LOGARITHM CANNOT BE REMOVED IN MAXIMUM NORM ERROR ESTIMATES FOR LINEAR FINITE ELEMENTS IN 3D

NATALIA KOPTEVA

Abstract. For linear finite element discretizations of the Laplace equation in three dimensions, we give an example of a tetrahedral mesh in the cubic domain for which the logarithmic factor cannot be removed from the standard upper bounds on the error in the maximum norm.

1. Introduction

Consider the problem

$$-\triangle u = f \quad \text{in } \Omega\subset\mathbb{R}^d, \qquad u=0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $d\in\{2,3\}$, and $\triangle$ is the Laplace operator (defined by $\triangle=\partial_x^2+\partial_y^2+\partial_z^2$ for $d=3$). It is well known that a standard linear finite element approximation u_h for (1.1) on a quasi-uniform mesh of diameter h is quasi-optimal in the sense that

$$\|u-u_h\|_{L_\infty(\Omega)} \lesssim h^2|\ln h|\,\|u\|_{W^2_\infty(\Omega)}, \tag{1.2}$$

see, e.g., [8, Theorem 2], [6, Theorem 3.1] and [9, Theorem 5.1], for, respectively, polygonal, convex polyhedral and smooth domains. It was also shown in [3, 2, 1] that when $d=2$, the logarithmic factor cannot be removed from (1.1). Surprisingly, it appears that this is still an open question for $d=3$. In this note, we address this by giving an example of a tetrahedral mesh in the cubic domain $\Omega=(0,1)^3$ such that, under certain conditions on f, one has $\|u-u_h\|_{L_\infty(\Omega)}\ge C^*h^2|\ln h|$, where a positive constant C^* depends on f, but not on h.

Our mesh is constructed using the two-dimensional triangulation used in a less known paper [1]. Furthermore, the general approach here also follows [1] in that, on a particular mesh, we estimate the error of a finite element method using its explicit finite difference representation.

The paper is organized as follows. In §2 we describe the mesh used in our example, give a finite difference representation of the resulting finite element method, and state the main results. Most of the proofs are deferred to §3.

Notation. We write $a\lesssim b$ when $a\le Cb$ with a generic constant C depending on Ω and f, but not on h. For any domain $\mathcal{D}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, the notation $C^{k,\alpha}(\bar{\mathcal{D}})$ is used for the standard Hölder space consisting of functions whose kth order partial derivatives are uniformly Hölder continuous with exponent $\alpha\in(0,1)$. We also use the standard space $L_\infty(\mathcal{D})$ and the related Sobolev space $W^2_\infty(\mathcal{D})$.

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2. Mesh description. Main results

Consider the standard linear finite element method with lumped-mass quadrature (for the case without quadrature, see Corollary 2.2). With the standard finite element space S_h of continuous piecewise-linear functions vanishing on $\partial\Omega$, the computed solution $u_h \in S_h$ is required to satisfy

$$\int_\Omega \nabla u_h \cdot \nabla v_h = \int_\Omega (f v_h)^I \qquad \forall v_h \in S_h, \tag{2.1}$$

where $v^I \in S_h$, for any $v \in C(\bar\Omega)$, denotes its standard piecewise-linear Lagrange interpolant.

Let $\Omega := (0,1)^3$ and define a tetrahedral mesh in Ω as follows. With an even integer N, set $h := N^{-1}$. Starting with the uniform rectangular grid $\{(x_i, y_j) = (ih, jh)\}_{i,j=0}^N$, define a triangulation of $(0,1)^2$ by drawing diagonals as on Fig. 1 (left) [1]. Note that each interior node is shared by 6 triangles, except for $(\frac12, \frac12)$, which is shared by 4 triangles.

Next, partition Ω into triangular prisms by constructing a tensor product of the two-dimensional triangulation in the (x,y)-plane and the uniform grid $\{z_k = kh\}$ in the z-direction. Finally, divide each triangular prism into three tetrahedra as on Fig. 1 (centre, right) using method A or method B. Note that for the resulting tetrahedral mesh to be well-defined, no prisms of the same type can share a vertical face. Such a well-defined mesh is generated if, for example, the shadowed triangles on Fig. 1 (left) correspond to method B of prism partition. Note also that it will be convenient to evaluate the local contributions to (2.1) associated with triangular prisms, the set of which (rather than of all tetrahedra) will be denoted $\mathcal{T}$.

