

SHISHKIN MESHES IN THE NUMERICAL SOLUTION OF SINGULARLY PERTURBED DIFFERENTIAL EQUATIONS

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Dedicated to G.I. Shishkin on the occasion of his 70th birthday.

Abstract. This article reviews some of the salient features of the piecewise-uniform Shishkin mesh. The central analytical techniques involved in the associated numerical analysis are explained via a particular class of singularly perturbed differential equations. A detailed discussion of the Shishkin solution decomposition is included. The generality of the numerical approach introduced by Shishkin is highlighted. The impact of Shishkin's ideas on the field of singularly perturbed differential is assessed in this selective review of his research output over the past thirty years.

Key Words. singularly perturbed problems, Shishkin mesh, Shishkin solution decomposition.

1. Introduction

This review paper addresses the numerical solution of computationally challenging singularly perturbed differential equations and, in particular, how this area of numerical analysis was enhanced by the contributions of the Russian mathematician Grigorii Ivanovich Shishkin. This review is not comprehensive in the sense that no attempt is made to give an overview of all current activities in the area, not even an overview of all the contributions made by Shishkin over the years (as this topic would warrant a monograph by itself). We aim to give a simple and self-contained presentation of those techniques by Shishkin that, in our opinion, have had most impact on the area. In particular, we shall describe the construction of Shishkin meshes and the Shishkin solution decomposition. We also aim to highlight the generality of Shishkin's approach, which is evidenced by a broad range of problems to which Shishkin has applied his methodology. Finally, we shall review some of the literature to demonstrate how Shishkin's ideas were employed and, furthermore, blended with other techniques by authors other than Shishkin.

Singularly perturbed differential equations are typically characterized by a small parameter ε multiplying some or all of the highest order terms in the differential equation. In general, the solutions of such equations exhibit multiscale phenomena. Within certain thin subregions of the domain, the scale of some partial derivatives is significantly larger than other derivatives. We call these thin regions of rapid change, boundary or interior layers, as appropriate. For small values of ε , an analytical approximation to the exact solution can be generated using the techniques of matched asymptotic expansions [38, 66, 100, 103]. Such asymptotic approximations identify the fundamental nature of the solution across the different scales.

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Classical computational approaches to singularly perturbed problems are known to be inadequate as they require extremely large numbers of mesh points to produce satisfactory computed solutions [26, 73]. Throughout the paper, we focus on robust numerical methods, also called uniformly convergent or parameter-uniform methods, that converge in the discrete maximum norm independently of the size of the singular perturbation parameter(s).

Shishkin has been producing publications on singularly perturbed problems since 1974; see [25]. Motivated by physical problems, he has developed over the intervening years a distinctive approach to both constructing and analysing appropriate parameter-uniform numerical methods for singularly perturbed problems. In general, he combines inverse-monotone finite difference schemes with layer-adapted piecewise-uniform meshes (often called Shishkin meshes by others) to produce a computed solution (with accuracy measured in the discrete maximum norm), whose global convergence is guaranteed independently of any singular perturbation parameter present in the problem.

Throughout his publications, Shishkin seeks out simplicity in the design of computational algorithms for singularly perturbed problems; in particular, simplicity in the design of the mesh. In §2, using a constant-coefficient ordinary differential equation, we identify the deficiencies associated with a uniform mesh. We then retrace the path followed by Shishkin and derive the necessary conditions for a piecewise-uniform mesh to support a uniformly convergent method. We also briefly discuss generalizations of Shishkin mesh in more than one dimension. In the context of simplicity, note that a piecewise-uniform mesh only differs from a uniform mesh at one or a few transition points.

A useful analytical technique developed by Shishkin, which is used in the numerical analysis associated with these layer-adapted meshes, is a particular solution decomposition. Note that this decomposition is not an asymptotic expansion, as there is no remainder term present. The decomposition technique involves defining some associated problems on extended domains in order to minimize the imposition of additional compatibility conditions and then employing Schauder-type estimates to determine bounds on the derivatives of each component in the decomposition. These a priori estimates on the derivatives of the exact solution are used both to identify rate constants in the layer functions (which are utilized in the mesh design) and in the error analysis [69, 70, 82, 94]. In §3, we first recall a solution decomposition by Bakhvalov, whose approach contains some of the key aspects of the Shishkin decomposition, and then describe a Shishkin decomposition for a linear convection–diffusion problem in two space dimensions.

In our opinion, a significant attribute of the Shishkin mesh is the fact that, once the location and width of all possible layers are identified, the same methodology is applicable to various different classes of singularly perturbed problems. In §4, we describe some problem classes for which a Shishkin mesh has been constructed and Shishkin's analysis technique has been applied, to emphasize the extent and generality of the Shishkin approach.

In the final section, we review the recent literature related to Shishkin's publications. We exclude papers authored or co-authored by Shishkin himself from this final section. We note that the increasing number of papers that involve a Shishkin mesh and/or a Shishkin decomposition is a clear indication of the significant impact that Shishkin's research has had on the area of numerical methods for singularly perturbed differential equations.

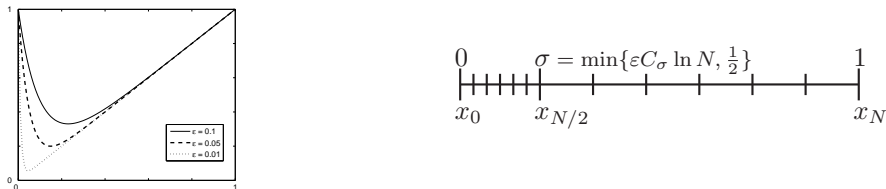


FIGURE 1. Solutions of problem (1) for various values of ε (left); piecewise-uniform Shishkin mesh (8) (right).

Notation: Let k be a non-negative integer, and $\lambda \in (0, 1)$ (unless it is explicitly stated that $\lambda = 1$). We use the standard spaces $C^k(\bar{\Omega})$ of functions whose derivatives up to order k are continuous in $\bar{\Omega}$, and $C^{k,\lambda}(\bar{\Omega})$, or $C^\lambda(\bar{\Omega})$ when $k = 0$, of Hölder continuous functions in $\bar{\Omega}$. Throughout the paper, we use the notation $\partial_x^k v$ and $\partial_y^m v$ for partial derivatives of any sufficiently smooth function $v(x, y)$, and $\Delta = \partial_x^2 + \partial_y^2$ for the Laplace operator. Furthermore, we let C denote a generic positive constant that may take different values in different formulas, but is always independent of the mesh and ε . A subscripted C (e.g., C_1) denotes a positive constant that is independent of N and ε and takes a fixed value. Notation such as $v = O(w)$ means $|v| \leq Cw$ for some C .

2. Shishkin mesh: transition point and accuracy of the computed solution

A Shishkin mesh is a piecewise uniform mesh (or a tensor-product version in more than one dimension). What distinguishes a Shishkin mesh from any other piecewise uniform mesh is the choice of the so-called transition parameter(s), which are the point(s) at which the mesh size changes abruptly.

By now, over 20 years since the Shishkin mesh was proposed [77, 78], many numerical analysts have at least heard the term “Shishkin mesh”, but perhaps not all of them fully appreciate how the choice of its transition parameter(s), and consequently, the mechanism of this mesh yields accuracy irrespective of how small the singular perturbation parameter is. In this section, we shall explain the influence of the transition parameter on the accuracy of the computed solution using a very simple one-dimensional example, and then briefly discuss generalizations of the Shishkin mesh into two dimensions.

2.1. One-dimensional example. Consider the following singularly perturbed two-point boundary-value problem

$$(1) \quad \varepsilon u'' + u' = 1 \quad \text{for } x \in (0, 1), \quad u(0) = u(1) = 1,$$

where $\varepsilon \in (0, 1]$ is a small parameter. The unique solution of this problem is given by

$$u(x) = x + \frac{w(x) - w(1)}{1 - w(1)}, \quad \text{where } w(x) := e^{-x/\varepsilon}.$$

When $\varepsilon \ll 1$, the term $w(1) = e^{-1/\varepsilon}$ becomes negligible, so $u(x) \approx x + e^{-x/\varepsilon}$. Thus there are **two different scales** involved in the solution u . On the scale of $x = O(1)$, the regular component x gradually changes over the interval $[0, 1]$, while on the scale of $x = O(\varepsilon)$ the layer component $e^{-x/\varepsilon}$ changes very rapidly in a small neighbourhood of the boundary point $x = 0$ to become negligible away from this point; see Figure 1 (left).

