MAXIMUM-NORM A POSTERIORI ERROR ESTIMATES FOR SINGULARLY PERTURBED REACTION-DIFFUSION PROBLEMS ON ANISOTROPIC MESHES*

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Abstract. Residual-type a posteriori error estimates in the maximum norm are given for singularly perturbed semilinear reaction-diffusion equations posed in polygonal domains. Linear finite elements are considered on anisotropic triangulations. The error constants are independent of the diameters and the aspect ratios of mesh elements and of the small perturbation parameter.

Key words. a posteriori error estimate, anisotropic triangulation, maximum norm, singular perturbation, reaction-diffusion.

AMS subject classifications. 65N15, 65N30.

1. Introduction. We consider finite element approximations to singularly perturbed semilinear reaction-diffusion equations of the form

$$Lu := -\varepsilon^2 \triangle u + f(x, y; u) = 0 \quad \text{for} \ \ (x, y) \in \Omega, \qquad u = 0 \quad \text{on} \ \ \partial\Omega, \tag{1.1}$$

posed in a, possibly non-Lipschitz, polygonal domain $\Omega \subset \mathbb{R}^2$. Here $0 < \varepsilon \leqslant 1$. We also assume that f is continuous on $\Omega \times \mathbb{R}$ and satisfies $f(\cdot; s) \in L_\infty(\Omega)$ for all $s \in \mathbb{R}$, and the one-sided Lipschitz condition $f(x, y; u) - f(x, y; v) \geqslant C_f[u - v]$ whenever $u \geqslant v$, with some constant $C_f \geqslant 0$. Then there is a unique solution $u \in W^2_\ell(\Omega) \subseteq W^1_q \subset C(\bar\Omega)$ for some $\ell > 1$ and $q > 2$ [6, Lemma 1]. We additionally assume that $C_f + \varepsilon^2 \geqslant 1$ (as a division by $C_f + \varepsilon^2$ immediately reduces (1.1) to this case).

Residual-type a posteriori error estimates in the maximum norm for this equation and its version in $\mathbb{R}^3$ were recently proved in [6] in the case of shape-regular triangulations. In the present paper, we restrict our consideration to Ω in $\mathbb{R}^2$ and linear finite elements, but our focus now shifts to more challenging anisotropic meshes, i.e. we allow mesh elements to have extremely high aspect ratios. (Figure 1.1 below illustrates permitted types of (semi-)anisotropic and isotropic mesh nodes.)

Even for the linear Laplace equation (which one gets from (1.1) if $\varepsilon = 1$, $f_u = 0$), we are aware of no such error estimates in the maximum norm on reasonably general triangulations under no mesh aspect ratio condition (e.g., [7, 15, 5, 17] assume shape regularity of mesh elements). But still of more interest are anisotropic meshes in the context of singularly perturbed differential equations (such as (1.1) with $\varepsilon \ll 1$). For such equations, the maximum norm is sufficiently strong to capture sharp boundary and interior layers in their solutions, while locally anisotropic meshes (fine and anisotropic in layer regions and standard outside) have been shown to yield reliable numerical approximations in an efficient way (see, e.g., [4, 8, 12, 18] and references therein). But such meshes are typically constructed a priori or by heuristic methods.

We discretize (1.1) using standard linear finite elements. Let $S_h \subset H^1_0(\Omega) \cap C(\bar\Omega)$ be a piecewise-linear finite element space relative to a triangulation $\mathcal{T}$, and let the computed solution $u_h \in S_h$ satisfy

$$\varepsilon^2 \langle \nabla u_h, \nabla v_h \rangle + \langle f^I_h, v_h \rangle = 0 \quad \forall\ v_h \in S_h, \qquad\qquad f_h(\cdot) := f(\cdot; u_h). \tag{1.2}$$

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Here $\langle \cdot, \cdot \rangle$ is the $L_2(\Omega)$ inner product, and f_h^I is the standard piecewise-linear Lagrange interpolant of f_h.

