$ and $N$). This is equivalent to stretching one of the coordinates. The algorithm and its analysis can be easily modified to accommodate this more general function.

### 4.3.4 The existence theorem

First we prove a preliminary result.

**Lemma 4.3.1.** *Let* $\{u_i^N\}$ *be the solution computed by our scheme* (4.3.47) *on an arbitrary mesh* $\{x_i\}$. *Then there exists a constant* $C$ *such that* $|D^- u_i^N| \le C(N + \varepsilon^{-1})$ *for* $i = 1, 2, \ldots, N$.

*Proof.* By the comparison/maximum principle, we have $|u_i^N| \le C$ for all $i$. As the mesh has $N$ subintervals, $h_m \ge N^{-1}$ for some $m$. These inequalities imply that $|D^- u_m^N| \le CN$.

Let $i \in \{1, 2, \ldots, N\}$ be arbitrary. Assume that $i \le m$, as the other case is similar. Now, by (4.3.47), we have

$$A^N u_m^N - A^N u_i^N = \sum_{j=i}^{m-1} \left(A^N u_{j+1}^N - A^N u_j^N\right) = -\sum_{j=i}^{m-1} h_{j+1} f_j \,.$$

Hence

$$|A^N u_i^N| \le |A^N u_m^N| + \|f\|_\infty \le \varepsilon |D^- u_m^N| + |u_m^N| + \|f\|_\infty \le C(\varepsilon N + 1).$$

Consequently, $\varepsilon |D^- u_i^N| \le C(\varepsilon N + 1) + |u_i^N|$, and the result follows. □

**Corollary 4.3.1.** *Let* $\{u_i^N\}$ *be the solution computed by our scheme* (4.3.47) *on an arbitrary mesh* $\{x_i\}$. *Then there exists a constant* $C$ *such that*

$$\sup_{x\in[0,1]} M^N(x) = \sup_{x\in[0,1]} \sqrt{1 + \left[(u^{(N)})'(x)\right]^2} \le C(N + \varepsilon^{-1}).$$

Now we can give our existence result.

**Theorem 4.25** (existence of a solution to the equidistribution problem)**.** *For each* $\varepsilon \in (0, 1]$ *and each positive integer* $N$, *there exists a mesh that provides a solution to the Equidistribution Problem on p. 164, i.e. that equidistributes the discrete arc-length monitor function in the sense of* (4.3.54), *for the scheme* (4.3.47) *applied to* (4.2.1).

*Proof.* One can regard steps 2 and 4 of the algorithm of §4.3.3 as a mapping

$$\Phi : (h_1, h_2, \ldots, h_N) \longmapsto (\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_N),$$

where $h_i$ and $\tilde{h}_i$ are the mesh-widths before and after regridding.

We claim that $\Phi : S_Q \longrightarrow S_Q$, where

$$S_Q = \left\{(h_1, h_2, \ldots, h_N) \in R^N : \ h_i \ge Q \quad \text{for} \quad i = 1, \ldots, N, \quad \sum_{i=1}^{N} h_i = 1\right\}$$

and $Q = Q(\varepsilon, N)$ satisfies $0 < Q < 1/N$. Note that these bounds on $Q$ imply that $S_Q$ is *nonempty*.

To prove this claim, let $\{u_i^N\}$ be the solution to (4.3.47) computed on the mesh with mesh-widths $h_1, h_2, \ldots, h_N$. Set $\ell_i = h_i \sqrt{1 + |D^- u_i^N|^2}$ for each $i$. By (4.3.57), our regridding is equivalent to

$$\int_{\tilde{x}_{i-1}}^{\tilde{x}_i} \sqrt{1 + \left[(u^N)'(x)\right]^2}\, dx = \frac{1}{N} \sum_i \ell_i \,, \qquad i = 1, \ldots, N.$$

Corollary 4.3.1 implies that

$$\int_{\tilde{x}_{i-1}}^{\tilde{x}_i} \sqrt{1+\left[(u^{(N)})'(x)\right]^2}\, dx \le C(N+\varepsilon^{-1})\tilde{h}_i \ ,$$

while

$$\frac{1}{N}\sum_i \ell_i \ge \frac{1}{N}\sum_i h_i = \frac{1}{N}.$$

Hence we have

$$C(N+\varepsilon^{-1})\tilde{h}_i \ge \frac{1}{N},$$

or

$$\tilde{h}_i \ge Q := \frac{1}{CN(N+\varepsilon^{-1})} \quad \text{for} \quad i=1,\dots,N. \tag{4.3.60}$$

Clearly $0 < Q < 1/N$ and we see that $\Phi$ maps $S_Q$ into itself.

The <u>*nonempty*</u> set $S_Q$ is <u>*convex*</u> and <u>*compact*</u>, and $\Phi$ is clearly <u>*continuous*</u>. By the *Brouwer fixed-point theorem* [37], the mapping $\Phi$ has a fixed point in $S_Q$. That is, there is a mesh on which the computed solution satisfies $\ell_i = \ell_j$ for all $i$ and $j$. □

**Remark 4.26.** To prove existence of a solution of the euquidistribution problem, we used the following property of our monitor function (given by Corollary 4.3.1):

- *There exists a quantity $Q(\varepsilon, N)$ such that for the solution $u_i^N$ of our difference scheme on an arbitrary mesh $\{x_i\}_{i=0}^N$ we have*

$$1 \le M^N(x) \le \frac{1}{Q(\varepsilon,N)}.$$

If a certain monitor function combined with a certain difference scheme enjoys the above property for some $Q(\varepsilon, N)$, then, imitating the proof of Theorem 4.25, one can show that the corresponding equidistribution problem also has a solution.

### 4.3.5 Accuracy of equidistributed and computed solutions

In the present §4.3.5 we consider the accuracy of solutions computed on two types of meshes: (i) meshes that yield a solution to (4.3.53); (ii) meshes obtained when our algorithm reaches step 5 and stops, i.e. meshes that yield a solution to (4.3.58). In other words, we investigate the accuracy of solutions to our Equidistribution Problem and Quasi-Equidistribution Problem respectively.

Our next lemma is crucial for these analyses.

**Lemma 4.3.2** (bound on length of polygonal solution curve)**.** *Let $\{u_i^N\}$ be the solution of* (4.3.47) *on an arbitrary mesh $\{x_i\}$. Let $L^N$ be the total arc-length along the solution curve $u^N(x)$. Then*

$$\boxed{1 \le L^N = \sum_{j=1}^{N} \sqrt{|u_j^N - u_{j-1}^N|^2 + |x_j - x_{j-1}|^2} \le C_1} \tag{4.3.61}$$

*where* $C_1 = 1 + 2\|f\|_\infty$.

*Proof.* Recall that $L^N = \sum_{i=1}^N h_i\sqrt{1+|D^-u_i^N|^2}$, so $L^N \ge \sum_i h_i = 1$. To prove the upper bound of the lemma, note that

$$L^N = \sum_{i=1}^N h_i\sqrt{1+|D^-u_i^N|^2} \le \sum_{i=1}^N h_i[1+|D^-u_i^N|] = 1 + \sum_{i=1}^N |u_i^N - u_{i-1}^N|.$$

Thus it suffices to prove that

$$\sum_{i=1}^N |u_i^N - u_{i-1}^N| \le 2\|f\|_\infty\,;$$

see [25, Appendix A] for the proof of this bound. ☐

Next we give a basic property of meshes generated by the algorithm.

**Corollary 4.3.2.** *Let* $\{x_i^{(k)}\}$ *be any mesh generated by the algorithm. Then for all* $i$ *we have*

$$h_i^{(k)} \le C_1 N^{-1}. \tag{4.3.62}$$

*Proof.* Since $C_1 \ge 1$, inequality (4.3.62) clearly holds true when $k = 0$, so assume that $k > 0$. By step 4 of the algorithm described in (4.3.57),

$$\int_{x_{i-1}^{(k)}}^{x_i^{(k)}} 1\,dx \le \frac{L^{(k-1)}}{N}\,.$$

Hence $h_i^{(k)} = x_i^{(k)} - x_{i-1}^{(k)} \le L^{(k-1)}/N \le C_1/N$, where we used Lemma 4.3.2. ☐

**Remark 4.27.** Since $L^N = \int_0^1 M^N(x)\,dx$, we observe that (4.3.61) is a discrete analogue of the upper bound in (4.3.51) (the lower bound is obvious since $1 \le M^N(x)$ ). Furthermore, to prove that a certain monitor function combined with a certain difference scheme enjoys (4.3.62) for some constant $C_1$, it suffices to show that this monitor function satisfies the following property:

- *There exists a positive constant* $C_1$ *such that for the solution* $u_i^N$ *of the difference scheme on an arbitrary mesh* $\{x_i\}_{i=0}^N$ *we have*

$$1 \le M^N(x); \qquad \int_0^1 M^N(x)\,dx \le C_1.$$

**Accuracy of equidistributed solution (4.3.53)**

The next result answers Question 2 of §4.3.2.