For problem (1) on a general mesh $\{x_i\}_{i=0}^N$ with the mesh size $h_i = x_i - x_{i-1}$, consider the conservative upwind finite-difference scheme

$$(2) \quad \frac{\varepsilon}{h_{i+1}} \left(\frac{U_{i+1} - U_i}{h_{i+1}} - \frac{U_i - U_{i-1}}{h_i} \right) + \frac{U_{i+1} - U_i}{h_{i+1}} = 1, \quad U_0 = U_N = 1.$$

(One attractive property of this scheme is that it satisfies a discrete maximum principle.) A calculation shows that this discrete problem has a unique solution

$$(3) \quad U_i = x_i + \frac{W_i - W_N}{1 - W_N}, \quad \text{where } W_i = \prod_{j=1}^i (1 + h_j/\varepsilon)^{-1}.$$

Clearly, the error $U_i - u(x_i)$ is directly related to $W_i - w(x_i)$, so we shall investigate this quantity on various meshes, starting with its value at the first internal mesh point x_1 , for which we get

$$W_1 - w(x_1) = (1 + \rho)^{-1} - e^{-\rho}, \quad \text{where } \rho := h_1/\varepsilon.$$

Note that

$$(4) \quad (1 + \rho)^{-1} - e^{-\rho} \geq 0.2 \quad \text{for } 2.1 \leq \rho \leq 3.$$

This means that, if a uniform mesh is used so $h_1 = N^{-1}$, for any value of N (no matter how large), then there exists a range of ε for which the error in w is greater than 0.2. More generally, for any mesh with any value of h_1 (no matter how small) chosen independently of ε , there exists a range of ε for which the error in w is greater than 0.2.

This observation brings us to the first requirement on the mesh that Shishkin imposed.

- For ε -uniform convergence in the discrete maximum norm, it is necessary that the first mesh interval h_1 satisfies

$$(5a) \quad h_1/\varepsilon \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Indeed, a Taylor series expansion shows that $|(1 + \rho)^{-1} - e^{-\rho}| \leq C\rho^2$, so condition (5a) at least yields $|W_1 - w(x_1)| \rightarrow 0$ as $N \rightarrow \infty$.

As we have already seen (this is also reflected by condition (5a)), one cannot get ε -uniform convergence for any standard finite difference scheme on a uniform mesh. If we want to have the mesh structure as simple as possible, then we might try to construct a suitable **piecewise-uniform mesh**. Thus we shall investigate a general piecewise-uniform mesh with N intervals on $[0, 1]$. Let the transition point $\sigma \in (0, 1)$, and divide each of $[0, \sigma]$ and $[\sigma, 1]$ into M and $N - M$ equal intervals of width $h = \sigma/M$ and $H = (1 - \sigma)/(N - M) = O(N^{-1})$ respectively:

$$\{x_i = ih \mid i = 0 \dots M, \quad x_i = \sigma + (i - N)H \mid i = M \dots N\}.$$

As the boundary layer occurs at $x = 0$, we choose $\sigma \leq \frac{1}{2}$ and $M \leq \frac{1}{2}N$; then $\frac{1}{2}N^{-1} \leq H \leq 2N^{-1}$. Note that the choice $\sigma = \frac{1}{2}$ and $M = \frac{1}{2}N$ yields a standard uniform mesh.

Recall the computed solution formula (3), in which W_i on this piecewise-uniform mesh now becomes

$$W_i = \begin{cases} (1 + h/\varepsilon)^{-i} & \text{for } i = 0, \dots, M, \\ (1 + h/\varepsilon)^{-M} (1 + H/\varepsilon)^{-(i-M)} & \text{for } i = M, \dots, N. \end{cases}$$

So under the necessary condition (5a), for $i \leq M$ we have $\rho = h/\varepsilon \rightarrow 0$ and

$$W_i = (1 + \rho)^{-i} = e^{-i \ln(1+\rho)} = e^{-x_i/\varepsilon} [1 + (x_i/\varepsilon) O(\rho)],$$

where we used $\ln(1 + \rho) = \rho[1 + O(\rho)]$, $i\rho = ih/\varepsilon = x_i/\varepsilon$ and also $e^{-(x_i/\varepsilon)O(\rho)} = 1 + (x_i/\varepsilon)O(\rho)$. Consequently,

$$|W_i - w(x_i)| = e^{-x_i/\varepsilon}(x_i/\varepsilon)O(\rho) \leq C\rho = Ch/\varepsilon \quad \text{for } i \leq M.$$

Next consider $i > M$, in particular, $i = M + 1$. A calculation, using $W_{M+1} = W_M(1 + H/\varepsilon)^{-1}$ and $W_M = e^{-\sigma/\varepsilon} + O(h/\varepsilon)$, yields

$$W_{M+1} - w(x_{M+1}) = e^{-\sigma/\varepsilon}[(1 + \rho')^{-1} - e^{-\rho'}] + O(h/\varepsilon), \quad \text{where } \rho' := H/\varepsilon.$$

As $\frac{1}{2}N^{-1} \leq H \leq 2N^{-1}$, again using (4), we see that for any value of N (no matter how large), there exists a range of ε for which the term in square brackets is greater than 0.2. To ensure the smallness of the error $W_{M+1} - w(x_{M+1})$, we therefore arrive at the second requirement on the mesh that was imposed by Shishkin.

- For ε -uniform convergence on the above piecewise-uniform mesh, it is necessary that the transition point σ satisfies

$$(5b) \quad \sigma/\varepsilon \rightarrow \infty \quad \text{as} \quad N \rightarrow \infty.$$

This condition immediately yields $|W_i - w(x_i)| \leq C(h/\varepsilon + e^{-\sigma/\varepsilon})$ for $i > M$ and hence for all i . This implies that under conditions (5), we have the error estimate

$$|U_i - u(x_i)| \leq C|W_i - w(x_i)| \leq C(h/\varepsilon + e^{-\sigma/\varepsilon}),$$

which can be rewritten as

$$(6) \quad |U_i - u(x_i)| \leq C(\hat{\sigma}M^{-1} + e^{-\hat{\sigma}}), \quad \text{where } \sigma =: \varepsilon\hat{\sigma}.$$

To minimize the right-hand side in this estimate, we choose M as large as possible, i.e. $M := O(N)$, e.g., $M := \frac{1}{2}N$. As the terms $\hat{\sigma}M^{-1}$ and $e^{-\hat{\sigma}}$ are increasing and decreasing, respectively, in $\hat{\sigma}$, they are made approximately of the same order by choosing $\hat{\sigma} = C_\sigma \ln N$ for some positive constant C_σ . Under this choice, the error estimate (6) becomes

$$(7) \quad |U_i - u(x_i)| \leq C(2C_\sigma N^{-1} \ln N + N^{-C_\sigma}) \leq CN^{-1} \ln N \quad \text{if } C_\sigma \geq 1.$$

That is, the considered numerical method is almost first-order convergent uniformly in the small parameter ε .

Note that the presented error analysis and, in particular, the error estimate (7) imply that if $C_\sigma < 1$, then the error will become only $O(N^{-C_\sigma})$, so the order of convergence will deteriorate. This theoretical conclusion has been confirmed by numerical experiments.

Note also that on more than one occasion, the authors have come across an intuitive point of view that were a piecewise-uniform mesh to be used, (i) σ should be $\bar{C}\varepsilon$, for some constant \bar{C} , and independent of N , and also that (ii) M should be rather small compared to $O(N)$, e.g., $M = 10$ independently of N . In view of (6), the choice (i) implies that $\hat{\sigma} = \bar{C}$ so $e^{-\hat{\sigma}} = e^{-\bar{C}}$, while the choice (ii) yields $\hat{\sigma}M^{-1} \geq CM^{-1} = C/10$; in both cases the error will not become smaller as $N \rightarrow \infty$, i.e. the numerical method will not be ε -uniformly convergent.

We now summarize the **Shishkin piecewise-uniform mesh parameters** for a slightly more general equation $\varepsilon u'' + a(x)u' = f(x)$ of type (1) with $a(x) > 0$:

$$(8) \quad M := \frac{1}{2}N, \quad \sigma = \varepsilon\hat{\sigma} := \min\{\varepsilon C_\sigma \ln N, \frac{1}{2}\}, \quad C_\sigma \geq p/\alpha$$

(see Figure 1 (right)). Here p is the order (of the local truncation error) of the method; e.g., for the first-order upwind scheme (2), we used $p = 1$. For problem (1) we used $\alpha := 1$; in general, for the equation $\varepsilon u'' + a(x)u' = f(x)$, the parameter α is such that $0 < \alpha < a(x)$. Note that the earlier choice of $\sigma = \varepsilon C_\sigma \ln N$ is changed

in (8) to $\sigma = \frac{1}{2}$, whenever $\varepsilon C_\sigma \ln N > \frac{1}{2}$. Thus for N sufficiently large (relative to $1/\varepsilon$) the mesh returns to a classical uniform mesh.