To roughly describe our results, assuming that anisotropic mesh elements are almost non-obtuse, our first estimator reduces to

$$\|u - u_h\|_{\infty\,;\Omega} \leqslant C\,\ell_h \max_{z\in\mathcal{N}}\Big(\min\{\varepsilon, H_z\}\,\big\|J_z\big\|_{\infty\,;\gamma_z} + \min\{\varepsilon^2, H_z^2\}\,\|\varepsilon^{-2} f_h^I\|_{\infty\,;\omega_z}\Big) \\ + C\,\|f_h - f_h^I\|_{\infty\,;\Omega}\,, \tag{1.3}$$

where C is independent of the diameters and the aspect ratios of elements in $\mathcal{T}$, and of ε (combine (7.2), (7.5), (5.2) with Lemma 8.1). Here $\mathcal{N}$ is the set of nodes in $\mathcal{T}$, J_z is the standard jump in the normal derivative of u_h across an element edge, ω_z is the patch of elements surrounding any $z \in \mathcal{N}$, γ_z is the set of edges in the interior of ω_z, $H_z = \mathrm{diam}(\omega_z)$, $\ell_h = \ln(2 + \varepsilon \underline{h}^{-1})$, and $\underline{h}$ is the minimum height of triangles in $\mathcal{T}$.

Note that if $\varepsilon = 1$, then (1.3) gives a standard a posteriori error bound, similar to the results in [7, 15, 17], only now we prove it for anisotropic meshes. Furthermore, (1.3) is almost identical with the a posteriori error estimate in [6], where the singularly perturbed case $\varepsilon \in (0, 1]$ is handled; by contrast, now we assume no shape regularity of the mesh.

An inspection of standard proofs for shape-regular meshes reveals that one obstacle in extending them to anisotropic meshes lies in the application of a scaled traced theorem when estimating the jump residual terms (this causes the mesh aspect ratios to appear in the estimator). Remark 6.2 sheds some light on our approach to addressing this technical difficulty.

It should be noted that the interior-residual term $\|\varepsilon^{-2} f_h^I\|_{\infty\,;\omega_z}$ in (1.3) is isotropic (unlike the other terms). In order to give a sharper (and more anisotropic in nature) bound for the interior-residual component of the error, we identify sequences of short edges that connect anisotropic nodes (see Figure 7.2, right). Under some additional assumptions on each such sequence (which we call a path), we prove that

$$\|u - u_h\|_{\infty\,;\Omega} \leqslant C\,\ell_h \Big[\max_{z\in\mathcal{N}}\Big(\min\{\varepsilon, H_z\}\big\|J_z\big\|_{\infty\,;\gamma_z}\Big) + \max_{z\in\mathcal{N}\setminus\mathcal{N}_{\text{paths}}}\Big(\min\{1, \varepsilon^{-2}H_z^2\}\|f_h^I\|_{\infty\,;\omega_z}\Big) \\ + \max_{z\in\mathcal{N}_{\text{paths}}}\Big(\min\{\varepsilon, H_z\}\,\min\{\varepsilon, h_z\}\|\varepsilon^{-2} f_h^I\|_{\infty\,;\omega_z} + \min\{1, \varepsilon^{-2}H_z^2\}\,\mathrm{osc}(f_h^I;\omega_z)\Big)\Big] \\ + C\,\|f_h - f_h^I\|_{\infty\,;\Omega}\,, \tag{1.4}$$

(combine (7.2), (7.16), (5.2) with Lemma 8.1). Here $\mathcal{N}_{\text{paths}}$ is the set of mesh nodes that appear in any path, and $h_z \simeq H_z^{-1}|\omega_z|$. As $h_z \ll H_z$ for anisotropic nodes, so (1.4) is clearly sharper than (1.3). (This is also evidenced by the numerical results in Section 6.4).