**Theorem 4.28** (accuracy of equidistributed solution)**.** *Let* $\{u_i^N\}$ *be the solution to* (4.3.47) *computed on a mesh* $\{x_i\}$ *that satisfies* (4.3.53). *Let* $u^N(x)$ *be the piecewise linear interpolant of* $\{(x_i, u_i^N)\}$. *Then*

$$\max_{0\le x\le 1} |u(x) - u^N(x)| \le \bar{C} C_1 N^{-1}. \tag{4.3.63}$$

*Proof.* (4.3.63) follows from (4.3.53) combined with (4.3.49) and (4.3.61). ☐

**Accuracy of computed solution (4.3.58) when the algorithm terminates**

We now analyze the accuracy of the solution computed on the final mesh generated by the algorithm.

**Theorem 4.29** (accuracy of solution when the algorithm terminates)**.** *Suppose that the algorithm reaches its stopping criterion and halts. Let the final mesh generated be* $\{x_i^*\}$*. Let* $\{u_i^*\}$ *be the discrete solution computed on this mesh, and let* $u^*(x)$ *be the piecewise linear interpolant of* $\{(x_i^*, u_i^*)\}$*. Then*

$$\max_{0\le x\le 1} |u(x) - u^*(x)| \le \bar{C}C_0C_1N^{-1}. \tag{4.3.64}$$

*Proof.* (4.3.64) follows from (4.3.58) combined with (4.3.49) and (4.3.61). ☐

### 4.3.6 How many iterations for $\varepsilon$-uniform accuracy?

In the present §4.3.6 we answer Question 4 of of §4.3.2.

For simplicity we assume that

$$\varepsilon \le N^{-1}, \tag{4.3.65}$$

which is not a restriction in practical situations. This inequality is used in the proof of Lemma 4.3.6. The analysis also works if, instead of (4.3.65), we have $\varepsilon \le \tilde{C}N^{-1}$ for some arbitrary but fixed constant $\tilde{C}$.

We also assume that $N$ is sufficiently large independently of $\varepsilon$ and of the iteration counter $k$.

Recall the decomposition (4.2.2),(4.2.3) of the solution $u(x)$ of our problem (4.2.1). Our next assumption is that the layer component $z(x)$ of this decomposition for some $C$ satisfies

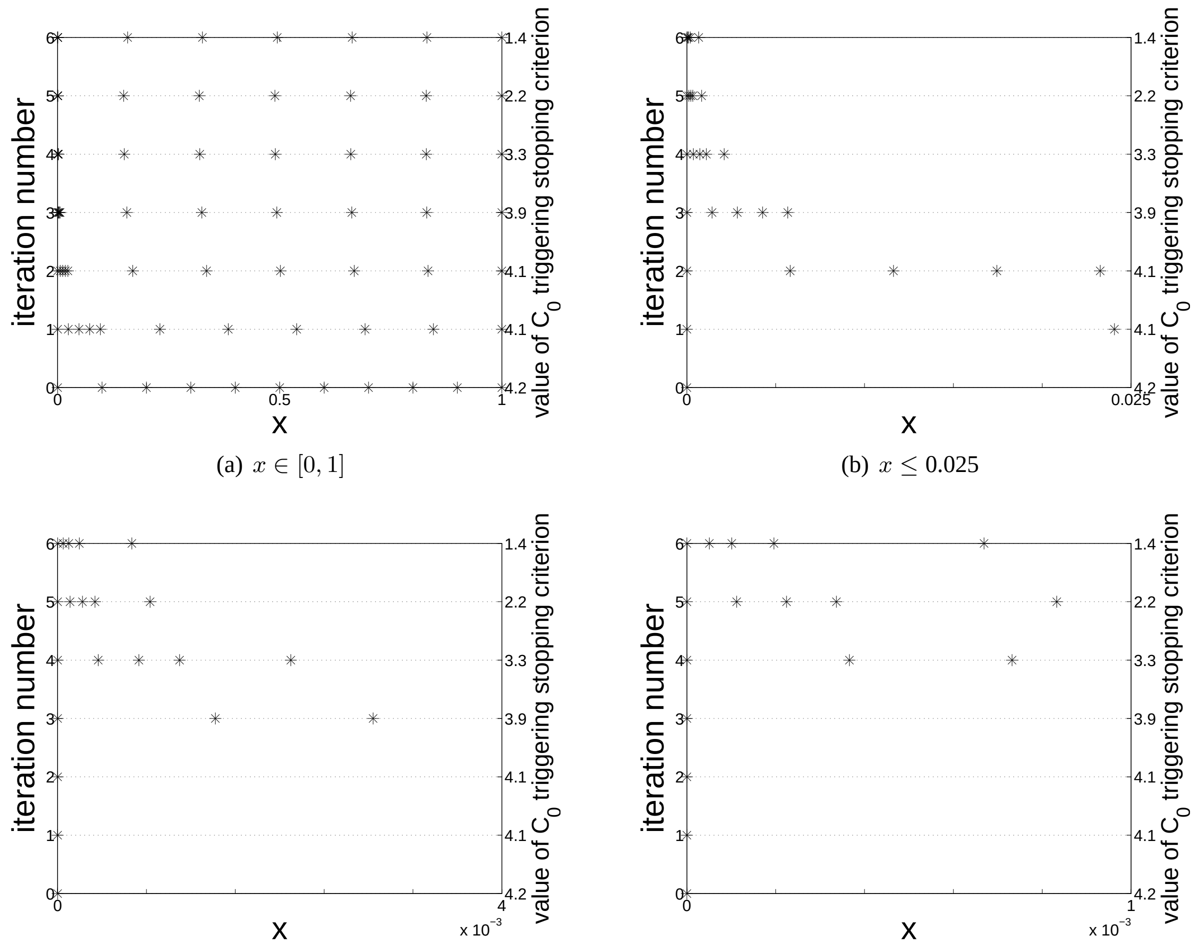
$$|z(0)| \ge C > 0. \tag{4.3.66}$$

This inequality is used in the proof of Lemma 4.3.3. If (4.3.66) were not true, then the solution of (4.2.1) would not have a boundary layer and the computed solution would be first-order accurate on any mesh with $\max_i h_i \le CN^{-1}$.

This entire §4.3.6 is devoted to the proof of our main convergence result, which shows that, starting from a uniform coarse mesh, the algorithm is nevertheless able to adjust the positions of its nodes so that eventually it yields a solution that is first-order accurate uniformly in $\varepsilon$.

**Theorem 4.30** (algorithm yields first-order accurate solution)**.** *Let* $N$ *be sufficiently large independently of* $\varepsilon$*. Then there exists a positive integer* $K$*, with* $K \le C(\ln(1/\varepsilon))/(\ln N)$*, such that* $\|e^{(K)}\|_\infty = \max_{0\le x\le 1} |e^{(K)}(x)| \le CN^{-1}$*, where* $e^{(k)}(x) := u^{(k)}(x) - u(x)$ *is the error in the* $k$*th solution* $u^{(k)}$ *computed by the algorithm.*

The proof of this theorem is divided into a series of intermediate results, and to further help the reader we group related material into subsections.




(a) $x \in [0,1]$ (b) $x \leq 0.025$

(c) $x \leq$ 4e-3 (d) $x \leq$ 1e-3

Figure 4.1: Mesh movement; pictures (b)-(d) enlarge boundary layer in picture (a); $\varepsilon = 10^{-4}$, $N = 10$.

