Local L^2 bounded projections in FEEC

Johnny Guzmán

Division of Applied Mathematics

Brown University

In collaboration with **Douglas Arnold**

Outline

- Finite element Exterior Calculus (FEEC)
- Importance of Bounded Projections
- Previous Work
- Construction of new projections

Finite Element Exterior Calculus

The Finite element exterior Calculus (FEEC) was coined by Arnold, Falk and Winther and developed in the following works.

- Arnold, Falk, and Winther, *Finite element exterior calculus, homological techniques, and applications*, (2006).
- Arnold, Falk, and Winther, *Finite element exterior calculus: from Hodge theory to numerical stability* , (2010).
- Arnold, *Finite Element Exterior Calculus*, (2018).

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- They generalize higher-order Nedelec finite elements to any dimension
- Develop finite element methods for the Hodge Laplacian
- Several applications are discussed
- There is strong connection between FEEC and Hassler Whitney's proof of de Rhams theorem.

Why do we need FEEC or Nedelec/Whitney finite elements: The Maxwell eigenvalue problem

Let $\Omega = [0,1]^2$ and consider the two functional spaces

$$H(\mathrm{curl}\,,\Omega) := \{u \in [L^2(\Omega)]^2 : \mathrm{curl}\, u \in L^2(\Omega)\}$$
$$\mathring{H}(\mathrm{curl}\,,\Omega) := \{u \in H(\mathrm{curl}\,,\Omega) : u \cdot t = 0 \text{ on } \partial\Omega\}.$$

Then, consider the Maxwell eigenvalue problem in two dimensions can be written as:

Find $u \in \mathring{H}(\mathrm{curl}\,,\Omega)$, $\lambda \in \mathbb{R}$ that satisfy

$$\int_\Omega \mathrm{curl}\, u\, \mathrm{curl}\, v = \lambda \int_\Omega u \cdot v \quad \forall v \in \mathring{H}(\mathrm{curl}\,,\Omega).$$

The non-zero eigenvalues are given by $\pi^2(m^2+n^2)$ for $m, n \in \mathbb{N}$.

Finite element approximation of the Maxwell eigenvalue problem

We start with a triangulation of Ω, $\mathcal{T}_h$. Then using this triangulation we can define a finite dimensional space $\mathring{V}_h \subset \mathring{H}(\mathrm{curl}\,, \Omega)$.

Find $u_h \in \mathring{V}_h$, $\lambda_h \in \mathbb{R}$ that satisfy

$$\int_\Omega \mathrm{curl}\, u_h \,\mathrm{curl}\, v_h = \lambda_h \int_\Omega u_h \cdot v_h \quad \forall v \in \mathring{V}_h.$$

One possible choice is

$$\mathring{V}_h = \{v \in [C(\Omega)]^2 : v|_T \in [\mathcal{P}^1(T)]^2, \forall T \in \mathcal{T}_h, v \cdot t = 0 \text{ on } \partial\Omega\}.$$

In other words, this is the space of vector-valued, **continuous piece-wise linear functions** defined $\mathcal{T}_h$ with tangential components vanishing on $\partial\Omega$. This choice, although at first glance seems natural, will lead to **inaccurate approximations** (see for example the work of Arnold, Falk and Winther or the work of Boffi).

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On the other hand, we can use the Nedelec/Whitney edge elements

$$V_h = \{v \in H(\text{curl}\,, \Omega) : v = a_T + b_T(y, -x)', a_T, b_T \in \mathbb{R}, \forall T \in \mathcal{T}_h, \}$$

and the space with boundary conditions

$$\mathring{V}_h = \{v \in V_h : v \cdot t = 0 \text{ on } \partial\Omega\}.$$

Convergence to the eigenvalues can be proved for this space.

Bounded Commuting Projections

In the center of the analysis of FEEC are bounded commuting projections. For simplicity, let us consider the three dimensional case with a contractible domain. We let $\Omega \subset \mathbb{R}^3$ be a contractible polyhedral domain. A de Rham sequence in Sobolev spaces is

$$H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}\,,\Omega) \xrightarrow{\text{curl}} H(\text{div}\,,\Omega) \xrightarrow{\text{div}} L^2(\Omega)$$

Let us use the notation $V^0 := H^1(\Omega)$, $V^1 = H(\text{curl}\,,\Omega)$, $V^2 := H(\text{div}\,,\Omega)$ and $V^3 := L^2(\Omega)$.