Note that our estimators are also useful for a more challenging parabolic version of (1.1). Indeed, plugging them (as error estimators for elliptic reconstructions) into the parabolic estimators [11] yields a posteriori error estimates for the parabolic case.

A posteriori error estimates for a problem of type (1.1) on anisotropic meshes are also given in [3, 9, 13, 14]. In [9, 3] the error is also estimated in the maximum norm, but the considered meshes have a tensor-product structure, while [14, 13] deal with general anisotropic meshes, but the error is estimated in a weaker energy norm.

The paper is organized as follows. In §2, we make basic assumptions on $\mathcal{T}$ and describe permitted mesh node types; for the latter, §3, gives a version of the scaled trace theorem. In §§4–5, the Green's function of a linearized problem is bounded in various norms, and then used to represent the error. Next, a simplified version of our analysis for partially structured anisotropic meshes is presented in §6, while §7 gives a posteriori error estimators for more general anisotropic meshes. We conclude the paper by bounding the Green's function interpolation error in the final §8.

Notation. We write $a \simeq b$ when $a \lesssim b$ and $a \gtrsim b$, and $a \lesssim b$ when $a \leqslant Cb$ with a generic constant C depending on Ω and f, but C does not depend on either ε or the diameters and the aspect ratios of elements in $\mathcal{T}$. Also, for $\mathcal{D} \subset \bar{\Omega}$, $1 \leqslant p \leqslant \infty$, and $k \geqslant 0$, let $\|\cdot\|_{p\,;\mathcal{D}} = \|\cdot\|_{L_p(\mathcal{D})}$ and $|\cdot|_{k,p\,;\mathcal{D}} = |\cdot|_{W_p^k(\mathcal{D})}$, where $|\cdot|_{W_p^k(\mathcal{D})}$ is the standard Sobolev seminorm with integrability index p and smoothness index k.

2. Basic triangulation assumptions. We shall use $z = (x_z, y_z)$, S and T to respectively denote particular mesh nodes, edges and elements, while $\mathcal{N}$, $\mathcal{S}$ and $\mathcal{T}$ will respectively denote their sets. For each $T \in \mathcal{T}$, let H_T be the maximum edge length and $h_T := 2H_T^{-1}|T|$ be the minimum height in T. For each $z \in \mathcal{N}$, let ω_z be the patch of elements surrounding any $z \in \mathcal{N}$, $\mathcal{S}_z$ the set of edges originating at z, and

$$H_z := \operatorname{diam}(\omega_z), \quad h_z := \max_{T\subset\omega_z} h_T, \quad \gamma_z := \mathcal{S}_z \backslash \partial\Omega, \quad \mathring{\gamma}_z := \{S \subset \gamma_z : |S| \lesssim h_z\}. \tag{2.1}$$

Throughout the paper we make the following **Triangulation Assumptions**.

- *Maximum Angle condition.* Let the maximum interior angle in any triangle $T \in \mathcal{T}$ be uniformly bounded by some positive $\alpha_0 < \pi$.
- *Local Coordinate condition.* For any $z \in \mathcal{N}$, let

$$|\sin\angle(S, \hat{S}_z)| \lesssim \tfrac{h_z}{|S|} \quad \forall\, S \subset \mathcal{S}_z, \quad \text{where } \hat{S}_z \in \mathcal{S}_z,\ |\hat{S}_z| = \max_{S\subset\mathcal{S}_z} |S| \tag{2.2}$$

  (i.e. here $\hat{S}_z$ is the longest edge in $\mathcal{S}_z$ (or any of such edges)).
- Also, let the number of triangles containing any node be uniformly bounded.

Note that the above conditions are automatically satisfied by shape-regular meshes.

Additionally, let each $z \in \mathcal{N}$ belong to one of the following **Mesh Node Types** (see also Figure 1.1), defined using a fixed small constant c_0 (to distinguish between anisotropic and isotropic elements).