**How does the algorithm work? An example**

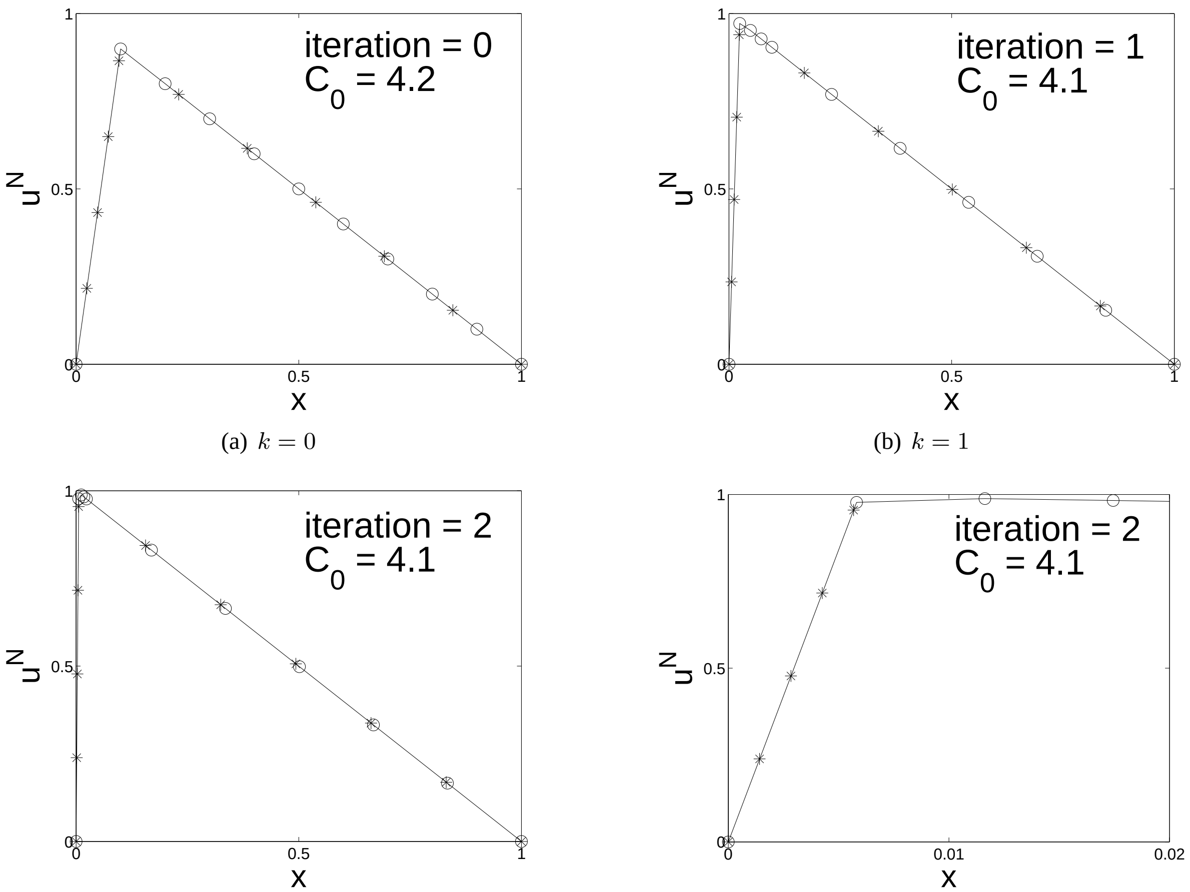
Before we start the proof of Theorem 4.30, consider a very simple example that will enable a better understanding of our algorithm.

**Example 4.3.1.** Let $u(x)$ solve the problem $-\varepsilon u'' - u' = 1, \quad u(0) = u(1) = 0$. One can easily check—see §4.2.1—that

$$u(x) \approx -e^{-x/\varepsilon} + (1 - x).$$

Figure 4.1, which should be read from bottom to top, shows the mesh after each iteration when our algorithm is applied to Example 4.3.1; $\varepsilon = 10^{-4}$, $N = 10$. Figure 4.2 shows intermediate computed solutions. Each of these meshes/computed solutions is labeled with the value of $C_0$ for which the stopping criterion (4.3.56) becomes an equation. And on Figure 4.3 we can see the final computed solution that corresponds to $C_0 \approx 1.4$ in the stopping criterion (4.3.56).

Figures 4.1 and 4.2 show that at least for the first 3 iterations, $h_1^{(k)}$ is divided into 4 equal mesh intervals, i.e.




(a) $k = 0$ (b) $k = 1$

(c) $k = 2$ (d) $k = 2$; boundary layer is enlarged

Figure 4.2: Intermediate computed solutions: points $\{(x_i^{(k)}, u_i^{(k)})\}$ marked by $\circ$ form the polygonal solution curve; points $*$ divide this curve into $N$ parts of equal arc-length, hence their projections form the new mesh $\{x_j^{(k+1)}\}$; $\varepsilon = 10^{-4}$, $N = 10$.

$$h_1^{(k+1)} \approx \frac{h_1^{(k)}}{4} \approx \frac{h_1^{(k)}}{0.4N}\,. \tag{4.3.67}$$

At iteration 4 the mesh size $h_1^{(k)}$ becomes $\mathcal{O}(\varepsilon)$, and starting from iteration 5 our mesh nodes form quite a reasonable layer-adapted mesh. Note that we also observe (4.3.67) when our algorithm works for other values of $\varepsilon$ and $N$.

To understand (4.3.67), examine graphs of intermediate computed solutions obtained using our algorithm on Figure 4.2. The total length of the polygonal solution curve does not change dramatically and remains $\approx 1 + \sqrt{2}$. Furthermore, until the first mesh node $x_1^{(k)}$ reaches the boundary layer, i.e. while $h_1^{(k)} > \varepsilon$, we have $u_1^{(k)} \approx 1$ and hence $\ell_1^{(k)} = M_1^{(k)} h_1^{(k)} \approx 1$. The remaining part of the solution curve has length $\approx \sqrt{2}$. Note that $\frac{1}{1+\sqrt{2}} \approx 0.4$, hence $40\%$ of the $10$ mesh nodes go into $[0, h_1^{(k)}]$, while the remaining mesh nodes are equidistributed on $[h_1^{(k)}, 1]$. This is consistent with (4.3.67).

Subdivision of $h_1^{(k)}$, which in general is described by an analogue of (4.3.67), will play a funda-

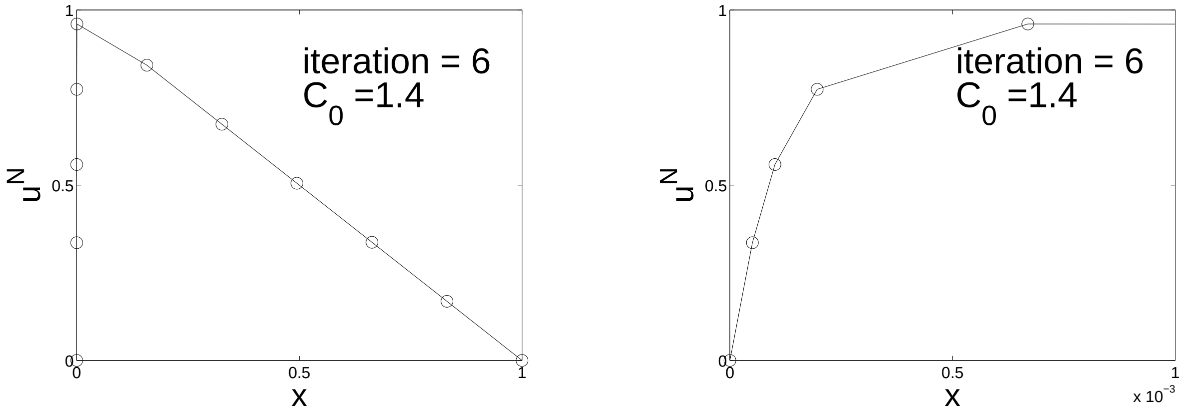



Figure 4.3: Final computed solution: $C_0 = 1.4 \leq 2$ (right-hand picture enlarges boundary layer); $\varepsilon = 10^{-4}$, $N = 10$.

mental role in our analysis.

### Notation and preliminary material

We introduce some notation that is used frequently later.

(i). Let $t_1$ and $t_2$ be arbitrary with $0 \leq t_1 < t_2 \leq 1$. Recall that $u^{(k)}(x)$ is the polygonal solution on the $k$th mesh $\{x_i^{(k)}\}$ generated by the algorithm. Define $L^{(k)}[t_1, t_2]$ to be the arc-length from $(t_1, u^{(k)}(t_1))$ to $(t_2, u^{(k)}(t_2))$ measured along the polygonal solution curve, i.e.

$$L^{(k)}[t_1, t_2] = \int_{t_1}^{t_2} \sqrt{1 + \left[(u^{(k)})'(x)\right]^2}\, dx. \tag{4.3.68}$$

In particular we note that $L^{(k)}[0, 1] = L^{(k)}$ in the notation of §4.3.3.

(ii). Define the parameter $\sigma = \sigma(\varepsilon, N)$ by

$$\sigma = \varepsilon \ln N \qquad \Leftrightarrow \qquad e^{-\sigma/\varepsilon} = N^{-1}. \tag{4.3.69}$$

This parameter is essentially the transition point $\tau$ between the fine and coarse meshes in the Shishkin mesh (4.2.28).

(iii). For $i = 1, \ldots, N$ and $k = 0, 1, \ldots,$ set

$$B_i^{(k)} = \prod_{j=1}^{i} \left(1 + \frac{h_j^{(k)}}{\varepsilon}\right)^{-1}. \tag{4.3.70}$$

Then

$$B_i^{(k)} \geq e^{-x_i^{(k)}/\varepsilon}, \tag{4.3.71}$$

and in fact the left-hand side here is a discrete approximation of the right-hand side on fine meshes. These follow from

$$\left(1+\frac{H}{\varepsilon}\right)^{-1} \geq e^{-H/\varepsilon}$$

and

$$\left(1+\frac{H}{\varepsilon}\right)^{-1} \approx e^{-H/\varepsilon} \qquad \text{if} \quad \frac{H}{\varepsilon} \ll 1,$$

where $H \in (0,1)$ is arbitrary. Furthermore, set

$$\lambda_i^{(k)} = \lambda_i^{(k)}(N,i,k) = \frac{C_1}{C_3 B_i^{(k)} N}\ , \tag{4.3.72}$$

where $C_1$ was defined in Lemma 4.3.2 and $C_3$ is defined below in Lemma 4.3.3. Many subsequent inequalities involve $\lambda_i^{(k)}$. It may help the reader to note that when $\lambda_i^{(k)}$ appears in our analysis, it usually turns out that $\lambda_i^{(k)} \leq C$ for some $C$.