Introduce the notation $U_{ijk} := u_h(x_i, y_j, z_k)$ for nodal values of u_h and the standard finite difference operators defined, for $t = x, y, z$, by

$$\delta_t^2 U_{ijk} := \frac{u_h(P_{ijk} + h\mathbf{i}_t) - 2u_h(P_{ijk}) + u_h(P_{ijk} - h\mathbf{i}_t)}{h^2}, \qquad P_{ijk} := (x_i, y_j, z_k),$$

where $\mathbf{i}_t$ is the unit vector in the t-direction. Now, we claim that the finite element method (2.1) can be rewritten, for $i, j, k = 1, \ldots, N-1$, as

$$\mathcal{L}^h U_{ijk} := -(\delta_x^2 + \delta_y^2 + \gamma_{ij}\delta_z^2) U_{ijk} = \gamma_{ij}\, f(x_i, y_j, z_k), \tag{2.2a}$$

$$\text{where} \quad \gamma_{ij} := \begin{cases} \frac23, & \text{if } i = j = \frac{N}{2}, \\ 1, & \text{otherwise,} \end{cases} \tag{2.2b}$$

subject to $U_{ijk} = 0$ for any $(x_i, y_j, z_k) \in \partial\Omega$.

Indeed, for a particular prism $T \in \mathcal{T}$, using the notation E_{lm} for the edge connecting vertices l and m (see Fig. 1), and assuming that E_{23} is parallel to the x-axis, a calculation shows that

$$\begin{aligned} \int_T \nabla u_h \cdot \nabla v_h = \tfrac13 |T| \Big\{ & 2(\partial_x u_h\, \partial_x v_h)\big|_{E_{23}} + (\partial_x u_h\, \partial_x v_h)\big|_{E_{2'3'}} \\ & + (\partial_y u_h\, \partial_y v_h)\big|_{E_{12}} + 2(\partial_y u_h\, \partial_y v_h)\big|_{E_{1'2'}} + \sum_{l=1}^{3} (\partial_z u_h\, \partial_z v_h)\big|_{E_{ll'}} \Big\}. \end{aligned} \tag{2.3}$$

Here we used the fact that each tetrahedron has an edge parallel to each of the coordinate axes. Also, within the prism T, each of such edges belongs to exactly 1 tetrahedron, except for E_{23} and $E_{1'2'}$, while each of the latter is shared by 2 tetrahedra.

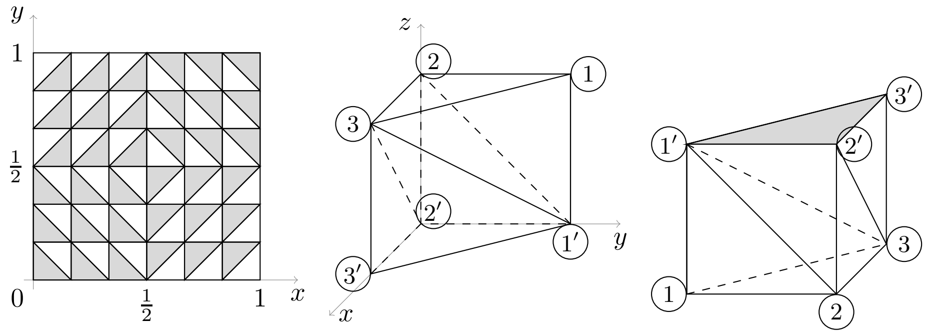

FIGURE 1. Two-dimensional triangulation in $(0,1)^2$ for $N = 6$ (left); partition of a triangular prism into 3 tetrahedra using method A (centre) and method B (right).

Next, set $v_h := \phi_{ijk}$ in (2.1), where $\phi_{ijk} \in S_h$ equals 1 at (x_i, y_j, z_k) and vanishes at all other mesh nodes. With this v_h, adding the contributions of (2.3) to the left-hand side of (2.1), one gets

$$\tfrac{1}{3}|T|\Big\{-6\delta_x^2 U_{ijk} - 6\delta_y^2 U_{ijk} - 6\gamma_{ij}\delta_z^2 U_{ijk}\Big\} = \tfrac{1}{12}|T|\Big\{24\gamma_{ij}\, f(x_i, y_j, z_k)\Big\}.$$

For the right-hand side here, we used the observation that each node (x_i, y_j, z_k) is shared by $24\gamma_{ij}$ tetrahedra. Clearly, the above relation immediately implies (2.2a).