After a triangulating Ω one can has the Whitney/Nedelec (lowest order) finite elements:

$$V_h^0(\Omega) \xrightarrow{\text{grad}} V_h^1 \xrightarrow{\text{curl}} V_h^2 \xrightarrow{\text{div}} V_h^3$$

V_h^0 are continuous piece-linears, V_h^1 are Nedelec edge elements, V_h^2 are the Raviart-Thomas finite elements, V_h^3 are piece-wise constants.

Commuting projections of Nedelec

Nedelec developed the canonical projections in this case and showed they are commuting $\tilde{\Pi}^i : \tilde{V}^i \to V_h^i$ for $0 \leq i \leq 3$. Here $\tilde{V}^i \subset V^i$ that consist of smoother functions.

The commuting diagram can be written as:

$$\begin{array}{ccccccc} \tilde{V}^0 & \xrightarrow{\text{grad}} & \tilde{V}^1 & \xrightarrow{\text{curl}} & \tilde{V}^2 & \xrightarrow{\text{div}} & \tilde{V}^3 \\ \downarrow \tilde{\Pi}^0 & & \downarrow \tilde{\Pi}^1 & & \downarrow \tilde{\Pi}^2 & & \downarrow \tilde{\Pi}^3 \\ V_h^0 & \xrightarrow{\text{grad}} & V_h^1 & \xrightarrow{\text{curl}} & V_h^2 & \xrightarrow{\text{div}} & V_h^3 \end{array}$$

$$\text{grad}\, \tilde{\Pi}^0 = \tilde{\Pi}^1 \text{grad}$$

$$\text{curl}\, \tilde{\Pi}^1 = \tilde{\Pi}^2 \text{curl}$$

$$\text{div}\, \tilde{\Pi}^2 = \tilde{\Pi}^3 \text{div}$$

Some examples of canonical projections:

The projection $\tilde{\Pi}^0$ is simply interpolating functions at the vertices of the triangulation. That is,

$$\tilde{\Pi}^0 v(x) = v(x) \quad \text{for all vertices } x.$$

The projection $\tilde{\Pi}^2$ is Raviart-Thomas projection

$$\int_f \tilde{\Pi}^2 v \cdot n \, ds = \int_f v \cdot n \, ds \quad \text{for all faces } f.$$

So, we see that $\tilde{\Pi}^i$ are not defined on the spaces V^i let alone bounded on in these spaces.

Bounded commuting projections

The work of Schoberl (2005), Christiansen and Winther (2008), Falk and Winther (2014), Ern et al. (2019) has lead to bounded commuting projections:

$$\begin{array}{ccccccc} V^0 & \xrightarrow{\text{grad}} & V^1 & \xrightarrow{\text{curl}} & V^2 & \xrightarrow{\text{div}} & V^3 \\ \downarrow \Pi^0 & & \downarrow \Pi^1 & & \downarrow \Pi^2 & & \downarrow \Pi^3 \\ V_h^0 & \xrightarrow{\text{grad}} & V_h^1 & \xrightarrow{\text{curl}} & V_h^2 & \xrightarrow{\text{div}} & V_h^3 \end{array}$$

$$\begin{aligned} \|\Pi^0 v\|_{H^1(\Omega)} &\leq C\|v\|_{H^1(\Omega)} \\ \|\Pi^1 v\|_{H(\text{curl},\Omega)} &\leq C\|v\|_{H(\text{curl},\Omega)} \\ \|\Pi^2 v\|_{H(\text{div},\Omega)} &\leq C\|v\|_{H(\text{div},\Omega)} \end{aligned}$$