(iv). Let $\{y_i^{(k)}\}$ be the solution to the problem

$$T^N y_i^{(k)} = f_i \quad \text{on} \ \{x_i^{(k)}\}, \qquad y_0^{(k)} = y(0), \quad y_N^{(k)} = y(1),$$

and let $\{z_i^{(k)}\}$ be the solution to the problem

$$T^N z_i^{(k)} = (Tz)_i = 0 \quad \text{on} \ \{x_i^{(k)}\}, \qquad z_0^{(k)} = z(0), \quad z_N^{(k)} = z(1),$$

These functions are discrete analogues of the functions $y$ and $z$ in (4.2.2).

**Lower bounds on discrete monitor functions**

**Lemma 4.3.3.** *For $i = 1, 2, \ldots, N$ and all $k$ there exist positive constants $C_2$ and $C_3$ such that if*

$$B_i^{(k)} \geq C_2 \varepsilon \quad \textit{for some } i\ , \tag{4.3.73}$$

*then*

$$M_i^{(k)} \geq \left| D^- u_i^{(k)} \right| \geq \frac{C_3 B_i^{(k)}}{\varepsilon}\ . \tag{4.3.74}$$

**Remark 4.31.** This lemma is the key to understanding of the mesh movement. Indeed, since the total arc-length $L^{(k)}$ is $\mathcal{O}(1)$ and does not change dramatically as the mesh nodes are moved—see Lemma 4.3.2—the number of mesh nodes to be moved into the region $[x_{i-1}^{(k)}, x_i^{(k)}]$ depends on $\ell_i^{(k)} = M_i^{(k)} h_i^{(k)}$ and is approximately equal to $\frac{\ell_i^{(k)}}{L^{(k)}} N$, while the local mesh size will be $h_j^{(k+1)} \approx h_i^{(k)} \frac{L^{(k)}}{\ell_i^{(k)}} N^{-1}$ $\leq h_i^{(k)} \frac{C_1}{\ell_i^{(k)}} N^{-1}$. The above lemma says that $\ell_i^{(k)} \geq C \frac{B_i^{(k)}}{\varepsilon} h_i^{(k)}$, and hence in the region $[x_{i-1}^{(k)}, x_i^{(k)}]$ the new local mesh size will be $h_j^{(k+1)} \leq C \frac{\varepsilon N^{-1}}{B_i^{(k)}}$.

**Remark 4.32.** It may help understanding Lemma 4.3.3, to compare it with its continuous analogue: *If $e^{-x/\varepsilon} \geq C_2 \varepsilon$ for some $x$, then $|u'(x)| \geq C_3 \varepsilon^{-1} e^{-x/\varepsilon}$.* Indeed, by (4.2.2), (4.2.3), (4.3.66), we have $u' = y' + z'$, where $|z'| \approx z(0)\varepsilon^{-1} e^{-x/\varepsilon}$, while $|y'| \leq C$. These observations imply the statement claimed.

*Proof.* It suffices to prove the lower bound for $|D^- u_i^{(k)}|$ in (4.3.74), since the bound for $M_i^{(k)}$ follows.

To simplify the presentation, we consider the proof for Example 4.3.1 only; for a more general case see [25, Lemma 5.2]. Clearly the computed solution $u_i^{(k)}$ can be split as $u_i^{(k)} = y_i^{(k)} + z_i^{(k)}$, where $y_i^{(k)} = 1 - x_i^{(k)}$, while $z_i^{(k)}$ satisfies $T^N z_i^{(k)} = 0,\ z_0^{(k)} = -1,\ z_N^{(k)} = 0$. Furthermore,

$$|D^- u_i^{(k)}| = |D^- z_i^{(k)} - 1| \geq |D^- z_i^{(k)}| - 1.$$

We claim that

$$|D^- z_i^{(k)}| \geq \frac{B_i^{(k)}}{2\varepsilon}. \tag{4.3.75}$$

Then (4.3.73) with $C_2 = 4$ implies (4.3.74) with $C_3 = 1/4$.

To show (4.3.75), introduce $Z_i = D^- z_i^{(k)}$. Then we have: $-\varepsilon D^+ Z_i - Z_{i+1} = 0$, which implies

$$Z_i = Z_{i-1} \left(1 + \frac{h_i^{(k)}}{\varepsilon}\right)^{-1} = Z_1 \frac{B_i^{(k)}}{B_1^{(k)}}.$$

Now it suffices to prove (4.3.75) for $i = 1$. One can easily check that for the discrete function $S_i = \prod_{j=1}^{i}(1 + \frac{h_j}{2\varepsilon})^{-1}$ (which is a slightly modified version of $B_i^{(k)}$), we have $T^N S_i \geq 0$. By the discrete maximum/comparison principle, this implies $0 \geq z_i^{(k)} \geq -S_i$ and hence

$$Z_1 = \frac{z_1^{(k)} - z_0^{(k)}}{h_1^{(k)}} \geq \frac{-S_1 + 1}{h_1^{(k)}} = \frac{1}{2\varepsilon}\left(1 + \frac{h_1^{(k)}}{2\varepsilon}\right)^{-1}.$$

This yields (4.3.75) for $i = 1$ and completes the proof. □

**Corollary 4.3.3.** *Let $N$ be sufficiently large independently of $\varepsilon$ and $k$. Then we have*

$$M_1^{(k)} \geq \left|D^- u_1^{(k)}\right| \geq \frac{C_3}{\varepsilon + h_1^{(k)}}\,. \tag{4.3.76}$$

*Proof.* Set $i = 1$ and note that

$$\frac{B_1^{(k)}}{\varepsilon} = \frac{1}{\varepsilon + h_1^{(k)}}\,.$$

Furthermore, by (4.3.62),(4.3.65), provided $N$ is sufficiently large, we get $B_1^{(k)}/\varepsilon \geq C_2$. Now Lemma 4.3.3 yields (4.3.76). □

### Subdivision of $h_1^{(k)}$

The next lemma shows that if the first mesh interval is too coarse, then at the next iteration the algorithm will subdivide it $\mathcal{O}(N)$ times.

**Lemma 4.3.4.** *Let $C_4$ be an arbitrary positive constant. For some $k \geq 0$, suppose that*

$$h_1^{(k)} \geq C_4\varepsilon\,. \tag{4.3.77}$$

*Then*

$$h_1^{(k+1)} \le C_5 h_1^{(k)}/N\,, \qquad \textit{where } C_5 = \left(\frac{1}{C_4}+1\right)\frac{C_1}{C_3}\,.$$

*Proof.* By Corollary 4.3.3, for $L^{(k)}[0,x_1^{(k)}] = \ell_1^{(k)} = M_1^{(k)} h_1^{(k)}$ we have

$$L^{(k)}[0,x_1^{(k)}] \ge \frac{C_3 h_1^{(k)}}{\varepsilon + h_1^{(k)}} = \frac{C_3}{\varepsilon/h_1^{(k)}+1} \ge \frac{C_3}{C_4^{-1}+1} \ge \frac{C_3 L^{(k)}/C_1}{C_4^{-1}+1}.$$

Here we also used (4.3.77) and Lemma 4.3.2. Then

$$L^{(k)}[0,x_1^{(k)}] \ge L^{(k)}/N,$$

which implies that $x_1^{(k+1)} < x_1^{(k)}$. Furthermore, by the construction of the algorithm,

$$L^{(k)}[0,x_1^{(k+1)}] = \frac{L^{(k)}}{N}, \qquad h_1^{(k+1)} = h_1^{(k)} \frac{L^{(k)}[0,x_1^{(k+1)}]}{L^{(k)}[0,x_1^{(k)}]}.$$

The second relation here follows from $L^{(k)}[0,x_1^{(k+1)}] = M_1^{(k)} h_1^{(k+1)}$ and $L^{(k)}[0,x_1^{(k)}] = M_1^{(k)} h_1^{(k)}$. Finally, we obtain

$$h_1^{(k+1)} = h_1^{(k)} \frac{L^{(k)}/\,N}{L^{(k)}[0,x_1^{(k)}]} \le \frac{h_1^{(k)}}{N}\frac{C_4^{-1}+1}{C_3/C_1} = h_1^{(k)}(C_5/N).$$

□

**Remark 4.33.** Imitating the proof of the above lemma, one can prove a more general statement: *If*

$$\frac{C_3 h_1^{(k)}}{C_1(\varepsilon + h_1^{(k)})} \ge N^{-1}, \tag{4.3.78}$$

*then*

$$h_1^{(k+1)} \le \frac{C_1}{C_3}(\varepsilon + h_1^{(k)})N^{-1}.$$

Indeed, under condition (4.3.78) we have

$$L^{(k)}[0,x_1^{(k)}] \ge \frac{C_3 h_1^{(k)}}{\varepsilon + h_1^{(k)}} \ge \frac{C_3 h_1^{(k)}}{(\varepsilon + h_1^{(k)})}\frac{L^{(k)}}{C_1} \ge \frac{L^{(k)}}{N}\,,$$

which yields

$$h_1^{(k+1)} = h_1^{(k)} \frac{L^{(k)}/\,N}{L^{(k)}[0,x_1^{(k)}]} \le \frac{h_1^{(k)}}{N}\frac{C_1(\varepsilon + h_1^{(k)})}{C_3 h_1^{(k)}}\,.$$

Intuitively one would expect Lemma 4.3.4 to apply over several iterations of the algorithm. This is indeed the case, as the next corollary shows.