On the mesh (8), it has been shown for a number of first- and second-order methods applied to equations similar to $\varepsilon u'' + a(x)u' = f(x)$ that the error in the discrete maximum norm is $O([N^{-1} \ln N]^p)$, where p is the order of the method [64, 10, 7, 97, 8].

In fact, if one uses a simple piecewise linear interpolant $U(x)$ of the computed solution U_i (i.e. the continuous function U that is linear on each $[x_{i-1}, x_i]$ and equal to U_i at each mesh node x_i), then one can show for $p \leq 2$ that

$$(9) \quad |U(x) - u(x)| \leq C(N^{-1} \ln N)^p \quad \text{for all } x \in [0, 1];$$

e.g., see [26] for (9) with $p = 1$ for simple upwinding. Indeed, this global error estimate follows from the corresponding nodal estimate $|U_i - u(x_i)| \leq C(N^{-1} \ln N)^p$, as $|U(x) - u^I(x)| \leq \max |U_i - u(x_i)|$, and one has the interpolation error estimate $|u^I(x) - u(x)| \leq C(N^{-1} \ln N)^p$ on a suitable Shishkin mesh, where u^I is the piecewise linear interpolant of the exact solution u . The error estimate (9) shows that by incorporating a Shishkin mesh into the numerical method, we obtain numerical approximations which are globally convergent uniformly in the small parameter.

A fortiori, parameter-uniform approximations to the scaled derivative can be easily generated on these piecewise-uniform meshes. In particular, if simple upwinding is used, then for the piecewise constant derivative U' of the computed-solution interpolant U , one can show [26] that

$$\varepsilon |U'(x) - u'(x)| \leq CN^{-1} \ln N \quad \text{for all } x \in [0, 1].$$

In view of (9) with $p = 1$, we have global convergence in an ε -weighted C^1 norm. An analogue of this result for a second-order numerical method is given in [11], where $O([N^{-1} \ln N]^2)$ -accurate approximations of the scaled derivatives $\varepsilon u'(x_i)$ were constructed. In fact, we note that away from layer regions, similar error estimates can be obtained for unscaled derivatives; see [32, 49], and also [44, 91] for two-dimensional convection-diffusion equations.

Remark 2.1. *For layer-adapted meshes other than of Shishkin type, we refer the reader to [55]. In particular, the first layer-adapted mesh is due to Bakhvalov [14]. As the mesh size in a Bakhvalov mesh changes gradually, such meshes yield slightly higher orders of convergence (typically $O(N^{-p})$ compared to $O([N^{-1} \ln N]^p)$ for Shishkin meshes); however at present, the simpler-to-construct and simpler-to-analyze piecewise-uniform Shishkin mesh has been used in the construction and analysis of robust numerical methods for a significantly wider set of singularly perturbed partial differential equations.*

2.2. Shishkin mesh in more than one dimension. In the previous subsection, we have seen that a piecewise-uniform mesh suffices to generate parameter-uniform numerical approximations to a solution of a singularly perturbed ordinary differential equation. More importantly, Shishkin established that piecewise-uniform meshes preserve this property in the context of a broad class of singularly perturbed partial differential equations.

For example, an appropriate tensor-product Shishkin mesh for a convection-diffusion problem in a rectangular domain is displayed in Figure 2; for a precise description of this mesh we refer the reader to Remark 3.2. A Shishkin mesh for a problem with a curvilinear boundary is displayed in Figure 3 and for a non-convex domain in Figure 4.

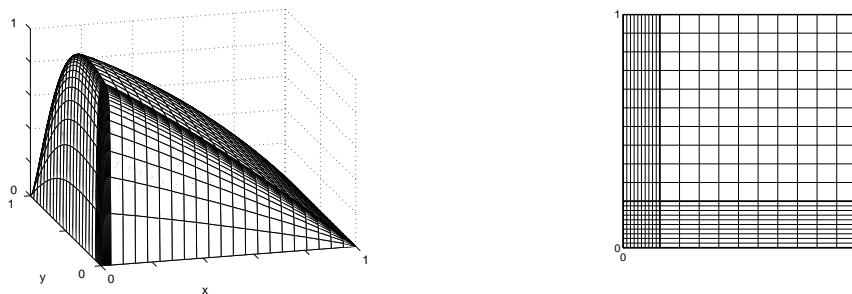


FIGURE 2. Boundary-layer solution of the convection-diffusion equation $\varepsilon\Delta u + 2\partial_x u + \partial_y u = 2.5 + y$ of type (13) for $\varepsilon = 10^{-2}$ (left); structure of a Shishkin mesh for this problem(right).

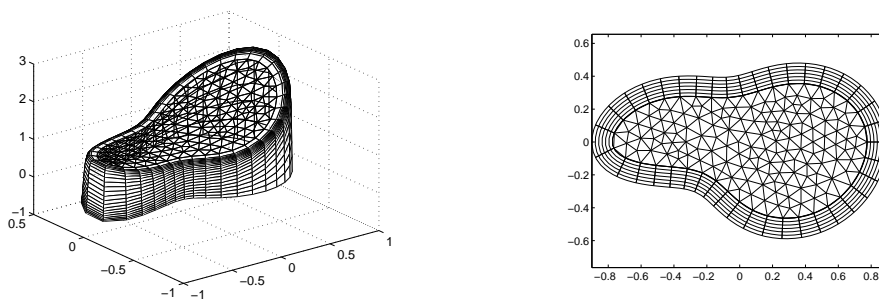


FIGURE 3. Boundary-layer solution of the reaction-diffusion equation $-\varepsilon^2\Delta u + u(u^2 - [x^2 + x + 1]^2) = 0$ for $\varepsilon = 10^{-2}$ (left); a curvilinear Shishkin mesh for this problem (right); see [46] for numerical analysis related to this problem.

To construct a Shishkin mesh for a certain singularly perturbed problem, one requires a priori information about the location and width (or scale) of any potential layers that can be present. Thus, at least, a crude asymptotic analysis is needed prior to the construction of a Shishkin mesh. The boundary layers are typically located where there is a disparity between the solution of the reduced problem (which is formally obtained by setting the small parameter to zero) and the boundary conditions. Interior layers might occur where the reduced solution is discontinuous. Then stretching transformations can be used to identify the layer widths as we shall now illustrate using an example.

Consider the equation, with $\beta_1 \geq 0, \beta_2 \geq 0$,

$$(10) \quad -\varepsilon\Delta u + y^{\beta_1}(1-y)^{\beta_2}\partial_x u = y^{\beta_1}(1-y)^{\beta_2}f(x,y) \quad \text{for } (x,y) \in \Omega = (0,1)^2$$

subject to the boundary condition $u = 0$ on $\partial\Omega$. Here ε is a small parameter and f is a smooth bounded function. Note that the solution \bar{u} of the reduced equation $\partial_x \bar{u} = f(x,y)$, subject to the boundary condition $\bar{u}(0,y) = 0$, does not necessarily vanish on the right edge $x = 1$ and the top and bottom edges $y = 0$ and $y = 1$. Consequently, the solution u of the original problem (10) will have boundary layers along these three edges.

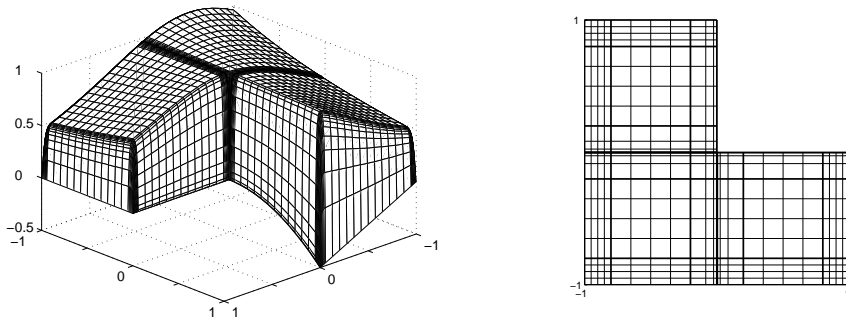


FIGURE 4. Boundary-layer solution of the reaction-diffusion equation $-\varepsilon^2 \Delta u + u = 2/(2 + x^2 - xy)$ in an L -shaped domain for $\varepsilon = 10^{-2}$ (left); structure of a Shishkin mesh for this problem (additionally graded near the corner of angle $3\pi/2$) (right); see [87, 9] for numerical analysis related to this problem.