Higher dimensions

In general dimension n one has the exterior derivatives d^k for $0 \leq k \leq n-1$ and one has

$$
\begin{array}{ccccccccccc}
V^0 & \xrightarrow{d^0} & V^1 & \xrightarrow{d^1} & V^2 & \longrightarrow & \cdots & \longrightarrow & V^{n-1} & \xrightarrow{d^{n-1}} & V^n \\
\downarrow \Pi^0 & & \downarrow \Pi^1 & & \downarrow \Pi^2 & & & & \downarrow \Pi^{n-1} & & \downarrow \Pi^n \\
V_h^0 & \xrightarrow{d^0} & V_h^1 & \xrightarrow{d^1} & V_h^2 & \longrightarrow & \cdots & \longrightarrow & V^{n-1} & \xrightarrow{d^{n-1}} & V_h^n
\end{array}
$$

Here the space $V^k = \{v \in L^2\Lambda^k(\Omega) : d^k v \in L^2\Lambda^{k+1}(\Omega)\}$. The commuting property can be expressed as:

$$\Pi^k d^k = \Pi^{k+1} d^{k+1}.$$

The bounded property is:

$$\|\Pi^k v\|_{V^k} \leq C \|v\|_{V^k}.$$

Why are Bounded commuting projections the key?

Let $v \in V^k$ and $d^k v = 0$. Then, there exists a $w \in V^{k-1}$ such that

P1) $d^{k-1} w = v$

P2) $\|w\|_{V^{k-1}} \le C \|v\|_{V^k}$

We would like the finite element spaces to have this property too.

Let $v_h \in V_h^k$ and $d^k v_h = 0$. Then, by P1 and P2 above there exists $w \in V^{k-1}$ such that $d^{k-1} w = v_h$ and $\|w\|_{V^{k-1}} \le C \|v_h\|_{V^k}$. We let $w_h = \Pi^{k-1} w$ then using the commuting property of Π^k

$$\text{P1h)} \quad d^{k-1} w_h = \Pi^k d^{k-1} w = \Pi^k v_h = v_h.$$

Using the bounded properties of Π^k we have

$$\text{P2h)} \quad \|w_h\|_{V^{k-1}} = \|\Pi^{k-1} w\|_{V^{k-1}} \le C \|w\|_{V^{k-1}} \le C \|v_h\|_{V^k}.$$

Our Contribution

Inspired by the work of Falk and Winther we define projections to the FEEC spaces with the following properties:

- Commute with the exterior derivative
- Are local
- Bounded in L^2
- Defined in any dimension

Ideas of construction

We let Δ_k denote all simplices of dimension k of our triangulation. The spaces $V_h^i := \text{span}\{\phi_\sigma : \sigma \in \Delta_i\}$. The ϕ_σ are the Whitney forms. They have the following property

$$\int_\tau \text{tr}_\tau \phi_\sigma = \delta_{\tau\sigma}$$

First we see that the canonical projection Π^i will be given by:

$$\Pi^i u = \sum_{\sigma \in \Delta_i} \left(\int_\sigma \text{tr}_\sigma u\right) \phi_\sigma.$$

Writing them in three dimensions using vector calculus we have:

$$\Pi^0 u = \sum_{\sigma \in \Delta_0} u(\sigma)\, \phi_\sigma, \quad \Pi^1 u = \sum_{\sigma \in \Delta_1} \left(\int_\sigma u \cdot t\, ds\right) \phi_\sigma$$

$$\Pi^2 u = \sum_{\sigma \in \Delta_2} \left(\int_\sigma u \cdot n\, ds\right) \phi_\sigma, \quad \Pi^3 u = \sum_{\sigma \in \Delta_3} \left(\int_\sigma u dx\right) \phi_\sigma.$$

Our Projections in 3D

$$\pi^0 u = \sum_{\sigma \in \Delta_0} \Big(\int_{\mathrm{es}(\sigma)} Z^0(\sigma) u \, dx \Big) \phi_\sigma, \quad \pi^1 u = \sum_{\sigma \in \Delta_1} \Big(\int_{\mathrm{es}(\sigma)} Z^1(\sigma) \cdot u \, dx \Big) \phi_\sigma$$

$$\pi^2 u = \sum_{\sigma \in \Delta_2} \Big(\int_{\mathrm{es}(\sigma)} Z^2(\sigma) \cdot u \, dx \Big) \phi_\sigma, \quad \pi^3 u = \sum_{\sigma \in \Delta_3} \Big(\int_{\mathrm{es}(\sigma)} Z^3(\sigma) u \, dx \Big) \phi_\sigma.$$

- Here $\mathrm{es}(\sigma)$ is the extended star of σ which is the union of all tetrahedra that intersect σ.
- The function $Z^i(\sigma)$ are weight functions that are bounded in L^2 and they are the key!