Now that our finite element method (2.1) is represented as a finite difference scheme (2.2), note that if γ_{ij} were equal to 1 for all i, j, we would immediately get the standard finite-difference error bound $\|u - u_h\|_{L_\infty(\Omega)} \lesssim h^2$ [7]. However, $\gamma_{\frac{N}{2},\frac{N}{2}} = \frac{2}{3} \neq 1$ results in a slightly worse convergence rate, consistent with (1.2).

Theorem 2.1. *Let $\Omega = (0,1)^3$ and $f := F(x,y)\sin(\pi z)$ in (1.1), with any $F \in C^{2,\alpha}([0,1]^2)$ subject to $F = 0$ at the corners of $(0,1)^2$, and $F(\frac{1}{2}, \frac{1}{2}) = \|F\|_\infty > 0$ (where $\|\cdot\|_\infty$ is used for the norm in $L_\infty((0,1)^2)$). Then $u \in C^{2,\alpha}(\bar\Omega) \subset W^2_\infty(\Omega)$, and there exists a positive constant C^* depending on F, but not on h, such that for the finite element approximation u_h obtained using (2.1) on the above tetrahedral mesh with a sufficiently small h, one has $\|u - u_h\|_{L_\infty(\Omega)} \geq C^* h^2 |\ln h|$.*

Corollary 2.2. *The result of Theorem 2.1 remains valid for a version of (2.1) without quadrature.*

Proof. Let $\bar u_h$ be the finite element solution obtained using linear finite elements without quadrature. Then $|u_h - \bar u_h| \lesssim h^2 |\ln h|^{2/3} \|f\|_{W^2_\infty(\Omega)}$; see [4, final 3 lines in Appendix A]. The desired assertion follows. ☐

Remark 2.3 (Additional solution smoothness does not improve the accuracy). Under the conditions of Theorem 2.1, for any $m \in \mathbb{N}$, one can choose $F \in C^{2m,\alpha}([0,1]^2)$ subject to $F = 0$ in small neighbourhoods of the corners of $(0,1)^2$. Then the theorem remains valid, while now $u \in C^{2m+2,\alpha}(\bar\Omega)$ (as an inspection of the proof of this theorem reveals; see, in particular, Lemma 3.1 and Remark 3.2). Thus, additional smoothness of the exact solution would not improve the accuracy of the finite element method.

3. Proof of Theorem 2.1

We split the proof into a number of lemmas, which involve an auxiliary function $w(x,y)$, as well as its finite difference approximation W_{ij}, defined by

$$\mathcal{M}w := -(\partial_x^2+\partial_y^2)w+\pi^2 w = F \quad \text{in } (0,1)^2, \tag{3.1}$$

$$\mathcal{M}^h W_{ij} := -(\delta_x^2+\delta_y^2)W_{ij}+\gamma_{ij}\,\pi^2 W_{ij} = \gamma_{ij}\,F(x_i,y_j), \tag{3.2}$$

the latter for $i,j=1,\ldots,N-1$, subject to $w=0$ and $W_{ij}=0$ on the boundary of $(0,1)^2$.

Lemma 3.1. *Under the conditions of Theorem 2.1 on f, the solution of problem (1.1) is $u=w(x,y)\sin(\pi z)$, where $w\in C^{2,\alpha}([0,1]^2)$ is a unique solution of (3.1) and $|\partial_x^4 w|+|\partial_y^4 w|\lesssim 1$ in $[0,1]^2$. Furthermore,*

$$\widetilde{F}(\tfrac12,\tfrac12)\ge \|F\|_\infty/\cosh(\tfrac12\pi), \qquad \textit{where } \widetilde{F}:=F-\pi^2 w. \tag{3.3}$$