To identify the width of the layer along the edge $y = 0$, in the vicinity of this edge we introduce the stretched variable $\eta = y/\varepsilon^p$ in the direction orthogonal to this edge, and then rewrite equation (10) in variables (x, η) as

$$(11) \quad -\varepsilon^{1-p\beta_1} \partial_x^2 u - \varepsilon^{1-2p-p\beta_1} \partial_\eta^2 u + \eta^{\beta_1} (1 - \varepsilon^p \eta)^{\beta_2} \partial_x u = \eta^{\beta_1} (1 - \varepsilon^p \eta)^{\beta_2} f(x, \varepsilon^p \eta).$$

For this equation to describe a suitable boundary-layer function, we need to impose the condition $1 - 2p - p\beta_1 = 0$. Otherwise, if $1 - 2p - p\beta_1 > 0$, then the first two terms in equation (11) become negligible for $\varepsilon \ll 1$ and we again get the reduced equation; if $1 - 2p - p\beta_1 < 0$, then the main term in equation (11) is $\varepsilon^{1-2p-p\beta_1} \partial_\eta^2 u$ and we effectively solve the equation $\partial_\eta^2 u = 0$, which cannot describe the boundary layer. Thus $1 - 2p - p\beta_1 = 0$ or $p = 1/(2 + \beta_1)$, i.e. the width (or scale) of the boundary layer along $x = 0$ is $O(\varepsilon^p) = O(\varepsilon^{1/(2+\beta_1)})$. A similar argument shows that the width of the layer is $O(\varepsilon^{1/(2+\beta_2)})$ along the edge $y = 1$ and is $O(\varepsilon)$ along the edge $x = 1$.

Consequently, a Shishkin mesh suitable for equation (10) is a piecewise-uniform tensor-product mesh $\{(x_i, y_j)\}_{i,j=0}^N$ with the transition points σ in the x -direction and $\tau_{1,2}$ in the y -direction chosen similarly to (8) as follows:

$$\sigma = \min\{\varepsilon C_\sigma \ln N, \frac{1}{2}\}, \quad \tau_k = \min\{\varepsilon^{1/(2+\beta_k)} C_\tau \ln N, \frac{1}{4}\} \quad \text{for } k = 1, 2,$$

for some positive constants C_σ and C_τ . The piecewise uniform mesh $\{x_i\}_{i=0}^N$ in the x -direction is obtained by dividing each of the intervals $[0, 1 - \sigma]$ and $[1 - \sigma, 1]$ into $\frac{1}{2}N$ equal subintervals, and the piecewise uniform mesh $\{y_j\}_{j=0}^N$ in the y -direction is obtained by dividing each of the intervals $[0, \tau_1]$, $[\tau_1, 1 - \tau_2]$ and $[\tau_2, 1]$ into $\frac{1}{4}N$, $\frac{1}{2}N$ and $\frac{1}{4}N$ equal subintervals, respectively. Note that C_τ is an arbitrary positive constant, while C_σ should be sufficiently large. In general, constants such as C_σ and C_τ might need to be sufficiently large, their choice requiring further asymptotic understanding of the problem; but any choice of those constants will typically yield ε -uniform convergence with a, possibly, lower than optimal order of convergence (similarly to what we observed in (7)). We refer the reader to [81] for a theoretical analysis of equation (10), and to [26] for a further discussion on the construction of Shishkin meshes for various singularly perturbed problems.

Thus, formal arguments, such as were used for equation (10), give sufficient information to **construct** a piecewise-uniform mesh. However, if one wishes to prove theoretical parameter-uniform convergence results, then it is required to establish a priori bounds on the derivatives of the solution, which identify how rapidly the layer functions decrease within the layer and also explicitly identify how the constants in these bounds depend on the singular perturbation parameter. This is the topic of the next section.

3. Shishkin solution decomposition

A crucial role in the convergence analysis used by Shishkin for singularly perturbed partial differential equations is played by first decomposing the exact solution into a sum of a regular and boundary/corner layer components. In this section, we discuss the main features of the Shishkin solution decomposition and then give an example of such a decomposition for a two-dimensional convection-diffusion problem.

3.1. Bakhvalov's solution decomposition: seeds of the Shishkin decomposition. The issue of obtaining a solution decomposition for singularly perturbed partial differential equations without imposing unnecessary compatibility conditions and smoothness restrictions on the data was addressed by Bakhvalov in 1969.

In the celebrated seminal paper [14], Bakhvalov, in particular, examined the following two-dimensional elliptic equation

$$Lu := \mu^2 u_{xx} + u_{yy} = f(x, y) \quad \text{for } (x, y) \in (-1, 1)^2$$

with a small parameter $\mu \in (0, 1]$, subject to the Dirichlet boundary conditions $u(\pm 1, y) = \phi_{\pm}(y)$, $u(x, \pm 1) = \omega_{\pm}(x)$ on the four edges. Here all the data ω_{\pm} , ϕ_{\pm} and f are in the Hölder space $C^{1,\lambda}$ for some $\lambda \in (0, 1]$. No other compatibility conditions are assumed besides the continuity of the solution on the boundary, i.e. $\omega_{\pm}(\pm 1) = \phi_{\pm}(\pm 1)$ and $\omega_{\pm}(\mp 1) = \phi_{\mp}(\pm 1)$.

Under these conditions only, it is proved in [14] that the error of the standard five-point difference scheme on a suitable layer-adapted mesh (constructed by Bakhvalov in this paper) is $O(N^{-(\lambda+1)} \ln^{\beta} N)$, where $\beta = 0$ for $\lambda < 1$ and $\beta = 1$ for $\lambda = 1$. This sharp error estimate was established by employing a solution decomposition, which we now describe.

Bakhvalov introduced the decomposition

$$u = u^0 + u^1 + u^2,$$

where the components u^0, u^1 are defined on the infinite strip $(-\infty, \infty) \times (-1, 1)$ by

$$Lu^0 = f, \quad Lu^1 = 0; \quad u^0(x, \pm 1) = 0, \quad u^1(x, \pm 1) = u(x, \pm 1).$$

Here the functions f and $\omega_{\pm}(x)$ are extended to the infinite strip and its boundary, respectively, in such a way that they have compact support. Hence for $x \in (-1, 1)$ we have

$$L(u^0 + u^1) = Lu, \quad (u^0 + u^1)(x, \pm 1) = u(x, \pm 1).$$

The boundary-layer component u^2 is defined in the original domain $[-1, 1]^2$ by

$$Lu^2 = 0 \text{ for } (x, y) \in (-1, 1)^2, \quad u^2(x, \pm 1) = 0, \quad u^2(\pm 1, y) = (u - u^0 - u^1)(\pm 1, y).$$

Using the fundamental solution of the Laplace equation, Bakhvalov analyzed each component in this solution decomposition, and thus derived sharp bounds on certain continuous and Hölder-type discrete derivatives of the solution.

We note that Bakhvalov's solution decomposition has the following features.

- The solution is decomposed into a regular component u^0 and components u^1 and u^2 that describe boundary layers and irregularities in the solution.

Indeed, u^0 is in $C^{3,\lambda}([-1, 1]^2)$. The component u^1 is in $C^{1,\lambda}([-1, 1]^2)$ and represents the irregularity in the solution due to insufficient smoothness of the boundary data ω_{\pm} , while the component u^2 is also in $C^{1,\lambda}([-1, 1]^2)$ and incorporates the boundary layer components in the x -direction, the irregularity in the solution due to insufficient smoothness of the boundary data ϕ_{\pm} , and the corner singularities due to insufficient corner compatibility conditions.

- Each of the three components satisfies an equation with the same differential operator L (unlike components in asymptotic expansions).
- Some components of the decomposition (u^0 and u^1) are defined on an extended domain, which facilitates estimation of their derivatives.
(Note that a similar domain extension appears, e.g., in an earlier paper [101] by Volkov, where it is used to define the smooth component in a solution decomposition for the equation $\Delta u = f$ posed in a rectangular domain.)

Thus one can clearly see the seeds of the Shishkin decomposition. The authors are not aware whether Shishkin was influenced by Bakhvalov's decomposition or created his decomposition technique independently. Whether it is the case or not, we note that Bakhvalov designed his decomposition for one particular problem with constant coefficients, while the Shishkin decomposition technique was applied to a wide class of elliptic and parabolic problems with variable coefficients.

Note that Bakhvalov invoked the *fundamental solution* of the differential operator to estimate derivatives of the decomposition components. This approach yields sharp estimates under minimal compatibility conditions. (Recently fundamental solutions were used in intricate solution decompositions for a variable-coefficient reaction-diffusion equation [5] and a constant-coefficient convection-diffusion equation [40].)