The weight functions Z^i: Property (1)

$$\int_{\mathrm{es}(\sigma)} Z^0(\sigma) u \, dx = u(\sigma) \qquad \forall \sigma \in \Delta_0, u \in V^0$$

$$\int_{\mathrm{es}(\sigma)} Z^1(\sigma) \cdot u \, dx = \int_\sigma u \cdot t \, ds \qquad \forall \sigma \in \Delta_1, u \in V^1$$

$$\int_{\mathrm{es}(\sigma)} Z^2(\sigma) \cdot u \, dx = \int_\sigma u \cdot n \, ds \qquad \forall \sigma \in \Delta_2, u \in V^2$$

$$\int_{\mathrm{es}(\sigma)} Z^3(\sigma) u \, dx = \int_\sigma u \qquad \forall \sigma \in \Delta_3, u \in V^3$$

The weight functions Z^i: Property (2)

$$\begin{aligned} -\operatorname{div} Z^1(\sigma) =& Z^0(\partial_1 \sigma) \quad \forall \sigma \in \Delta_1(\sigma) \\ \operatorname{curl} Z^2(\sigma) =& Z^1(\partial_2 \sigma) \quad \forall \sigma \in \Delta_2(\sigma) \\ -\operatorname{grad} Z^3(\sigma) =& Z^2(\partial_3 \sigma) \quad \forall \sigma \in \Delta_3(\sigma). \end{aligned}$$

Let $\sigma = [x_0, \ldots, x_k]$ be an oriented k-simplex with vertices x_i then the boundary operator is given by

$$\partial_k \sigma = \sum_{j=0}^{k} (-1)^j [x_0, \ldots, \widehat{x_j}, \ldots, x_k].$$

Here $\widehat{\cdot}$ represents omission.

The weight functions Z^i: Property (3) and (4)

(3) The support of $Z^i(\sigma)$ is in $\mathrm{es}(\sigma)$.

(4) They are bounded in L^2 with the correct scaling.

Chief tool to construct Z^i

The fundamental tool is existence of a regular right inverse of the exterior derivative in the kernel space. This is proved on Lipschitz domains in:

- M. Costabel and A. McIntosh. *On Bogovskii and regularized Poincare integral operators for de Rham complexes on Lipschitz domains*, 2010.

Let D be a contractible Lipschitz domain in 3d.

(1) Let $u \in \mathring{H}(\mathrm{curl}\,, D)$ and $\mathrm{curl}\, u = 0$ on D then there exists $\rho \in \mathring{H}^1(D)$ such that $\mathrm{grad}\, \rho = u$.

(2) Let $u \in \mathring{H}(\mathrm{div}\,, D)$ and $\mathrm{div}\, u = 0$ on D then there exists $\rho \in [\mathring{H}^1(D)]^3$ such that $\mathrm{curl}\, \rho = u$.

(3) Let $u \in L^2(D)$ and $\int u\, dx = 0$ then there exists $\rho \in [\mathring{H}^1(D)]^3$ such that $\mathrm{div}\, \rho = u$.

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We need the results of Costabel and McIntosh with $D = \mathrm{es}(\sigma)$.

Furthermore, for the right bounds of the weight functions Z^i we need the right that the right inverse on Costabel and McIntosh are bounded correctly. In particular, we need to see how the constants of the bounds behave.

In a recent paper we attempted to do this, but only handled some cases. So in the current work we list the bound as an assumption.

J. Guzmán and A. J. Salgado, *Estimation of the continuity constants for Bogovskii and regularized Poincare integral operators*, 2021.