**Corollary 4.3.4.** *Let $C_4$ be an arbitrary positive constant. Then there exists a non-negative integer $K' = K'(C_4)$ such that we have*

$$C_4\varepsilon \le h_1^{(k)} \qquad \textit{and} \qquad h_1^{(k+1)} \le C_5 h_1^{(k)}/N \qquad \textit{for} \quad k = 0,1,\ldots,K'-1$$

*(where $C_5$ was defined in Lemma 4.3.4), but*

$$h_1^{(K')} < C_4\varepsilon. \tag{4.3.79}$$

*Furthermore, if $N > C_5$, then*

$$K' \leq \frac{-\ln(C_4C_5\varepsilon)}{\ln(N/C_5)} = \frac{\ln(1/(C_4C_5)) + \ln(1/\varepsilon)}{\ln N - \ln C_5} \leq \frac{C\ln(1/\varepsilon)}{\ln N}\,. \tag{4.3.80}$$

*Proof.* If $h_1^{(0)} < C_4\varepsilon$ for some constant $C_4$, then we are done, so assume that $h_1^{(0)} \geq C_4\varepsilon$. Initially $h_1^{(0)} = N^{-1}$. Hence for all $k$ such that $h_1^{(m-1)} \geq C_4\varepsilon$ for $m = 1, 2, \ldots, k$, by Lemma 4.3.4 and induction we have

$$h_1^{(k)} \leq \frac{C_5 h_1^{(k-1)}}{N} \leq \left(\frac{C_5}{N}\right)^k \frac{1}{N}.$$

This inequality remains valid for successive $k$ until a value $K'$ is reached for which $h_1^{(K')} < C_4\varepsilon$. For $N > C_5$, it is easy to verify that $(C_5/N)^k/N < C_4\varepsilon$ is equivalent to

$$k > \frac{-\ln(C_4C_5\varepsilon)}{\ln(N/C_5)} - 1,$$

and (4.3.80) follows. □

**Remark 4.34.** Let $C_8 := C_1(1 + C_4)/C_3$. From Remark 4.33 and (4.3.79) it follows that either $h_1^{(K')} < C_8\varepsilon N^{-1}$ or $h_1^{(K'+1)} < C_8\varepsilon N^{-1}$.
Indeed, assume that $C_8\varepsilon N^{-1} \leq h_1^{(K')} \leq C_4\varepsilon$. Then

$$\frac{C_3 h_1^{(K')}}{C_1(\varepsilon + h_1^{(K')})} = \frac{C_3 h_1^{(K')}/\varepsilon}{C_1(1 + h_1^{(K')}/\varepsilon)} \geq \frac{C_3C_8N^{-1}}{C_1(1 + C_4)} = N^{-1},$$

and, by Remark 4.33, we have

$$h_1^{(K'+1)} \leq \frac{C_1}{C_3}(\varepsilon + h_1^{(K')})N^{-1} \leq \frac{C_1}{C_3}(1 + C_4)\,\varepsilon N^{-1}.$$

**Remark 4.35.** On the other hand, an inspection of (4.3.60) shows that (without assuming (4.3.65)) there exists $C$ such that for every mesh $\{\tilde{h}_i\}$ computed by the algorithm, one has

$$\min_i \tilde{h}_i \geq \frac{C\varepsilon}{N(\varepsilon N + 1)}\,.$$

As (4.3.65) is satisfied in the present section, this bound implies that $h_1^{(k)} \geq C\varepsilon N^{-1}$ for all $k$, so our upper bound on $h_1^{(K')}$ (or $h_1^{(K'+1)}$) is sharp.

### Properties of projections of polygonal solution curve

This subsection contains two crucial corollaries that shed light on the lengths of certain mesh intervals generated inside the boundary layer. It may help the reader if we point out that a subscript $i$ usually corresponds to the mesh $\{x_i^{(k)}\}$, while a subscript $j$ usually corresponds to the mesh $\{x_j^{(k+1)}\}$.

**Lemma 4.3.5.** *Suppose that inequality* (4.3.73) *holds true for some $i$ and $k$.*

(i) *For any $[t_1, t_2] \subseteq [0, x_i^{(k)}]$ such that $L^{(k)}[t_1, t_2] = L^{(k)}/N$, we have*

$$|t_2 - t_1| \le \varepsilon \lambda_i^{(k)}. \tag{4.3.81}$$

(ii) *If*

$$x_i^{(k)} \ge \varepsilon \lambda_i^{(k)}, \tag{4.3.82}$$

*then*

$$L^{(k)}[0, x_i^{(k)}] \ge L^{(k)}/N. \tag{4.3.83}$$

**Remark 4.36.** Inequality (4.3.73) means that $x_i^{(k)}$ is, roughly speaking, in the boundary-layer region. The above lemma deals with projections of the polygonal solution curve. Part (ii) says that under condition (4.3.82), there will be at least one mesh node $x_1^{(k+1)}$ in $[0, x_i^{(k)}]$. Part (i) is designed to treat $[t_1, t_2] = [x_{j-1}^{(k+1)}, x_j^{(k+1)}]$—recall that $L^{(k)}[x_{j-1}^{(k+1)}, x_j^{(k+1)}] = L^{(k)}/N$—and gives an upper bound for the local mesh size $h_j^{(k+1)} = t_2 - t_1$.

*Proof.* First note that, by (4.3.68), for $[t_1, t_2] \subseteq [0, x_i^{(k)}]$ we obtain

$$L^{(k)}[t_1, t_2] = \int_{t_1}^{t_2} \sqrt{1 + [(u^{(k)})'(x)]^2}\, dx > |t_2 - t_1| \min_{1 \le j \le i} |D^- u_j^{(k)}|.$$

For $j \le i$ one clearly has $B_j^{(k)} \ge B_i^{(k)}$, so by (4.3.73) we have $B_j^{(k)} \ge C_2 \varepsilon$ for $j = 1, 2, \ldots, i$. Now, by Lemma 4.3.3, we get

$$L^{(k)}[t_1, t_2] \ge |t_2 - t_1| \frac{C_3 B_i^{(k)}}{\varepsilon}. \tag{4.3.84}$$

(i) Suppose that $[t_1, t_2] \subseteq [0, x_i^{(k)}]$ and $L^{(k)}[t_1, t_2] = L^{(k)}/N$. Then inequality (4.3.84), Lemma 4.3.2, and (4.3.72) immediately yield

$$|t_2 - t_1| \le \frac{\varepsilon L^{(k)}[t_1, t_2]}{C_3 B_i^{(k)}} = \frac{\varepsilon L^{(k)}}{C_3 B_i^{(k)} N} \le \varepsilon \lambda_i^{(k)}.$$

(ii) Now suppose instead that (4.3.82) holds true. Setting $[t_1, t_2] = [0, x_i^{(k)}]$ in (4.3.84) and then using (4.3.72) and Lemma 4.3.2, we obtain

$$L^{(k)}[0, x_i^{(k)}] \ge x_i^{(k)} \frac{C_3 B_i^{(k)}}{\varepsilon} \ge C_3 \lambda_i^{(k)} B_i^{(k)} = C_1/N \ge L^{(k)}/N.$$ □

This lemma has the following two corollaries.

