17th Annual Workshop on Numerical Solution of Problems with Layer Phenomena

11th–13th November, 2020
University of Limerick

17th Annual Workshop on Numerical Solution of Problems with Layer Phenomena

Dedicated to the memory of Piet Hemker

Wednesday 11th to Friday 13th November 2020

Wednesday 11th November

3:00 pm **Opening of Workshop**

3:30 pm Eugene O'Riordan, Dublin City University, Ireland

4:00 pm Róisín Hill, National University of Ireland Galway, Ireland

4:30 pm Martin Stynes, Beijing Computational Science Research Center, China

5:00 Virtual reception. (Please supply your own chosen beverage.)

Thursday 12th November

11:00 am Christos Xenophontos, University of Cyprus

11:30 am Alexander Zadorin, Sobolev Institute of Mathematics, Novosibirsk, Russia

12:00 pm Nikolai Nefedov, Lomonosov Moscow State University, Russia

Lunch Break

3:00 pm Scott MacLachlan, Memorial University of Newfoundland, Canada

3:30 pm Faiza Alssaedi, National University of Ireland Galway, Ireland

4:00 pm Gabriel Barrenechea, Univeristy of Strathclyde, Scotland, UK

Friday 13th November

11:00 am José Luis Gracia, University of Zaragoza, Spain

11:30 am Vladimir Volkov, Lomonosov Moscow State University, Russia

12:00 pm Natalia Kopteva, University of Limerick, Ireland

Closing of Workshop

Abstracts

Numerical solution of fourth-order real and complex-valued singularly perturbed problems

Faiza Alssaedi & Niall Madden *(National University of Ireland Galway)*

We are interested in the numerical solution of two families of fourth-order singularly perturbed problems. First, we consider real-valued reaction-diffusion equations. Our model differential equation is

$$-\varepsilon u^{(4)}(x) + au''(x) - bu(x) = f(x) \quad \texttt{on} \quad \Omega := (0,1). \tag{1}$$

The coefficient functions, a and b, and right-hand side function, f, are real-valued functions on the interval Ω.

The second set of problems we consider are complex-valued; it is similar to but a, b and f may be complex-valued. Thus, so too is the solution u.

As ever, ε is a positive, real-valued parameter, and we assume $0 < \varepsilon \leq 1$, but typically have that $\varepsilon \ll 1$. That is, the differential equations are singularly perturbed nature.

Both problems we consider are equipped with the boundary conditions

$$u(0) = u''(0) = 0, \qquad u(1) = u''(1) = 0. \tag{2}$$

When solving these so-called "simply supported" problems, it is typical to transform them into (weakly) coupled systems of second-order reaction-diffusion equations, expressed in terms of the unknowns u and $w = u''$.

When analysing a finite element method for solving this system, it is usually assumed that the coupling matrix of the coupled system is pointwise coercive, which is sometimes referred to as to as "positive definite but not necessarily symmetric"; see, e.g., [1]. However, we show that the standard transformation above does not yield a coupling matrix that satisfies the coercivity condition.

This motivates us to propose a new transformation, first for the real-valued problem, which involves a coefficient-depended parameter. We shall show how to, under reasonable assumptions, determine the value of the parameter in the transformation that ensures that the coupling matrix is coercive.

We then show how to extend the ideas to the complex-value problem, so that it can be rendered as as a system of four second-order real-valued problems, with a coercive coupling matrix on the zero-order terms.

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Divergence-free finite element methods for an inviscid flow model

Gabriel Barrenechea, *(University of Strathclyde, Scotland, UK)*

Abstract: In this talk I will review some recent results on the stabilisation of linearised incompressible inviscid flows (or, with a very small viscosity). The partial differential equation is a linearised incompressible equation similar to Euler's equation, or Oseen's equation in the vanishing viscosity limit. In the first part of the talk I will present results on the well-posedness of the partial differential equation itself. From a numerical methods' perspective, the common point of the two works is the aim of proving the following type of estimate:

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2} \leq C h^{k+\frac{1}{2}} |\boldsymbol{u}|_{H^{k+1}} \,, \tag{3}$$

where $\boldsymbol{u}$ is the exact velocity and $\boldsymbol{u}_h$ is its finite element approximation. In the estimate above, the constant C is independent of the viscosity (if the problem has a viscosity), and, more importantly, independent of the pressure. This estimate mimics what has been achieved for stabilised methods for the convection-diffusion equation in the past. Nevertheless, up to the best of our knowledge, had only been achieved for Oseen's equation using equal-order elements, and assuming a regular pressure.