(1) *Anisotropic Nodes*, whose set is denoted by $\mathcal{N}_{\rm ani}$, are such that

$$h_z < c_0 H_z, \qquad h_T \simeq h_z \text{ and } H_T \simeq H_z \quad \forall\, T \subset \omega_z. \tag{2.3}$$

Note that the above implies that $\mathcal{S}_z$ contains at most two edges of length $\lesssim h_z$

(2) *Semi-Anisotropic Nodes*, whose set is $\mathcal{N}_{\rm s.ani}$, are such that $z \notin \mathcal{N}_{\rm ani}$ and

$$\begin{gathered} h_z < c_0 H_z, \qquad |\angle(S, \hat{S}_z)| \lesssim \tfrac{h_z}{|S|} \quad \forall\, S \subset \mathcal{S}_z, \\ H_T \simeq H_z \text{ and } h_T \simeq h_z \quad \text{or} \quad H_T \simeq h_T \simeq h_z \quad \forall\, T \subset \omega_z. \end{gathered} \tag{2.4}$$

(Compared to (2.2), the angle condition here implies that all edges S in $\mathcal{S}_z$ of length $|S| \simeq H_z$ lie inside a sector of angle $\simeq \frac{h_z}{H_z}$ centered at z.)

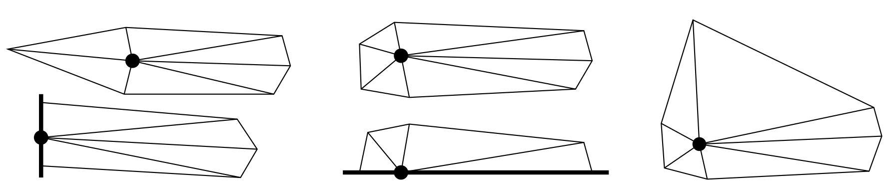


FIG. 1.1. *Examples of anisotropic nodes $z \in \mathcal{N}_{\rm ani}$ (left), semi-anisotropic nodes $z \in \mathcal{N}_{\rm s.ani}$ (centre), an isotropic node $z \in \mathcal{N}_{\rm iso}$ (right), and a node $z \in \mathcal{N}_{\rm ani} \cap \mathcal{N}^*_{\partial\Omega}$ (bottom left).*

(3) *Isotropic Nodes*, whose set is denoted by $\mathcal{N}_{\rm iso}$, are such that

$$\begin{gathered} c_0 H_z \leqslant h_z \leqslant H_z, \qquad\quad h_T \simeq H_T \ \text{ or } \ H_T \simeq H_z \quad\ \forall\ T \subset \omega_z, \\ \forall\ S \in \mathcal{S}_z \quad \exists\ \tilde{T} \subset \omega_z \ \ : \ \ S \subset \partial\tilde{T}, \quad h_{\tilde{T}} \simeq H_{\tilde{T}}, \end{gathered} \tag{2.5}$$

where $\tilde{T}$ is not necessarily in $\mathcal{T}$, but can be constructed as above using S as one of its edges. By this definition, at least one T in ω_z is isotropic and satisfies $h_T \simeq H_T \simeq H_z$, while some T in ω_z may be anisotropic, and others, being isotropic, may have $H_T \ll h_z \simeq H_z$ (see Figure 1.1, right). Note that if z is surrounded only by shape-regular elements, then it is isotropic.

(1*) One typically expects anisotropic elements near the boundary to be aligned along it. To distinguish the boundary nodes for which this is not the case, we introduce a special set of boundary nodes $\mathcal{N}^*_{\partial\Omega}$ as follows:

$$\mathcal{N}^*_{\partial\Omega} := \big\{\ z \in \partial\Omega \cap \mathcal{N}\backslash\mathcal{N}_{\rm iso} \ \ : \ \ |\mathcal{S}_z \cap \partial\Omega| \lesssim h_z \quad \text{or} \quad z \text{ is a corner of } \Omega\ \big\}. \tag{2.6}$$

It will be assumed throughout the paper that $\mathcal{N}^*_{\partial\Omega} \cap \mathcal{N}_{\rm s.ani} = \varnothing$ so $\mathcal{N}^*_{\partial\Omega} \subset \mathcal{N}_{\rm ani}$.