**Corollary 4.3.5.** *Suppose that for some $i$ and $k$, inequalities* (4.3.73) *and* (4.3.82) *are satisfied. On* *the next mesh* $\{x_j^{(k+1)}\}$ *generated by the algorithm, if for some* $j \geq 1$ *we have*

$$x_{j-1}^{(k+1)} \leq x_i^{(k)} - \varepsilon\lambda_i^{(k)}, \tag{4.3.85}$$

*then*

$$h_j^{(k+1)} \leq \varepsilon\lambda_i^{(k)}. \tag{4.3.86}$$

*Proof.* First, (4.3.83) implies that there is a unique point $x^* \in [0, x_i^{(k)})$ such that $L^{(k)}[x^*, x_i^{(k)}] = L^{(k)}/N$. Hence (4.3.81) yields $x_i^{(k)} - x^* \leq \varepsilon\lambda_i^{(k)}$. But then it follows from (4.3.85) that $x_{j-1}^{(k+1)} \leq x^*$, so

$$L^{(k)}[x_{j-1}^{(k+1)},\ x_i^{(k)}] \geq L^{(k)}/N.$$

By the construction of the algorithm,

$$L^{(k)}[x_{j-1}^{(k+1)},\ x_j^{(k+1)}] = L^{(k)}/N.$$

Consequently $x_j^{(k+1)} \leq x_i^{(k)}$. Thus we can take $[t_1, t_2] = [x_{j-1}^{(k+1)},\ x_j^{(k+1)}]$ in (4.3.81) to get

$$h_j^{(k+1)} = x_j^{(k+1)} - x_{j-1}^{(k+1)} \leq \varepsilon\lambda_i^{(k)};$$

i.e. (4.3.86) holds true. □

**Corollary 4.3.6.** *Suppose that for some* $i$ *and* $k$*, inequality* (4.3.73) *is satisfied. On the next mesh* $\{x_j^{(k+1)}\}$ *generated by the algorithm, suppose that for some* $j \geq 1$ *we have*

$$x_j^{(k+1)} \leq x_i^{(k)}. \tag{4.3.87}$$

*Then* (4.3.86) *holds true.*

*Proof.* By the construction of the algorithm,

$$L^{(k)}[x_{j-1}^{(k+1)},\ x_j^{(k+1)}] = L^{(k)}/N,$$

and by (4.3.87) we have $[x_{j-1}^{(k+1)},\ x_j^{(k+1)}] \subseteq [0, x_i^{(k)}]$. Thus we can take $[t_1, t_2] = [x_{j-1}^{(k+1)},\ x_j^{(k+1)}]$ in (4.3.81) to get $h_j^{(k+1)} = x_j^{(k+1)} - x_{j-1}^{(k+1)} \leq \varepsilon\lambda_i^{(k)}$. □

### A sufficient condition for accuracy

In this subsection we show that if for some $k \geq 0$ the mesh $\{x_i^{(k)}\}$ satisfies the following condition, then starting from iteration $k+1$, the error is bounded by $CN^{-1}$.

**Condition 4.37.** *For every* $i$ *such that* $x_{i-1}^{(k)} \leq \sigma - C_6\varepsilon$ *we have*

$$h_i^{(k)} \leq C_7\varepsilon. \tag{4.3.88}$$

The following lemma is a version of [25, Lemma 5.5].

**Lemma 4.3.6.** *Suppose for some $k \geq 0$ the mesh $\{x_i^{(k)}\}$ satisfies Condition 4.37, where $C_6$ and $C_7$ are arbitrary constants satisfying*

$$C_6 > C_7 \geq C_1/(C_2C_3), \qquad e^{(C_6-C_7)} > C_2(1+C_1). \tag{4.3.89}$$

(i) *Then the meshes $\{x_i^{(k+m)}\}$, where $m = 0, 1, 2, \ldots$, also satisfy Condition 4.37 with $C_6$ replaced by $C_6 + mC_7$.*

(ii) *Furthermore, there exists $C$ such that*

$$\|e^{(k+1)}(\cdot)\|_\infty = \|u^{(k+1)}(\cdot) - u(\cdot)\|_\infty \leq CN^{-1}, \tag{4.3.90}$$

*where $e^{(k+1)}(x) := u^{(k+1)}(x) - u(x)$ is the error in the computed solution on the $(k+1)$st mesh.*

(iii) *In fact, for any positive integer $m$ we have $\|e^{(k+m)}\|_\infty \leq CN^{-1}$ for $m = 1, 2, \ldots$, where $C = C(m)$.*

*Proof.* (i) Let $I$ be the largest value of $i$ for which $x_{i-1}^{(k)} \leq \sigma - C_6\varepsilon$, so

$$x_{I-1}^{(k)} \leq \sigma - C_6\varepsilon < x_I^{(k)} \leq \sigma - (C_6 - C_7)\varepsilon. \tag{4.3.91}$$

In order to apply Corollary 4.3.5 with $i = I$, we check its hypotheses. By (4.3.71) and (4.3.91) one has

$$\begin{aligned} B_I^{(k)} &\geq \exp\left(-x_I^{(k)}/\varepsilon\right) \\ &\geq \exp\left(-[\sigma - (C_6 - C_7)\varepsilon]/\varepsilon\right) \geq \frac{1}{N} e^{(C_6-C_7)}. \end{aligned} \tag{4.3.92}$$

Thus, by (4.3.89) and (4.3.65), we get

$$B_I^{(k)} \geq \frac{1}{N} C_2 \geq C_2\varepsilon, \tag{4.3.93}$$

We have proved (4.3.73) for the case $i = I$.

Next we verify (4.3.82) when $i = I$. By (4.3.91), we get

$$x_I^{(k)} > \sigma - C_6\varepsilon = \varepsilon(\ln N - C_6) \geq C_7\varepsilon$$

for $N$ sufficiently large. But we have

$$\varepsilon\lambda_I^{(k)} = \frac{C_1\varepsilon}{C_3 B_I^{(k)} N} \leq \frac{C_1\varepsilon}{C_2C_3} \leq C_7\varepsilon. \tag{4.3.94}$$

by (4.3.93) and (4.3.89). Thus (4.3.73) and (4.3.82) are satisfied when $i = I$, provided $N$ is sufficiently large. Observe then that (4.3.94) implies $\lambda_I^{(k)} \leq C_7$. Now Condition 4.37 for the mesh $x_j^{(k+1)}$ with $C_6$ replaced by $C_6 + C_7$ follows from Corollary 4.3.5. Indeed,

$$x_{j-1}^{(k+1)} \leq \sigma - (C_6 + C_7)\varepsilon = [\sigma - C_6\varepsilon] - C_7\varepsilon$$

implies, by (4.3.91) and (4.3.94), that

$$x_{j-1}^{(k+1)} \leq x_I^{(k)} - \varepsilon\lambda_I^{(k)}.$$

Hence, by Corollary 4.3.5, we have

$$h_j^{(k+1)} \leq C_7\varepsilon. \tag{4.3.95}$$

(ii) *Case A.* $x_{j-1}^{(k+1)} \leq \sigma - (C_6 + C_7)\varepsilon$. From (4.3.95) it follows that

$$x_j^{(k+1)} \leq \sigma - C_6\varepsilon.$$

Let $i$ be such that $x_j^{(k+1)} \in [x_{i-1}^{(k)}, x_i^{(k)}]$. Then $x_{i-1}^{(k)} \leq x_j^{(k+1)} \leq \sigma - C_6\varepsilon$, so $i \leq I$ by definition of $I$ and we also have

$$h_i^{(k)} \leq C_7\varepsilon \tag{4.3.96}$$

by our hypotheses. Since $i \leq I$, by $B_i^{(k)} \geq B_I^{(k)}$ and (4.3.93) we can apply Corollary 4.3.6 to obtain

$$h_j^{(k+1)} \leq \varepsilon\lambda_i^{(k)} = \frac{C_1\varepsilon}{C_3B_i^{(k)}N}\,. \tag{4.3.97}$$

Using (4.3.71) now yields

$$\frac{h_j^{(k+1)}}{\varepsilon}\exp\left(-\frac{x_{j-1}^{(k+1)}}{\varepsilon}\right) \leq \frac{C_1}{C_3N}\exp\left(\frac{x_i^{(k)} - x_{j-1}^{(k+1)}}{\varepsilon}\right). \tag{4.3.98}$$

But $h_j^{(k+1)} \leq C_7\varepsilon$ by (4.3.94) and (4.3.97), which together with (4.3.96) implies that $x_i^{(k)} - x_{j-1}^{(k+1)} \leq 2C_7\varepsilon$. From (4.3.98) we therefore get

$$\frac{h_j^{(k+1)}}{\varepsilon}\exp\left(-\frac{x_{j-1}^{(k+1)}}{\varepsilon}\right) \leq CN^{-1}. \tag{4.3.99}$$

*Case B.* $x_{j-1}^{(k+1)} > \sigma - (C_6 + C_7)\varepsilon$. By (4.3.69), we have

$$\begin{aligned}\exp\left(-\frac{x_{j-1}^{(k+1)}}{\varepsilon}\right) &= N^{-1}\exp\left(\frac{\sigma - x_{j-1}^{(k+1)}}{\varepsilon}\right) \\ &\leq N^{-1}e^{(C_6+C_7)} \leq CN^{-1}.\end{aligned} \tag{4.3.100}$$

Combining bounds (4.3.99) and (4.3.100) for cases A and B, for $j = 1, \ldots, N$ we get

$$\min\left\{\frac{h_j^{(k+1)}}{\varepsilon},\ 1\right\}\exp\left(-\frac{x_{j-1}^{(k+1)}}{\varepsilon}\right) \leq CN^{-1}. \tag{4.3.101}$$

Now (4.3.90) follows from (4.2.23b).

(iii) Repeat the proof of part (ii) replacing $C_6$ with $[C_6 + (m-1)C_7]$. □

**Proof of Theorem 4.30: main idea**

A complete rigorous proof of Theorem 4.30 is given in [25, §5.6]. Since the analysis is quite technical, here we only present the underlying idea.