I will first present results of discretisations using $H(div)$-conforming spaces, such as Raviart-Thomas, or Brezzi-Douglas-Marini where an estimate of the type (3) is proven (besides an optimal estimate for the pressure). In the second part of the talk I will move on to H^1-conforming divergence-free elements, with the Scott-Vogelius element as the prime example. In this case, due to the H^1-conformity, the need of an extra control of the vorticity equation, and some appropriate jumps, appears. So, a new stabilised finite element method adding control on the vorticity equation is proposed. The method is independent of the pressure gradients, which makes it pressure-robust and leads to pressure-independent error estimates such as (3). Finally, some numerical results will be presented and the present approach will be compared to the classical residual-based SUPG stabilisation.

This work is a collaboration with N. Ahmed (Gulf University for Science and Technology, Kuwait), E. Burman (UCL, UK), J. Guzmán (Brown, USA), and A. Linke and C. Merdon, from WIAS, Berlin.

Numerical approximations to singularly perturbed convection-diffusion parabolic problems with a discontinuous initial condition

José Luis Gracia, (*Institute of Mathematics and Applications and University of Zaragoza, Spain)*

This is a companion talk to the presentation *Singularly perturbed convection-diffusion parabolic problems with a discontinuous initial condition* by the same authors. Some representative test examples of singularly perturbed parabolic problems of convection-diffusion type with a discontinuous initial condition are examined. The solutions are approximated using a numerical/analytical scheme. The solution u of each test example is decomposed into the sum of two components $u = s + y$. The analytical function s associated with the initial discontinuity is identified and separated off from u, while the component y, whose initial condition is a continuous function, is approximated with a finite difference scheme defined on a piecewise uniform Shishkin mesh. In approximating these test problems, it is shown that it is crucial to identify if the convective coefficient depends solely on the time variable or not. The parameter-uniform global convergence of the numerical approximations for all the test problems is discussed.

This research is joint work with **Eugene O'Riordan** from *the Dublin City University and the University of Limerick, Ireland.*

Generating layer-adapted meshes using MPDEs

Róisín Hill & Niall Madden *(National University of Ireland, Galway)*

We consider the numerical solution, by finite elements methods, of singularly-perturbed reaction-diffusion equations whose solutions exhibit boundary layers. Our model problem is

$$-\varepsilon^2 \Delta u + bu = f, \quad \in \Omega \subseteq \mathbb{R}^d, \quad \text{with } u|_{\partial\Omega} = 0, \tag{4}$$

where $d = 1, 2$. Here b and f are given functions with $b \geq \beta^2 > 0$ and $0 < \varepsilon \ll 1$. Our interest lies in developing parameter-robust methods, where the quality of the solution is independent of the value of the perturbation parameter, ε. One way to achieving this is to use layer resolving methods based on meshes that concentrate their mesh points in regions of large variations in the solution.

The graded *a priori* Bakhvalov mesh resolves layers and yields parameter robust solutions [1]. A Bakhvalov mesh for (4), with $d = 1$ and b and f such that the solution has one layer, near $x = 0$, can be generated by equidistributing the mesh density function

$$\rho(x) = \max\left\{1, K\frac{\beta}{\varepsilon}\exp\left(-\frac{\beta x}{\sigma\varepsilon}\right)\right\}. \tag{5}$$

Here K is a positive constant that determines the proportion of mesh points that resolve the layer and $\sigma > 0$ is a constant that depends on the underlying method, see [2].

We investigate the use of Mesh PDEs (MPDEs), as first presented in [3], to generate layer resolving meshes that yield parameter robust solutions to (4). Our chosen one-dimensional MPDE is

$$(\rho(x)x'(\xi))' = 0 \quad \text{for } \xi \in (0, 1), \tag{6}$$

and impose boundary conditions that fix the end points of the resulting mesh. When ρ is as defined in (5), the solution to (6) yields a Bakhvalov mesh.