The Shishkin decomposition technique is simpler (although may require additional compatibility conditions) and therefore has been applied to wider classes of problems. This became possible due to the inclusion by Shishkin of one more key ingredient:

- For general variable-coefficient partial differential equations, the classical Schauder a priori bounds were employed by Shishkin to estimate the derivatives of some components in his solution decomposition.

For example, for the regular problem

$$[\Delta + a_1 \partial_x + a_2 \partial_y]z = f \quad \text{for } (x, y) \in \Omega, \quad z = 0 \quad \text{for } (x, y) \in \partial\Omega.$$

posed in a smooth domain $\Omega \subset \mathbb{R}^2$, with sufficiently smooth coefficients a_1, a_2 , we have the Schauder-type estimate (see, e.g., [51, (1.11), p. 110])

$$(12) \quad \|z\|_{C^{2+k,\lambda}(\bar{\Omega})} \leq C^* (\|f\|_{C^{k,\lambda}(\bar{\Omega})} + \max_{\bar{\Omega}} |z|), \quad k = 0, 1, \dots$$

Here the constant $C^* = C^*(k)$ does not depend on the size of the domain Ω ; this is important when using these bounds in the context of singularly perturbed problems (see an example in §3.2). Furthermore, if $z \in C^{k+2,\lambda}(\bar{\Omega})$, then (12) holds true even if Ω is a *rectangular* domain [58, Theorem 3.1].

To illustrate the Shishkin decomposition, in the next subsection we shall apply it to a convection-diffusion problem in the unit square.

3.2. Shishkin decomposition for a convection-diffusion problem. In this section, we discuss the Shishkin decomposition ideas in relation to the singularly perturbed elliptic problem

$$(13a) \quad Lu = \varepsilon \Delta u + a_1 \partial_x u + a_2 \partial_y u = f \quad \text{for } (x, y) \in \Omega := (0, 1)^2,$$

$$(13b) \quad u = 0 \quad \text{for } (x, y) \in \partial\Omega.$$

Here ε is a small positive parameter, and a_1, a_2 are sufficiently smooth positive coefficients that satisfy

$$(13c) \quad a_1(x, y) > \alpha_1 > 0, \quad a_2(x, y) > \alpha_2 > 0 \quad \text{for } (x, y) \in \bar{\Omega}.$$

Furthermore, we assume the standard corner compatibility conditions

$$(13d) \quad f(0, 0) = f(0, 1) = f(1, 0) = f(1, 1) = 0.$$

A typical solution of this problem exhibits boundary layers along the left edge $x = 0$ and the bottom edge $y = 0$; an example is displayed in Figure 2 (left).

Detailed decompositions for this problem are presented in [6, 58, 69]. Here we mainly follow [69], but will use a number of results from [58].

From [34, Theorem 3.2], if $f \in C^{1,\lambda}(\bar{\Omega})$, then $u \in C^{3,\lambda}(\bar{\Omega})$ if and only if $f = 0$ at the four corners, i.e. (13d) is satisfied.

Remark 3.1. *Note that a similar result was established by Volkov [101] for the equation $\Delta u = f$ in a rectangular domain. For our equation (13a), this result can also be deduced from Volkov [101] and Kondrat'ev [41] as follows. As $f \in C^{1,\lambda}(\bar{\Omega}) \subset W_2^1(\Omega)$, the asymptotic expansion of type [41, expansion (5.13)] for u implies that $u \in C^{1,\lambda}(\bar{\Omega})$. Rewrite (13a) as $\Delta u = F$ with $F := \varepsilon^{-1}[-a_1 \partial_x u - a_2 \partial_y u + f]$. Then $F \in C^{0,\lambda}(\bar{\Omega})$ and, in view of (13b), condition (13d) is equivalent to its analogue for F , which, by [101, Theorem 3.1], is necessary and sufficient for $u \in C^{2,\lambda}(\bar{\Omega})$. Now $F \in C^{1,\lambda}(\bar{\Omega})$, so repeating the above argument, we deduce that condition (13d) is necessary and sufficient for $u \in C^{3,\lambda}(\bar{\Omega})$.*

Next, using the stretching transformations $\xi = x/\varepsilon, \eta = y/\varepsilon$ and the Schauder-type bound (12) for the corresponding stretched square domain $(0, 1/\varepsilon)^2$, one gets $|\partial_\xi^k \partial_\eta^m u| \leq C$, where C is independent of ε , which immediately implies the estimate

$$(14) \quad |\partial_x^k \partial_y^m u| \leq C \varepsilon^{-(k+m)} \quad \text{for } (x, y) \in \bar{\Omega}, \quad 0 \leq k + m \leq 3;$$

see, e.g., [58, Theorem 3.2] for details.

Sharper bounds on the solution can be derived by decomposing the solution into the sum

$$(15) \quad u = v + w_1 + w_2 + w_{12}.$$

Here v is the regular component, w_1 (w_2) is a boundary layer component associated with the left edge $x = 0$ (bottom edge $y = 0$), and w_{12} is a corner layer component associated with the outflow corner $(0, 0)$. The decomposition into regular and layer components is defined so that $Lv = f$ and for each layer component satisfies the homogeneous differential equation $Lw_{1,2,12} = 0$. (Note that in [58], the solution of (13) is decomposed in a similar manner, but the layer functions are constructed such that $Lw_{1,2,12} = O(\varepsilon) \neq 0$.)

Lemma 3.1 (Regular component). *Assume that a_1, a_2 are smooth and let f satisfy*

$$(16) \quad f \in C^{5,\lambda}(\bar{\Omega}) \quad \text{and} \quad \partial_x^k \partial_y^m f(1, 1) = 0 \quad \text{for } 0 \leq k + m \leq 4.$$

Then there exists a function $v \in C^{3,\lambda}(\bar{\Omega})$ such that $Lv = f$ for $(x, y) \in \Omega$, $v = 0$ for $(x, y) \in \partial\Omega_{\text{in}} := \{(x, 1) | 0 \leq x \leq 1\} \cup \{(1, y) | 0 \leq y \leq 1\}$, and

$$(17) \quad |\partial_x^k \partial_y^m v| \leq C[1 + \varepsilon^{2-(k+m)}] \quad \text{for } (x, y) \in \bar{\Omega}, \quad 0 \leq k + m \leq 3.$$

Proof. Define an extended rectangular domain $\Omega^* := (-1, 1)^2$ such that $\Omega \subset \Omega^*$ and the inflow boundary $\partial\Omega_{\text{in}}$ is part of the boundary of the extended domain, i.e. $\partial\Omega_{\text{in}} \subset \partial\Omega^*$. We define smooth extensions of the functions a_1, a_2 and f to the domain Ω^* such that they coincide with their prototype functions in Ω . On the extended domain Ω^* the regular component v is defined to be

$$v := v_0 + \varepsilon v_1 + \varepsilon^2 v_2,$$

where the reduced solution v_0 and the second term v_1 are the solutions of the first-order problems

$$[a_1 \partial_x + a_2 \partial_y] v_0 = f \quad \text{for } (x, y) \in \Omega^*, \quad v_0(1, y) = v_0(x, 1) = 0;$$

$$[a_1 \partial_x + a_2 \partial_y] v_1 = -\Delta v_0 \quad \text{for } ((x, y) \in \Omega^*, \quad v_1(1, y) = v_1(x, 1) = 0;$$

and the remainder term v_2 is the solution of the elliptic problem

$$Lv_2 = -\Delta v_1 \quad \text{for } (x, y) \in \Omega^*, \quad v_2 = 0 \quad \text{for } (x, y) \in \partial\Omega^*.$$

By this construction, the regular component v satisfies $Lv = f$ in Ω and $v = 0$ on the inflow boundary $\partial\Omega_{\text{in}}$. Thus it remains to show that $v \in C^{3,\lambda}(\bar{\Omega})$ and establish the desired estimates for its derivatives.