First, we make the simplifying assumption that

$$\varepsilon \sim (C_5 N^{-1})^m \qquad \text{for some integer } \ m \geq 2.$$

We understand this in the sense that $C^{-1} \leq \varepsilon/(C_5 N^{-1})^m \leq C$ for some constant $C > 1$. We also assume a slightly modified version of Lemma 4.3.4:

$$h_1^{(k)} \sim h_1^{(k-1)}(C_5/N) \sim C_5^{-1}(C_5 N^{-1})^{k+1}, \qquad k = 0, \dots, m-1.$$

Then, by (4.3.69), we have

$$h_1^{(m-2)} \sim (C_5 N^{-1})^{m-1} \sim \varepsilon N \gg \sigma, \qquad h_1^{(m-1)} \sim (C_5 N^{-1})^m \sim \varepsilon \ll \sigma,$$

and consequently

$$h_1^{(m-1)} \ll \sigma \ll h_1^{(m-2)}.$$

Observe that if $x_j^{(m-1)} \leq \sigma$ for some $j$, then $x_j^{(m-1)} \leq h_1^{(m-2)}$, which, by the construction of our algorithm, implies

$$h_j^{(m-1)} = h_1^{(m-1)} \leq C_7 \varepsilon$$

for some sufficiently large constant $C_7$. In addition, we choose $C_7$ to satisfy (4.3.89). Now the mesh $\{x_i^{(m-1)}\}$ satisfies Condition 4.37 even for $C_6 = 0$, hence for some sufficiently large constant $C_6$ that satisfies (4.3.89). Finally, by Lemma 4.3.6(ii), we have

$$\left\| e^{(m)}(\cdot) \right\|_\infty = \left\| u^{(m)}(\cdot) - u(\cdot) \right\|_\infty \leq C N^{-1}. \qquad \square$$

**Remark 4.38.** Lemma 4.3.6(iii) and our argument show in fact that $\|e^{(k)}\|_\infty \leq CN^{-1}$ for $k = m, m+1, \dots, m+I'$, where $I'$ is some positive integer, but we expect that this inequality holds true for *all* $k \geq m$.

### 4.3.7 Numerical results

We present numerical results for two test problems. The first is the linear problem

$$-\varepsilon u''(x) - (pu)'(x) = f(x) \qquad x \in (0,1), \qquad u(0) = u(1) = 1, \tag{4.3.102a}$$

in which $p(x) = (x+1)^3$,

$$u(x) = \frac{1}{p(x)} \exp\left( -\frac{1}{\varepsilon} \int_0^x p(t)\, dt \right) + \exp(-x/2), \tag{4.3.102b}$$

and the right-hand side $f$ is chosen so that (4.3.102) is satisfied. While this means that $f$ depends on $\varepsilon$, the dependence is harmless and does not affect our theoretical results.

Table 4.1: Accuracy and speed of algorithm applied to the linear problem (4.3.102).

| | $\varepsilon = 1$ | | $\varepsilon = 10^{-2}$ | | | $\varepsilon = 10^{-8}$ | | | $\max_{\varepsilon=10^{-k},\, k=0,\dots,8}$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| $N$ | $\|e\|_{\infty,d}$ | $K$ | $\|e\|_{\infty,d}$ | $K$ | $\frac{K \ln N}{\ln(1/\varepsilon)}$ | $\|e\|_{\infty,d}$ | $K$ | $\frac{K \ln N}{\ln(1/\varepsilon)}$ | $\|e\|_{\infty,d}$ | $\|e\|_{\infty}$ |
| 32 | 4.25e-3 | 1 | 7.64e-2 | 2 | 1.51 | 8.55e-2 | 7 | 1.32 | 8.76e-2 | 1.13e-1 |
| | 1.00 | | 0.92 | | | 0.87 | | | 0.90 | 0.91 |
| 64 | 2.12e-3 | 1 | 4.04e-2 | 2 | 1.81 | 4.69e-2 | 6 | 1.35 | 4.71e-2 | 5.99e-2 |
| | 1.00 | | 1.03 | | | 0.96 | | | 0.97 | 0.76 |
| 128 | 1.06e-3 | 1 | 1.97e-2 | 1 | 1.05 | 2.40e-2 | 5 | 1.32 | 2.41e-2 | 3.53e-2 |
| | 1.00 | | 0.97 | | | 1.03 | | | 0.97 | 1.19 |
| 256 | 5.27e-4 | 1 | 1.01e-2 | 1 | 1.20 | 1.18e-2 | 4 | 1.20 | 1.23e-2 | 1.55e-2 |
| | 1.00 | | 0.97 | | | 0.92 | | | 0.98 | 0.74 |
| 512 | 2.63e-4 | 1 | 5.16e-3 | 1 | 1.35 | 6.26e-3 | 4 | 1.35 | 6.26e-3 | 9.27e-3 |

The second test problem is quasi-linear and is taken from [20, 26, 31]:

$$-\varepsilon u'' - (e^u)' + (\pi/2)\sin(\pi x/2)e^{2u} = 0 \quad \text{for } x \in (0,1), \qquad u(0) = u(1) = 0. \tag{4.3.103}$$

Note that $-\pi/2 \le u(\cdot) \le 0$, so on taking $b(x,u) = e^u$, one has $e^{-\pi/2} \le b_u(\cdot,\cdot) \le 1$ for the range of values of interest. The exact solution $u$ of (4.3.103) is unavailable, so we shall compare our computed solution $u^N$ with the approximate solution

$$u_A(x) = -\ln\left[(1+\cos(\pi x/2))\left(1 - e^{-x/2\varepsilon}/2\right)\right].$$

It is known [31, p. 137] that for all $x$ one has $u(x) = u_A(x) + \gamma(x)$, where $|\gamma(\cdot)| \le C\varepsilon$ for some $C$. In order that the comparison of $u^N$ with $u_A(x)$ be valid, we consider only the case $\varepsilon \le 10^{-5}$ in this test problem.

Numerical results for these problems are presented in Tables 4.1 and 4.2. As is customary for finite difference methods, errors are measured in the discrete maximum norm defined by $\|e\|_{\infty,d} = \max_{x_i} |u(x_i) - u^N(x_i)|$. This norm does not provide a reliable indicator of the accuracy of the computed solution if the mesh fails to resolve the boundary layer (for example, if $\varepsilon << N^{-1}$ and the mesh is uniform, then $\|e\|_{\infty,d}$ is small even though $\|e\|_\infty = \mathcal{O}(1)$). For this reason we also give for each $N$ the maximum of $\|e\|_\infty$ over all values of $\varepsilon$; this reliable measure of the error conclusively demonstrates the convergence of the algorithm uniformly in $\varepsilon$. The closeness of the values of $\|e\|_\infty$ and $\|e\|_{\infty,d}$ also shows that the algorithm does succeed in resolving the boundary layer.

Under each error ($\|e\|_\infty$ or $\|e\|_{\infty,d}$) is given the rate of convergence, expressed as a power of $N$ and computed in the standard way. These rates approach the value 1 as $N$ increases, confirming Theorem 4.29; the irregular nature of the convergence of $\|e\|_\infty$ to 1 is presumably due to the nonlinear nature of the adaptive algorithm. For each $\varepsilon$ and $N$, Tables 4.1 and 4.2 also list $K$, the number of iterations needed before the stopping criterion (4.3.56) is satisfied. We took $C_0 = 2$ in (4.3.56); values of $C_0$ close to 1 yielded slightly more accurate solutions but required many more iterations of the algorithm. The column $K \ln N/(\ln(1/\varepsilon))$ in the tables measures the ratio between the actual number of iterations and the predicted number of iterations $\mathcal{O}(\ln(1/\varepsilon)/(\ln N))$ guaranteeing first-order accuracy

Table 4.2: Accuracy and speed of algorithm applied to the quasi-linear problem (4.3.103).

| | $\varepsilon = 10^{-5}$ | | | $\varepsilon = 10^{-8}$ | | | $\max_{\varepsilon=10^{-k},\, k=5,\ldots,8}$ | |
|---|---|---|---|---|---|---|---|---|
| $N$ | $\Vert e\Vert_{\infty,d}$ | $K$ | $\frac{K \ln N}{\ln(1/\varepsilon)}$ | $\Vert e\Vert_{\infty,d}$ | $K$ | $\frac{K \ln N}{\ln(1/\varepsilon)}$ | $\Vert e\Vert_{\infty,d}$ | $\Vert e\Vert_{\infty}$ |
| 32 | 5.63e-2 | 5 | 1.51 | 5.36e-2 | 7 | 1.32 | 5.71e-2 | 1.01e-1 |
| | 0.91 | | | 0.87 | | | 0.91 | 1.42 |
| 64 | 3.01e-2 | 4 | 1.44 | 2.94e-2 | 6 | 1.35 | 3.03e-2 | 3.77e-2 |
| | 0.99 | | | 0.92 | | | 0.94 | 0.47 |
| 128 | 1.51e-2 | 3 | 1.26 | 1.55e-2 | 5 | 1.32 | 1.58e-2 | 2.71e-2 |
| | 0.88 | | | 1.04 | | | 0.95 | 1.44 |
| 256 | 8.20e-3 | 3 | 1.44 | 7.54e-3 | 4 | 1.20 | 8.20e-3 | 1.00e-2 |
| | 1.17 | | | 0.84 | | | 0.97 | 0.78 |
| 512 | 3.65e-3 | 2 | 1.08 | 4.20e-3 | 4 | 1.35 | 4.20e-3 | 5.82e-3 |

(see Theorem 4.30). We not only see that the algorithm requires very few iterations, even for small $\varepsilon$, but also that our estimate $\mathcal{O}(\ln(1/\varepsilon)/(\ln N))$ of the number of iterations is sharp.