Of course, this means that one must solve (6) numerically. The most obvious fixed point method converges, but extremely slowly, at a rate that depends adversely on N and ε. So we devise a scheme based on h-refinement that ensures rapid convergence of the fixed point method.

We then present generalisations of (6) for generating meshes in more complicated settings, including a two-dimensional problem with space-varying ε, along with supporting numerical results, generated using FEniCS [4]. The algorithms, code, and results, are fully described in [5].

Acknowledgements Supported by the Irish Research Council (GOIPG/2017/463).

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Upper and lower solutions in the numerical analysis of semilinear singularly perturbed differential equations: rectangular domain with zero-order compatibility conditions

Natalia Kopteva *(University of Limerick, Ireland)*

In this talk, we shall discuss an extension of the method of differential inequalities, which is a well-established tool in the asymptotic analysis of singularly perturbed differential equations [7], to the error analysis of numerical approximations of such equations. For the case of singularly perturbed semilinear ordinary differential equations, this approach was used in [3, 8, 6], while an elliptic equation of this type in a smooth domain was addressed in [5].

We shall first recall the basics of this approach in the context of a 1d semilinear reaction-diffusion equation following [6] and a similar problem posed in a smooth 2d domain [5]. In the main part of the talk, we shall look into this problem in a square domain. The asymptotic analysis of the latter problem in the linear case was first addressed by V.F. Butuzov in 1973 [2]. Compared to smooth domains, one now additionally deals with corner layers. The semilinear case of this problem will be considered under the assumptions made in [4], which imply that the solution is in the Hölder space $C^{1,\lambda}(\bar{\Omega})$. For this problem, new error bounds for its numerical approximations will be obtained using upper and lower discrete solutions. Note that the latter are constructed employing the methodology from [1] for the linear case under the zero-order compatibility conditions at the corners. In future, this approach will be used for the more challenging case of polygonal domains.

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Parameter-robust preconditioners for singularly perturbed convection-diffusion equations

Scott MacLachlan & Niall Madden & Thai Anh Nhan *(Memorial University of Newfoundland)*

Embedded in many algorithms for the simulation of problems in computational fluid dynamics is the need to accurately and efficiently model boundary layers in the flow. In this talk, we consider the simpler model of a scalar convection-diffusion equation, where the structure of the arising boundary layers is well-known in existing theory. This structure leads to layer-adapted meshes on which standard finite-difference discretizations give approximation of the solution that is independent of the singular perturbation parameter. We propose preconditioners that also reflect the layer-adapted mesh structure, resulting in "parameter-robust" discretizations and solvers for convection-diffusion equations. In this talk, we will present theory for the one-dimensional case, along with heuristics and numeri

Periodic and Stationary Solutions of Nonlinear Reaction-Diffusion Problems with Singularly Perturbed Boundary Conditions

Nikolay Nefedov *(Department of Mathematics, Faculty of Physics, Lomonosov Moscow State University, 119899 Moscow, Russia)*

We present an extension of asymptotic method of differential inequalities (see [1] and references there in) to new classes of problems with singularly perturbed boundary conditions. We illustrate our results by consideration of boundary and interior layer type stationary solutions of the reaction-diffusion problem

$$\begin{cases} \varepsilon^2 \dfrac{\partial^2 v}{\partial x^2} - \dfrac{\partial v}{\partial t} = f(v, x, \varepsilon), \quad x \in (-1; 1), \quad t > 0; \\ \varepsilon \dfrac{dv}{dx}(\mp 1, t) = u^{(\mp)}, \; v(x, 0) = v_{init}(x, \varepsilon) \end{cases} \tag{7}$$

and its extension to multidimensional in space variable case. Hear the source function f is sufficiently smooth.