The extensions of a_1, a_2, f can be constructed so that appropriate compatibility conditions for v_2 are satisfied at the three artificially introduced corners of Ω^* . The additional compatibility conditions (16) assumed at the fourth inflow corner $(1, 1)$ of Ω^* suffice [58, Theorem 4.1] for $v_0 \in C^{5,\lambda}(\bar{\Omega}^*)$, $v_1 \in C^{3,\lambda}(\bar{\Omega}^*)$, $\Delta v_0(1, 1) = \Delta v_1(1, 1) = 0$ and consequently $v_2 \in C^{3,\lambda}(\bar{\Omega}^*)$. Hence $v \in C^{3,\lambda}(\bar{\Omega})$. Moreover, as the components v_0, v_1 do not depend on ε , we have $|\partial_x^k \partial_y^m v_{0,1}| \leq C$ in $\bar{\Omega}$. Combining this with the analogue of (14) for v_2 in Ω^* yields $|\partial_x^k \partial_y^m v| \leq C[1 + \varepsilon^2 \cdot \varepsilon^{-(k+m)}] = C[1 + \varepsilon^{2-(k+m)}]$. \square

The above lemma implies that implicit boundary values can be specified at the outflow boundary $\partial\Omega \setminus \partial\Omega_{\text{in}}$ so that the solution of $Lv = f$ has first and second order derivatives which are uniformly bounded throughout the domain. The only distinction between the problems that u and v satisfy is in the outflow boundary conditions.

Now we shall investigate $u - v$ for which we have $L[u - v] = 0$ in Ω and $u - v = 0$ if $x = 0$ or $y = 0$. The function $u - v$ represents the layer components in the solution. We shall split this function into three components associated with two boundary layers and a corner layer.

Lemma 3.2 (Boundary-layer components). *Under the conditions of the previous lemma, there exists two functions $w_i \in C^{3,\lambda}(\bar{\Omega})$, $i = 1, 2$, such that $Lw_i = 0$ for $(x, y) \in \Omega$ and $w_i(0, y) = w_i(x, 0) = 0$. Furthermore, $w_1 = u - v$ for $x = 0$, $w_2 = u - v$ for $y = 0$, and for $(x, y) \in \bar{\Omega}$ and $1 \leq k \leq 3$ we have*

$$(18) \quad |w_1| \leq Ce^{-\alpha_1 x/\varepsilon}, \quad |w_2| \leq Ce^{-\alpha_2 y/\varepsilon},$$

$$(19) \quad |\partial_x^k w_1| + |\partial_y^k w_2| \leq C\varepsilon^{-k}, \quad |\partial_y^k w_1| + |\partial_x^k w_2| \leq C(1 + \varepsilon^{1-k}).$$

Proof. It suffices to establish the existence and the desired bounds for w_1 , as the bounds for w_2 can be established in an analogous manner.

Define a second extended domain $\Omega^{**} = (0, 1) \times (-1, 1)$ such that the inflow boundary $\partial\Omega_{\text{in}}$ and the edge $\{(0, y) | 0 \leq y \leq 1\}$ are part of the extended boundary $\partial\Omega^{**}$. The boundary layer component w_1 is defined as the solution of

$$\begin{aligned} Lw_1 &= 0 && \text{for } (x, y) \in \Omega^{**}, \\ w_1(0, y) &= -v(0, y) && \text{for } y \in (-1, 1), \\ w_1 &= 0 && \text{for } (x, y) \in \partial\Omega^{**} \setminus \{(0, y) | -1 \leq y \leq 1\}; \end{aligned}$$

thus $w_1 = u - v$ for $\{(0, y) | 0 \leq y < 1\}$. The function $v(0, y)$ in the boundary condition coincides with v from Lemma 3.1 for $0 \leq y \leq 1$ and is smoothly extended to $\{(0, y) | -1 \leq y < 0\}$ so that $\partial_y^k v(0, -1) = 0$ for $k = 0, 1, 2$. Then, by [34, Theorem 3.2], we have $w_1 \in C^{3,\lambda}(\bar{\Omega}^{**})$ as the compatibility conditions at the four corners of Ω^{**} are all satisfied. In particular, the compatibility condition at $(0, 1)$ is satisfied due to $f(0, 1) = 0$ from (13d).

Since $Lw_1 = 0$, while, by (13c), we have $Le^{-\alpha_1 x/\varepsilon} \leq 0$, use a maximum principle to deduce the first desired bound

$$|w_1(x, y)| \leq Ce^{-\alpha_1 x/\varepsilon} \quad \text{for } (x, y) \in \bar{\Omega}^{**}.$$

In view of $w_1 \in C^{3,\lambda}(\bar{\Omega}^{**})$, an analogue of (14) applies to w_1 in Ω^{**} and implies another desired bound

$$|\partial_x^k w_1| \leq C\varepsilon^{-k}.$$

Sharper bounds on the derivatives of w_1^* in the direction normal to the side $x = 0$ can be derived as follows. Consider the following expansion of w_1

$$w_1(x, y) = v(0, y)\phi(x, y) + \varepsilon z_1(x, y),$$

where

$$\varepsilon\phi_{xx} + a_1(0, y)\phi_x = 0 \quad \text{for } x \in (0, 1), \quad \phi(0, y) = 1, \quad \phi(1, y) = 0.$$

A calculation shows that $z_1 = 0$ on $\partial\Omega^{**}$, and $Lz_1(x, y) = g(x, y)$ in Ω^{**} , where

$$|g(x, y)| \leq C\varepsilon^{-1}(1 + x/\varepsilon)e^{-x a_1(0, y)/\varepsilon} \leq C\varepsilon^{-1}e^{-\alpha_1 x/\varepsilon}.$$

By the maximum/comparison principle, it follows now that

$$|z_1(x, y)| \leq Ce^{-\alpha_1 x/\varepsilon} \leq C \quad \text{for } (x, y) \in \Omega^{**}.$$

Next, using the stretching transformations $\xi = x/\varepsilon$, $\eta = y/\varepsilon$ and the Schauder-type bound (12) for the corresponding stretched domain $\hat{\Omega}^{**} = [0, 1/\varepsilon] \times [-1/\varepsilon, 1/\varepsilon]$, we deduce that

$$|\partial_\xi^k \partial_\eta^m z_1| \leq C(\varepsilon \|g(\varepsilon\xi, \varepsilon\eta)\|_{C^{1,\lambda}(\hat{\Omega}^{**})} + \max_{\hat{\Omega}^{**}} |z_1|) \leq C \quad \text{for } 0 \leq k + m \leq 3.$$

This implies that

$$\|\partial_x^k \partial_y^m z_1\|_\Omega \leq C\varepsilon^{-k-m}, \quad 0 \leq k + m \leq 3.$$

The remaining assertion on w_1 follows. \square

Finally, consider the corner layer component $w_{12} = u - (v + w_1 + w_2)$, which is defined on the original domain Ω as follows:

$$(20) \quad \begin{aligned} Lw_{12} &= 0 && \text{for } (x, y) \in \Omega, \\ w_{12}(x, 0) &= -w_1(x, 0), \quad w_{12}(0, y) = -w_2(0, y), \quad w_{12}(1, y) = w_{12}(x, 1) = 0. \end{aligned}$$

Lemma 3.3 (Corner-layer component). *Under the conditions of Lemma 3.1, the function $w_{12} \in C^{3,\lambda}(\bar{\Omega})$, and for $(x, y) \in \bar{\Omega}$ we have*

$$(21) \quad |w_{12}| \leq C e^{-\alpha_1 x/\varepsilon} e^{-\alpha_2 y/\varepsilon}; \quad |\partial_x^k \partial_y^m w_{12}| \leq C \varepsilon^{-k-m}, \quad 1 \leq k+m \leq 3.$$

Proof. Recall that $u, v, w_1, w_2 \in C^{3,\lambda}(\bar{\Omega})$ and $L(u-v) = Lw_1 = Lw_2 = 0$. Also note that $u - (v + w_1 + w_2) = 0$ on $\partial\Omega$. Thus $u - (v + w_1 + w_2)$ solves problem (20), i.e. $w_{12} = u - (v + w_1 + w_2) \in C^{3,\lambda}(\bar{\Omega})$. From the comparison principle and the bounds on w_1 and w_2 established above, we deduce that

$$|w_{12}(x, y)| \leq C e^{-\alpha_1 x/\varepsilon} e^{-\alpha_2 y/\varepsilon} \quad \text{for } (x, y) \in \Omega$$

and a version of (14) also applies to the derivatives of w_{12} . \square

Thus we have constructed a Shishkin solution decomposition (15) and estimated the derivatives of its components. Sharper pointwise bounds on the derivatives of the layer components, which contain decaying exponential factors, can be derived using local Schauder-type estimates [51, p.110, (1.12) and (1.13)] in place of the global bound [51, p.110, (1.11)].