Proof. The regularity of w and the bounds on its pure fourth partial derivatives follow from [11, Remark 4 in §8]. For (3.3), in view of $F(\frac12,\frac12)=\|F\|_\infty$, it suffices to show that

$$\|F\|_\infty-\pi^2 w(x,y)\ge B(x):=\|F\|_\infty\frac{\cosh(\pi(x-\frac12))}{\cosh(\frac12\pi)}\,.$$

The latter is obtained by an application of the maximum principle for the operator $\mathcal{M}$ as $\mathcal{M}(\|F\|_\infty-\pi^2 w)\ge 0=\mathcal{M}B$, while $\|F\|_\infty-\pi^2 w=\|F\|_\infty\ge B$ on $\partial\Omega$. $\square$

Remark 3.2. In Lemma 3.1, we have $w\in C^{2,\alpha}([0,1]^2)$ rather than $w\in C^{4,\alpha}([0,1]^2)$, as the latter requires additional corner compatibility conditions on F [11, Theorem 3.1], while bounded fourth pure partial derivatives of w are sufficient for the finite-difference-flavoured analysis that yields the crucial bound (3.4) below.

Lemma 3.3. *For U_{ijk} of (2.2) and W_{ij} of (3.2) one has $|U_{ijk}-W_{ij}\sin(\pi z_k)|\lesssim h^2$.*

Proof. First, note that $-\delta_z^2[\sin(\pi z_k)]=\lambda_h\sin(\pi z_k)$ where $\lambda_h:=\frac{4}{h^2}\sin^2(\frac12\pi h)=\pi^2+O(h^2)$ [7, §II.3.2]. Combining this with (2.2a) and (3.2) yields

$$\begin{aligned}\mathcal{L}^h[W_{ij}\sin(\pi z_k)] &= \big[-(\delta_x^2+\delta_y^2)W_{ij}+\gamma_{ij}\lambda_h W_{ij}\big]\sin(\pi z_k)\\ &= \underbrace{\gamma_{ij}F(x_i,y_j)\sin(\pi z_k)}_{=\mathcal{L}^h U_{ijk}}+O(h^2),\end{aligned}$$

where we also used $|W_{ij}|\le \pi^{-2}\|F\|_\infty$. The desired result follows by an application of the discrete maximum principle for the operator $\mathcal{L}^h$. $\square$

Lemma 3.4 ([1])**.** *Let w solve (3.1), and $\widetilde{W}_{ij}$ satisfy $-(\delta_x^2+\delta_y^2)\widetilde{W}_{ij}=\gamma_{ij}\widetilde{F}(x_i,y_j)$, subject to $\widetilde{W}_{ij}=0$ at the boundary nodes. Then there exists a constant C_1, independent of h and $\widetilde{F}$, such that*

$$w(\tfrac12,\tfrac12)-\widetilde{W}_{\frac N2,\frac N2}\ge C_1 h^2|\ln h|\,\widetilde{F}(\tfrac12,\tfrac12)-O(h^2). \tag{3.4}$$

Proof. Recalling that $\widetilde{F}=F-\pi^2 w$, rewrite (3.1) as $-(\partial_x^2+\partial_y^2)w=\widetilde{F}$. Now $\widetilde{W}_{ij}$ may be considered a finite difference approximation of w, for which (3.4) is obtained in [1]. Note that C_1 is independent of h and $\widetilde{F}$ (as it is related to the discrete Green's function for the operator $-(\delta_x^2+\delta_y^2)$; see also Remark 3.5). $\square$

Remark 3.5 (C_1 in (3.4) [1]). It is noted in [1] that $\widetilde{W}_{ij}$ of Lemma 3.4 allows the representation $\widetilde{W}_{ij} = \mathring{W}_{ij} - \frac{1}{3}h^2 G_{ij}\,\widetilde{F}(\frac{1}{2},\frac{1}{2})$, where $-(\delta_x^2+\delta_y^2)\mathring{W}_{ij} = \widetilde{F}(x_i,y_j)$, subject to $\mathring{W}_{ij}=0$ at the boundary nodes, while G_{ij} is the discrete Green's function for the operator $-(\delta_x^2+\delta_y^2)$ associated with the node $(\frac{N}{2},\frac{N}{2})$. To be more precise, $-(\delta_x^2+\delta_y^2)G_{ij}$ equals h^{-2} if $i=j=\frac{N}{2}$ and 0 otherwise, subject to $G_{ij}=0$ at the boundary nodes. Furthermore, there is a constant $C_1>0$ independent of h (as well as of $\widetilde{F}$) such that $G_{\frac{N}{2},\frac{N}{2}} \ge 3C_1|\ln h|$. (For the latter, Andreev [1] uses [5, (16)]; see also [10, Lemma 6] for a similar result in the finite element context.) Finally, for $\mathring{W}_{ij}$, one has a standard finite-difference error bound $|\mathring{W}_{ij} - w(x_i,y_j)| \lesssim h^2$. The above observations yield (3.4).