We also present the extension of our results for periodic parabolic problem with singularly perturbed boundary conditions

$$\begin{aligned} & \varepsilon^2 \left(\triangle u - \frac{\partial v}{\partial t} \right) - f(u, x, t, \varepsilon) = 0, \\ & (x, t) \in D_t := \{(x, t) \in R^3 : x \in D, t \in R\}, \\ & \varepsilon \frac{\partial u(x, t, \varepsilon)}{\partial n} = u_\Gamma(x, t), \quad x \in \Gamma, t \in R, \\ & u(x, t, \varepsilon) = u(x, t + T, \varepsilon), \quad x \in \bar{D}, t \in R, \end{aligned} \tag{8}$$

Problems of the types (7) and (8) have a lot of applications (see [3], [4] and references therein). The conditions of the stability or instability stationary solutions are presented. In particular, we prove the stability of stationary solutions with of non monotone boundary layer. The study conducted in this work gives an answer about local and non-local attraction domain of the stable stationary solutions or stable periodic solutions. The results are used for the stabilization of unstable solutions of the corresponding Dirichlet problems. They suggest an numerical approach of computing unstable solutions of various applied problems by numerical methods of stationing.

ACKNOWLEDGMENTS.
This work was supported by the Russian Fund of Basic Research (project N 19–01–000327).

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Singularly perturbed convection-diffusion parabolic problems with a discontinuous initial condition

Eugene O'Riordan, *(Dublin City University and University of Limerick, Ireland)*

A singularly perturbed parabolic problem of convection-diffusion type with a discontinuous initial condition is examined. A particular complimentary error function is identified which matches the discontinuity in the initial condition. The difference between this analytical function and the solution of the parabolic problem is approximated numerically. If the convective coefficient only depends on time, then a standard layer-adapted method can be used. Otherwise, a coordinate transformation is used so that a layer-adapted mesh can be aligned to the interior layer present in the solution. Numerical analysis is presented for the associated numerical method, which establishes that the numerical method is a parameter-uniform numerical method.

This research is joint work with **Jose Luis Gracia** from the *University of Zaragoza, Spain.*

A weighted and balanced finite element method for singularly perturbed reaction-diffusion problems

Martin Stynes & Niall Madden *(Beijing Computational Science Research Center)*

A new finite element method (FEM) is presented for a general class of singularly perturbed reaction-diffusion problems $-\varepsilon^2\Delta u+bu=f$ posed on bounded domains $\Omega\subset\mathbb{R}^k$ for $k\geq 1$, with homogeneous Dirichlet boundary condition $u=0$ on $\partial\Omega$. The method is shown to be quasioptimal (on arbitrary meshes and for arbitrary conforming finite element spaces) with respect to a weighted balanced norm. A robust (i.e., independent of ε) almost first-order error bound is obtained for the particular case where Ω is the unit square in $\mathbb{R}^2$ and the FEM comprises piecewise bilinears on a Shishkin mesh. Some illustrative numerical results are presented.

Asymptotic solution of the coefficient inverse problems for Burgers type equations

Vladimir Volkov & Nikolay Nefedov
(Lomonosov Moscow State University, Faculty of Physics, Moscow, Russia)

Here some ways to use asymptotic analysis for solving the coefficient inverse problems are considered. The main idea is based on the fact that asymptotic approach allows to reduce the original nonlinear singularly perturbed problem to a set of more simple problems while receiving fairly accurate qualitative and quantitative description of the solution. These reduced problems do not contain small parameters and have a lower spatial dimension (and may be not differential, but algebraic), so it can significantly simplify the procedure for solving some classes of inverse problems.

First way to use asymptotic analysis – optimization of the numerical calculations in traditional gradient methods by using some special meshes. These methods require to solve the direct and adjoint singularly perturbed (may be, multi-dimensional) problems for PDEs on each iteration of the optimization process and it may needs million iterations in practical applications.

Asymptotic approach allows to extract a priory information about some features of the solutions and gives the possibility to generate special meshes according to these features for optimization numerical calculations. Some of these results were published in [1].