In summary, our numerical results confirm our theoretical results.

Two figures are provided to aid the reader's understanding of the meshes computed by the algorithm when solving the linear problem (4.3.102). Figure 4.4, which should be read from bottom to top, shows the mesh after each iteration. Each of these meshes is labeled with the value of $C_0$ for which the stopping criterion (4.3.56) becomes an equation; one can deduce how many iterations the algorithm would have taken to converge given a priori a value of $C_0$. Recall that we took $C_0 = 2$ in the computations of this section. One can see from Figure 4.4 that taking any smaller value of $C_0$ will increase the number of iterations without significantly changing the mesh.

Figure 4.5 plots mesh-generating functions for the Bakhvalov and Shishkin meshes (a mesh $\{x_i\}_{i=0}^N$ is constructed from a mesh-generating function $x(t) : [0,1] \to [0,1]$ by the formula $x_i = x(i/N)$ for $i = 0, \ldots, N$) and the discrete analogue of such a function for the final computed mesh corresponding

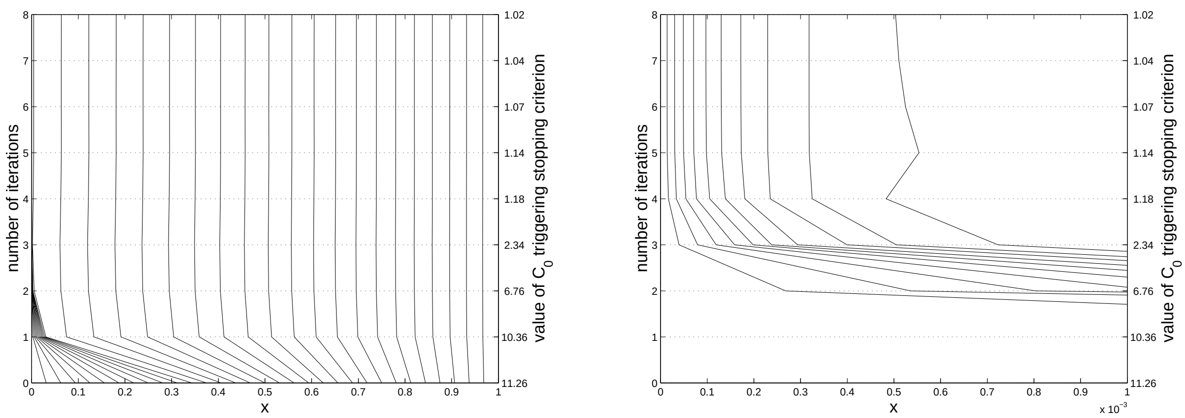



Figure 4.4: Mesh movement (right-hand picture enlarges boundary layer); $\varepsilon = 10^{-4},\ N = 32$.

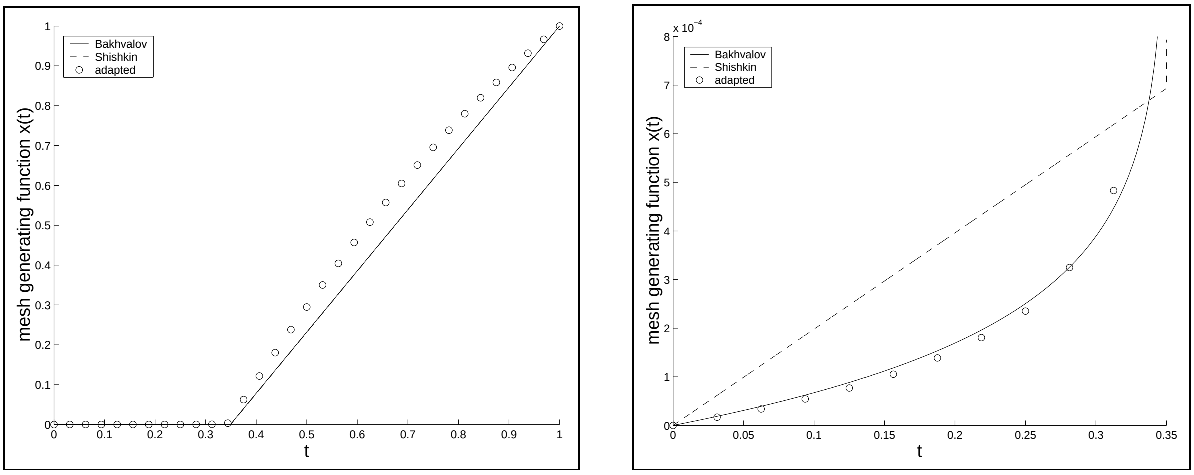



Figure 4.5: Comparison of adapted and a priori meshes (right-hand picture enlarges boundary layer); $\varepsilon = 10^{-4},\ N = 32$.

to $C_0 = 2$ (where only the discrete points $(i/N,\, x_i)$ are plotted). For the two special meshes, the user-chosen parameters were set in such a way that the transition from the fine to the coarse meshes agreed with a similar transition observed in the adaptive mesh. (In the Bakhvalov mesh (4.2.27) we choose the parameters $b = 0.35$ and $\lambda = 2$. In the Shishkin mesh (4.2.28) we take $\lambda = 2$, and we place $35\%$ of the nodes in the boundary layer, instead of the usual $50\%$, so that its mesh-generating function comes as close as possible to the mesh computed by the algorithm.) The left-hand side of Figure 4.5 shows that the Bakhvalov and Shishkin meshes coincide, but the meshes can easily be distinguished when the layer region is magnified. Figure 4.5 shows that the computed adaptive mesh is very close to a Bakhvalov mesh and is different in nature from a Shishkin mesh.

### 4.3.8 Possible generalizations

We believe that the analysis presented in §4.3 may be applicable to other differential equations, other discretizations, and other monitor functions. Hence now we shall point out the key ingredients of this analysis.

- First we recall Remark 4.26, which is concerned with existence of a solution to a discrete equidistribution problem. If a certain monitor function enjoys the following property, then, imitating the proof of Theorem 4.25, one can show that the corresponding fully discrete equidistribution problem has a solution:

  *There exists a quantity $Q(\varepsilon, N)$ such that for the discrete solution $u_i^N$ on an arbitrary mesh $\{x_i\}_{i=0}^N$ we have*

  $$1 \le M^N(x) \le \frac{1}{Q(\varepsilon, N)}\,.$$

  Here the lower bound can be always obtained for any positive monitor function by scaling. Note that the upper bound allows dependence on $N$ and $\varepsilon$, hence such a quantity $Q$ should exist for any reasonable monitor function.

- The next property is more restrictive. However, we believe this property reasonable and desirable; see Remark 4.27:

  *There exists a positive constant $C_1$ such that for the computed solution $u_i^N$ on an arbitrary mesh $\{x_i\}_{i=0}^N$ we have*

  $$1 \le M^N(x); \qquad \int_0^1 M^N(x)\,dx \le C_1. \tag{4.3.104}$$

  Furthermore, this property immediately implies that for any mesh produced by the moving mesh algorithm we have

  $$h_i^{(k)} \le C_1 N^{-1};$$

  see Corollary 4.3.2.

- The most natural way to formulate a suitable equidistribution problem seems to be based on a certain, preferably sharp, a posteriori error estimate of the type

  $$\|\text{error}\| \le C \max_i |h_i M_i|^p,$$

  where $p$ is usually the order of the numerical method. If such an estimate is available, combining it with (4.3.104), for the solution of the corresponding (quasi-)equidistribution problem we obtain

  $$\|\text{error}\| \le C N^{-p}.$$

  E.g., we used the a posteriori error estimate (4.3.49), where $p = 1$ and the monitor function $M_i$ is the arc-length monitor function (4.3.54). Combining this with (4.3.61), we obtained Theorems 4.28 and 4.29.

- Finally, we recall the most subtle property that we used, which is a sharp lower bound of the monitor function on an arbitrary mesh, such as given by Lemma 4.3.3 for the arc-length monitor function. Note that a *lower* bound for the monitor function enables us to get *upper* bounds for the local mesh sizes of the intermediate meshes produced by the algorithm, and hence allows a very good understanding of how the algorithm will work.

In summary, we believe that the above four ingredients, although they are not stated completely rigourously here, are sufficient to extend our moving-mesh method analysis to other problems and other 1d monitor functions; see [8] for a more detailed discussion.

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