Remark 3.6. It was also pointed out in [1] that $\widetilde{W}_{ij} = w_h(x_i,y_j)$, where w_h is a linear finite element solution for $-(\partial_x^2+\partial_y^2)w = \widetilde{F}$ obtained using the two-dimensional triangulation of $(0,1)^2$ shown on Fig. 1 (left).

Lemma 3.7. *Under the conditions of Theorem 2.1 on F, for the solutions of* (3.1) *and* (3.2) *with a sufficiently small h, one has* $\max_{i,j=0,\dots,N}|w(x_i,y_j)-W_{ij}| \ge C_0 h^2|\ln h|$, *with a positive constant C_0 that depends on F, but not on h.*

Proof. Set $C_0 := 4\pi^{-2}C_1\|F\|_\infty/\cosh(\frac{1}{2}\pi)$ and $e_{ij} := \widetilde{W}_{ij} - W_{ij}$, where C_1 and $\widetilde{W}_{ij}$ are from Lemma 3.4. Note that $-(\delta_x^2+\delta_y^2)e_{ij} = \gamma_{ij}\pi^2[W_{ij} - w(x_i,y_j)]$ (in view of $\widetilde{F} = F - \pi^2 w$). Also, for the auxiliary $B_{ij} := \frac{1}{2}\pi^2 C_0 h^2|\ln h|\{\frac{1}{4}-(x_i-\frac{1}{2})^2\}$, note that $-(\delta_x^2+\delta_y^2)B_{ij} = \pi^2 C_0 h^2|\ln h|$. We now prove the desired bound by contradiction. Assume that $\max_{i,j}|w(x_i,y_j)-W_{ij}| < C_0h^2|\ln h|$. Then $-(\delta_x^2+\delta_y^2)[B_{ij}\pm e_{ij}] > 0$. Now, an application of the discrete maximum principle yields $B_{ij}\pm e_{ij} \ge 0$. So $|e_{ij}| \le B_{ij}$, so $|\widetilde{W}_{ij} - W_{ij}| = |e_{ij}| \le \frac{1}{8}\pi^2 C_0 h^2|\ln h|$. Combining this with (3.4), one concludes that

$$w(\tfrac{1}{2},\tfrac{1}{2}) - W_{\frac{N}{2},\frac{N}{2}} \ge \underbrace{\left\{C_1\widetilde{F}(\tfrac{1}{2},\tfrac{1}{2}) - \tfrac{1}{8}\pi^2 C_0\right\}}_{\ge \frac{1}{8}\pi^2 C_0} h^2|\ln h| - O(h^2) \ge C_0 h^2|\ln h|$$

for a sufficiently small h, where we also used $C_1\widetilde{F}(\frac{1}{2},\frac{1}{2}) \ge \frac{1}{4}\pi^2 C_0$ (in view of (3.3)). The above contradicts our assumption that $\max_{i,j}|w(x_i,y_j)-W_{ij}| < C_0h^2|\ln h|$. □

Proof of Theorem 2.1. It now suffices to combine the findings of Lemmas 3.1, 3.3 and 3.7. In particular, $u(x_i,y_j,\frac{1}{2}) - U_{i,j,\frac{N}{2}} = w(x_i,y_j) - W_{i,j} + O(h^2)$, by Lemmas 3.1 and 3.3. So, in view of Lemma 3.7, one gets $\|u-u_h\|_{L_\infty(\Omega)} \ge C^*h^2|\ln h|$ with any fixed positive constant $C^* < C_0$. □

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Department of Mathematics and Statistics, University of Limerick, Limerick, Ireland
E-mail address: `natalia.kopteva@ul.ie`