Remark 3.2. *Once a Shishkin decomposition has been constructed, the choice of transition points for a piecewise-uniform tensor-product Shishkin mesh is obvious. Similarly to (8), we choose the transition points σ_1 and σ_2 in the x - and y -direction, respectively, by*

$$\sigma_k = \min\{\varepsilon C_k \ln N, \frac{1}{2}\}, \quad C_k \geq p/\alpha_k \quad \text{for } k = 1, 2,$$

where p is the order of the difference operator, e.g., $p = 1$ for simple upwinding. Then the piecewise uniform meshes $\{x_i\}_{i=0}^N$ and $\{y_j\}_{j=0}^N$ are obtained by dividing each of the intervals $[0, \sigma_k]$ and $[\sigma_k, 1]$ into $\frac{1}{2}N$ equal subintervals, where $k = 1$ for the x -direction and $k = 2$ for y -direction.

Now, if the continuous problem (13) is discretized by a finite difference scheme $L^N U_{ij} = f(x_i, y_j)$ for $(x_i, y_j) \in \Omega$, with an inverse-monotone operator L^N , e.g., using simple upwinding, then the Shishkin decomposition facilitates the numerical analysis as follows. The discrete solution is decomposed as $U = V + W_1 + W_2 + W_{12}$ similarly to the exact-solution decomposition (15), where $L^N U = f$, $L^N W_{1,2,12} = 0$ for $(x_i, y_j) \in \Omega$, and $V_{ij} = v(x_i, y_j)$, $W_{1,2,12;ij} = w_{1,2,12}(x_i, y_j)$ at the boundary meshnodes $(x_i, y_j) \in \partial\Omega$.

Using the bounds (17) on the derivatives of the regular component v and classical stability and consistency arguments, for the error in the regular component one gets $|V_{ij} - v(x_i, y_j)| \leq CN^{-1}$. Next, from the pointwise bound (18) on the continuous layer function w_1 , one can deduce that $|w_1(x_i, y_j)| \leq CN^{-p}$ for $i \geq \frac{1}{2}N$. Given that $L^N W_1 = 0$, discrete barrier functions can be constructed to show that $|W_{1;ij}| \leq CN^{-p}$ for $i \geq \frac{1}{2}N$. Hence $|W_{1;ij} - w_1(x_i, y_j)| \leq CN^{-p}$ for $i \geq \frac{1}{2}N$, i.e. outside the fine-mesh region in the x -direction. For $i < \frac{1}{2}N$ (where the mesh step in the x -direction depends on ε), stability and consistency arguments combined with (19) yield $|W_{1;ij} - w_1(x_i, y_j)| \leq CN^{-1} \ln N$. As $p = 1$, we get $|W_{1;ij} - w_1(x_i, y_j)| \leq CN^{-1} \ln N$ at all mesh nodes. In a similar manner, at all mesh nodes one also gets $|W_{2;ij} - w_2(x_i, y_j)| \leq CN^{-1} \ln N$ and $|W_{12;ij} - w_{12}(x_i, y_j)| \leq CN^{-1} \ln N$ (the latter bound here is obtained using (21)). Thus, invoking a Shishkin decomposition for problem (13), we have established that $|U_{ij} - u(x_i, y_j)| \leq CN^{-1} \ln N$ for all $(x_i, y_j) \in \bar{\Omega}$.

4. Generality of the Shishkin approach: broad range of problems analyzed

In our opinion, one significant feature of the Shishkin mesh and the Shishkin approach to the solution decomposition and the convergence analysis is the fact that the same methodology is applicable to broad classes of singularly perturbed problems. To illustrate this observation, we now identify some of the classes of problems within the field of singularly perturbed partial differential equations for which piecewise-uniform meshes have been successfully used by Shishkin to construct and analyze robust numerical methods.

We start with monographs [82, 94], in which Shishkin considers the following **general problem classes**:

- elliptic reaction-diffusion equations

$$[\varepsilon^2 L_2 - c(x)] u(x) = f(x) \quad \text{for } x \in \Omega \subset \mathbb{R}^n;$$
- elliptic convection-diffusion equations

$$(22) \quad [\varepsilon L_2 + L_1] u(x) = f(x) \quad \text{for } x \in \Omega \subset \mathbb{R}^n;$$
- parabolic reaction-diffusion equations

$$[\varepsilon^2 \tilde{L}_2 - \tilde{c}(x, t) - p(x, t) \partial_t] u(x, t) = f(x, t) \quad \text{for } x \in \Omega \subset \mathbb{R}^n, \quad t \in (0, T];$$
- parabolic convection-diffusion equations

$$[\varepsilon \tilde{L}_2 + \tilde{L}_1 - p(x, t) \partial_t] u(x, t) = f(x, t) \quad \text{for } x \in \Omega \subset \mathbb{R}^n, \quad t \in (0, T].$$

Here ε is a small positive parameter, and we use the notation $x = (x_1, \dots, x_n)$. The differential operator L_2 is a general second-order elliptic operator of the form

$$L_2 = \sum_{s,k=1}^n a_{sk}(x) \partial_{x_s} \partial_{x_k} + \sum_{s=1}^n b_s(x) \partial_{x_s} - c^2(x),$$

for which there is a positive constant α such that

$$\alpha^{-1} \sum_{s=1}^n \zeta_s^2 \leq \sum_{s,k=1}^n a_{sk}(x) \zeta_s \zeta_k \leq \alpha \sum_{s=1}^n \zeta_s^2 \quad \text{for } x \in \bar{\Omega}, \quad (\zeta_1, \dots, \zeta_n) \in \mathbb{R}^n.$$

The differential operator L_1 is the first-order hyperbolic operator of the form

$$L_1 = \sum_{s=1}^n b_s^1(x) \partial_{x_s} - c^1(x), \quad \text{where } \sum_{s=1}^n (b_s^1(x))^2 \geq \beta > 0 \quad \text{for } x \in \bar{\Omega}.$$

It is also assumed that

$$c(x) \geq \gamma > 0, \quad c^2(x) \geq 0, \quad c^1(x) \geq 0 \quad \text{for } x \in \bar{\Omega}.$$

In the above relations β and γ are some positive constants. The differential operators \tilde{L}_2 and \tilde{L}_1 used in the parabolic equations are defined similarly to L_2 and L_1 , respectively, but the coefficients in these operators are functions of (x, t) , while

$$p(x, t) \geq \gamma > 0, \quad \tilde{c}(x, t) \geq 0 \quad \text{for } x \in \bar{\Omega}, \quad t \in (0, T].$$

All the coefficients present in these equations are assumed to be bounded and sufficiently smooth.

Shishkin discretizes the above equations combining inverse-monotone finite difference methods with appropriate piecewise-uniform tensor-product meshes, and establishes ε -uniform convergence of such discretizations. The space domain $\Omega \subset \mathbb{R}^n$ is typically such that one can easily introduce a tensor-product mesh, e.g., an n -dimensional box, or n -dimensional analogues of the two-dimensional L -shaped

domain (see Figure 4 (right)). Shishkin also considers piecewise-smooth domains, in which case he employs the overlapping Schwartz domain decomposition method; the curvilinear parts of the boundary are locally straightened so that each subdomain is transformed into an n -dimensional box, in which the equation in the transformed variables is discretized.

It is worth noting that these problems are posed in an arbitrary number of space dimensions. The highest order operators are not just the simple Laplacian operator, but are in fact general elliptic second order operators. Hence, mixed second order derivatives are admitted in these classes. Shishkin has also studied these problem classes assuming weak regularity and minimal compatibility conditions; see [82, 94] and references therein.

Characteristic layers. We particularly emphasize results by Shishkin for characteristic layers. For example, consider the equation

$$(23) \quad -\varepsilon \Delta u + b(x, y) \partial_x u = f(x, y) \quad \text{for } (x, y) \in \Omega = (0, 1)^2,$$

with $b(x, y) \geq \beta > 0$, subject to Dirichlet boundary conditions. Here the data is sufficiently regular and satisfies sufficient compatibility at the corners. Then, in general, a regular exponential layer of width $O(\varepsilon)$ will appear in the vicinity of the right edge $x = 1$ and characteristic boundary layers of width $O(\sqrt{\varepsilon})$ will appear in the vicinity of the bottom and top edges $y = 0$ and $y = 1$.