Second way – asymptotic analysis makes it possible to reveal more simple connections between the input data and the parameters of the inverse problem (coefficients in the equation, boundary and initial conditions, etc.), which have to be determined. So, there is no need to repeatedly solve the direct and adjoint singularly perturbed problems because finding an unknown coefficient in the equation can be reduced to solving some algebraic equation, or to a simple differentiation of the solution or the observed data [2], [3]. In general, asymptotic analysis allows to reduce with certain accuracy the original inverse problem $A_\varepsilon q = f$ for the partial differential equation to a simpler problem $A_0 q = f$, which is often not a differential equation, but algebraic formula associating the observed data to the coefficients in the equation to be determined.

The main ideas of our approach demonstrated on the asymptotic solution of some coefficient inverse problems for Burger's type equation with moving front type solution [3]. We show, that if there is an experimental possibility to fix properties of the moving front, finding of the unknown coefficient in the equation or boundary conditions can be reduced to solving of some algebraic equation.

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Isogeometric analysis for singularly perturbed high-order, two-point boundary value problems of reaction-diffusion type

Christos Xenophontos *(University of Cyprus)*

We consider two-point singularly perturbed boundary value problems of order $2\nu \in \mathbb{Z}^+$, and the approximation of their solution using isogeometric analysis. In particular, we use a Galerkin formulation with B-splines as basis functions, defined using appropriately chosen knot vectors. Under the assumption of analytic data, we prove robust exponential convergence in the energy norm, independently of the singular perturbation parameter, as the polynomial degree of the B-splines is increased. Numerical examples illustrating the theory are also presented.

Approaches to calculating derivatives in the presence of a boundary layer

Zadorin A.I., *(Sobolev Institute of Mathematics, Siberian Branch, Russian Academy of Sciences, Novosibirsk, Russia)*

A problem of approximation the derivatives of functions with large gradients in the boundary layer is investigated. The application of classical formulas of numerical differentiation on the uniform grid to such functions leads to significant errors.

Consider the following approaches:

1) Fitting the numerical differentiation formula to the boundary layer component. We suppose that function $u(x)$ has a form:

$$u(x) = p(x) + \gamma \Phi(x), \ x \in [a, b], \tag{9}$$

where $p(x)$ is regular component, $\Phi(x)$ is known boundary layer component with large gradients. On a uniform grid with nodes $x_1, ..x_k$ we construct an interpolation formula with k nodes that is exact on the boundary layer component:

$$L_{\Phi,k}(u, x) = L_{k-1}(u, x) + \frac{[x_1, \dots, x_k]u}{[x_1, \dots, x_k]\Phi} \Big[\Phi(x) - L_{k-1}(\Phi, x) \Big], \tag{10}$$

where $[x_1, \dots, x_k]u$ is the divided difference for a function $u(x)$ and $L_{k-1}(u, x)$ is Lagrange polynomial with $(k-1)$ interpolation nodes $x_1, ..x_{k-1}$. Differentiating this interpolant, we obtain formulas for numerical differentiation that are exact for $\Phi(x)$. For some k we prove that the error is uniform with respect to the singular component $\Phi(x)$. Publicated in [1]–[3].

2) Application of classical formulas of numerical differentiation based on the differentiation of the Lagrange polynomial on Shishkin and Bakhvalov meshes. The following estimate of the error is obtained on Shishkin mesh

$$\varepsilon^n |u^{(n)}(x) - L_k^{(n)}(u, x)| \le C \Big(\frac{\ln N}{N} \Big)^{k-n}, \quad n \ge 0, \ x \in [x_1, x_k],$$

where C does not depend on small parameter ε, N - number of nodes of mesh Ω. Here $[x_1, x_k]$ is subinterval with k nodes. To prove this estimate we use Shishkin decomposition in the presence of an exponential boundary layer. Publicated in [4]–[5].

3) Applying a cubic spline on the Shishkin mesh. By differentiating such a spline, the first and second derivatives are found as smooth functions. The error estimate is uniform in a small parameter [6].

4) Applying an exponential spline on the uniform grid. An analog of a cubic spline is constructed in such a way that the spline is exact on the singular component responsible for the growth of the function in the boundary layer. Differentiating the spline gives an approximation of the derivatives. The error is uniform in a small parameter [7].

This work was supported by the Russian Foundation for Basic Research, project 20-01-00650.

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