The asymptotic analysis of this problem is very intricate and serves to illustrate the complexity of its solutions [38, chap. 4], [74] (see also [82, Chapter IV] and [40] for pointwise estimates of solution derivatives). In contrast, a Shishkin mesh combined with an inverse-monotone difference scheme captures globally accurate numerical approximations to the exact solutions not only of (23), but also of its analogues in n dimensions. In particular, in [82, Chapt. IV, §1] Shishkin considers equation (22) in an n -dimensional box, subject to Dirichlet boundary conditions, with the coefficients of L_1 satisfying, for some $1 \leq p < n$,

$$b_s^1 \geq \beta > 0 \quad \text{for } s = 1, \dots, p; \quad b_s^1 = 0 \quad \text{for } s = p + 1, \dots, n.$$

For this equation, under minimal regularity assumptions, it is shown that one obtains convergence of orders almost $1/14$ or $1/18$ (with a logarithmic factor; the order depends on the regularity assumed). Furthermore, for the two-dimensional equation (23), under stronger assumptions on the regularity of the solution, one can establish convergence of $O(N^{-1} \ln^2 N)$, where N is the number of meshnodes in the x - and y -directions [70]. Note that the Shishkin mesh used in [70] is constructed exactly as in §2.2 for equation (10) in the case of $\beta_1 = \beta_2 = 0$.

Shishkin also establishes a **higher-order convergence** of $O(N^{-2} \ln^2 N)$ [92] for a version of (23) with variable diffusion coefficients and a reaction term:

$$-\varepsilon [a_1(x, y) \partial_x^2 + a_2(x, y) \partial_y^2] u + b(x, y) \partial_x u + c(x, y) u = f(x, y).$$

This equation was considered in the square domain $(0, 1)^2$, with $a_{1,2} \geq \alpha > 0$, $b(x, y) \geq \beta > 0$ and $c(x, y) \geq 0$, subject to Dirichlet boundary conditions. A Shishkin mesh was constructed exactly as in §2.2 for equation (10) in the case of $\beta_1 = \beta_2 = 0$. For ε sufficiently large compared to the maximum mesh size, central differencing was employed in the entire domain as in this case it yields an inverse-monotone method. Otherwise, the domain was divided into four subdomains. In the regular-exponential-layer subdomain along the edge $x = 1$, central differencing was used as in this subdomain the mesh size in the x -direction is negligible compared to ε so the discrete maximum principle holds. In the characteristic-layer subdomains

along the edges $y = 0$ and $y = 1$, the almost-second-order-accurate computed solution was obtained combining the upwind discretization of the parabolic operator $-\varepsilon a_2(x, y) \partial_y^2 + b(x, y) \partial_x + c(x, y)$ with a defect-correction technique. Similarly, in the interior subdomain, the upwind discretization of the reduced hyperbolic operator $b(x, y) \partial_x + c(x, y)$ was combined with a defect-correction technique.

In the case of regular exponential layers, an alternative approach to using layer-adapted meshes is to use fitted operator methods on uniform meshes [37, 73]. However, in the case of problems with characteristic layers, Shishkin established that no fitted operator method on a uniform mesh exists for such problems [76, 80]; see also [82, §II.1.3], [64, Chap.14], [63].

Systems of singularly perturbed differential equations. Since 2003, there has been a growing attention in the literature to layer-adapted meshes in the numerical solution of systems of singularly perturbed reaction-convection-diffusion equations; see the review paper [60]. Interestingly, the earliest papers on this topic seem to be by Bakhvalov [14] and by Shishkin [83, 85, 86]; see also [94, Chap. 13] and references therein for his more recent results. We note that Shishkin typically considers systems of partial differential equations, rather than ordinary differential equations, posed in strip domains or rectangular domains; the equations are coupled via the reaction terms (i.e. weakly coupled) and involve **multiple small parameters**.

We conclude this section noting that in his extensive set of publications, Shishkin has applied his methodology to a variety of problems, some of which have not been discussed here (so we refer the reader to the recent monograph [94]). In particular, he has constructed and analyzed numerical methods for semi-linear [27, 28] and quasi-linear [29] equations, problems on unbounded domains [93], multi-parameter problems [67], [94, Chap. 8], problems with discontinuous data [68, 77, 78, 79] and mixed derivatives [23]. He has also developed parameter-uniform algorithms for the benchmark problems of Burgers [88] equation, the Blasius [26, 89] and Prandtl [26, 90] problems associated with modelling flow past a flat plate, and the Black-Scholes equation [53] from financial mathematics. In his research, he has invoked additional computational techniques such as domain decomposition [61, 84], defect-correction [35, 36] and Richardson extrapolation [75] methods. In our opinion, the generality of Shishkin's approach is clearly evidenced by this significant body of research, which has yielded wide appreciation within the research community.

5. Impact on the area

A search of the research literature indicates that there is an increasing number of papers that invoke a Shishkin mesh and/or a Shishkin decomposition. In this section we review some of this literature to highlight how Shishkin's ideas were employed and, furthermore, blended with other techniques by authors other than Shishkin. This sample of papers is not a comprehensive survey. Instead, we have selected some papers to identify various research paths that have evolved around the analysis and implementation of the Shishkin and other layer-adapted meshes. We refer the reader to other sources [55, 72, 73, 95] for more extensive reviews of layer-adapted meshes.

By the mid 1990's the Shishkin mesh had become established as a powerful ingredient in computational methods for singularly perturbed partial differential equations. Furthermore, this ignited an interest in the broad area of fitted/layer-adapted/a priori-adapted meshes and fuelled a re-examining of the Bakhvalov mesh

and other layer-adapted meshes [55, 72, 73, 102]. Shishkin meshes have been employed to create parameter-uniform numerical methods for singularly perturbed problems having additional singularities due to, for example, the geometry of the domain [9, 46] or discontinuous data [16]. Shishkin meshes have also been employed within the context of coupled systems [24, 39, 60] of singularly perturbed differential equations.

Shishkin typically considers an inverse-monotone method on a Shishkin mesh and establishes its uniform convergence in the discrete maximum norm by combining the solution decomposition (into regular and layer components) procedure with the classical numerical analysis techniques of discrete maximum principle/barrier function and truncation error analysis. Versions of this overall approach have been employed by many researchers to create robust numerical methods and establish their convergence for various singularly perturbed problems; see, for example, [15, 21, 44, 57, 97, 98]. Several publications have studied the Shishkin decomposition and related expansions [21, 22, 58]. In a series of recent publications, Andreev has produced Shishkin-type [4, 6] and Bakhvalov-type [5] decompositions and consequently established parameter uniform convergence assuming minimal or no compatibility conditions at the corners of rectangular domains.

Achieving higher order convergence on Shishkin meshes and preserving the monotonicity properties of the computed solutions has been studied in several papers [18, 19, 20, 32, 44, 65]. Others have studied the potential increase in the order of convergence of non-monotone methods, such as classical central differencing for ordinary differential equations [7, 47, 52] and partial differential equations [42, 43]. Alternative approaches to the error analysis of finite difference methods on Shishkin meshes and other layer-adapted meshes have involved discrete Green's functions and energy inequalities. The discrete Green's function approach in this context has been developed in one dimension [10, 7] and two dimensions [3]. Note that stronger stability of one-dimensional inverse-monotone discrete operators was established in [8] and further investigated in [1, 2]. Energy inequalities were employed to establish almost-second-order convergence in the discrete maximum norm for parabolic [42] and elliptic [43] convection-diffusion problems.

Shishkin normally generates parameter-uniform numerical methods using a finite difference methodology. Combining the flexibility of a finite element framework with the benefits of layer-adapted meshes has been of particular interest to many researchers [13, 54, 96, 105]; see also [73, §3.5.2]. Stabilizing higher order discretizations of convection-diffusion problems on Shishkin meshes by using streamline-diffusion [17, 31, 45, 56, 59, 98, 99], discontinuous and continuous interior penalty methods [30, 104], or local projection algorithms [62] continues to attract the attention of many researchers.

Shishkin advocates the use of inverse-monotone numerical methods. The Shishkin mesh is highly anisotropic, but the solving of the associated linear systems is not computationally expensive, especially when monotone methods are employed [12, 33, 71]. It should be noted that non-monotone numerical methods (such as central differencing or standard Galerkin) may significantly increase the computational cost [26, chap 9]; but this can be rectified by using stabilized, although not necessary inverse-monotone, methods (e.g., streamline-diffusion finite elements).

Note that an appropriate Shishkin mesh can be utilized as a benchmark mesh to test alternative approaches to mesh construction such as adaptive mesh generation techniques. For example, see [15, 48, 50] for comparisons of adaptive meshes with Shishkin and Bakhvalov meshes. Although any adaptive mesh will be graded in the

layer regions and so, in this respect, will be very different from a Shishkin mesh, the local mesh sizes of the two meshes can be quite similar within the layer regions. In this sense, one expects a suitable adaptive algorithm to produce a similar mesh to a Shishkin mesh and, furthermore, similar (or even smaller) errors in the computed solution.

In conclusion, we observe that the number of papers that use Shishkin meshes in various contexts is rapidly increasing, which clearly indicates that Shishkin's ideas have had a significant impact on the area of singularly perturbed differential equations.

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