Remark 2.1. Node types (1)–(3) cover most practical situations, although they by no means exhaust all possible configurations of, possibly anisotropic, mesh elements.

3. Scaled trace bounds. In this section, we formulate a version of the scaled trace theorem using the scaled $W_1^1(D)$ norm

$$|||v|||_D := (\text{diam} D)^{-1}\|v\|_{1\,;D} + \|\nabla v\|_{1\,;D}\,.$$

In particular, in view of $\text{diam}(\omega_z) = H_z$ and $\text{diam}(T) \simeq H_T$,

$$|||v|||_{\omega_z} = H_z^{-1}\|v\|_{1\,;\omega_z} + \|\nabla v\|_{1\,;\omega_z}\,, \qquad |||v|||_T \simeq H_T^{-1}\|v\|_{1\,;T} + \|\nabla v\|_{1\,;T}\,.$$

Note that as $\text{diam}(T) \leqslant \text{diam}(\omega_z)$ for any $T \subset \omega_z$, so

$$|||v|||_{\omega_z} \leqslant \sum_{T\subset\omega_z} |||v|||_T\,, \tag{3.1}$$

while for any anisotropic node, in view of (2.3), one in fact has

$$|||v|||_{\omega_z} \simeq \sum_{T\subset\omega_z} |||v|||_T\,. \tag{3.2}$$

LEMMA 3.1. *For any node $z \in \mathcal{N}$ of type* (2.3), (2.4), *or* (2.5), *and any function $v \in W_1^1(\omega_z)$, one has*

$$\|v\|_{1\,;\mathring{\gamma}_z} + \frac{h_z}{H_z}\|v\|_{1\,;\gamma_z\backslash\mathring{\gamma}_z} \lesssim \sum_{T\subset\omega_z} |||v|||_T\,, \tag{3.3}$$

where γ_z and $\mathring{\gamma}_z$ are from (2.1). Furthermore, for any segment $\bar{S}_z \subset \omega_z$ that originates at z and satisfies $|\bar{S}_z| \simeq H_z$, one has

$$\frac{h_z}{H_z}\|v\|_{1\,;\bar{S}_z} \lesssim |||v|||_{\omega_z}\,. \tag{3.4}$$

Proof. First recall that for any edge S of any triangle $T \in \mathcal{T}$

$$\frac{1}{|S|}\,\|v\|_{1\,;S} \lesssim \frac{1}{|T|}\,\|v\|_{1;T} + \frac{\mathrm{diam}(T)}{|T|}\,\|\nabla v\|_{1;T}\,.$$

Here $\mathrm{diam}(T) \simeq H_T$ so $\frac{1}{|S|}\|v\|_{1\,;S} \lesssim \frac{H_T}{|T|}|\!|\!|v|\!|\!|_T$, while $|T| \simeq h_T H_T$ so

$$\frac{h_T}{|S|}\,\|v\|_{1\,;S} \lesssim |\!|\!|v|\!|\!|_T \qquad \text{for } \; S \subset \partial T. \tag{3.5}$$

Now consider each node type separately. If z is an anisotropic node (2.3), then one has $h_T \simeq h_z$ in (3.5) for any edge S of any $T \subset \omega_z$. This immediately implies (3.3) in view of $|S| \simeq h_z$ for any $S \in \mathring{\gamma}_z$ and $|S| \simeq H_z$ for any $S \in \gamma_z \backslash \mathring{\gamma}_z$.

Next, let z be a semi-anisotropic node (2.4). If $S \in \gamma_z$ is an edge of $T \subset \omega_z$ such that $H_T \simeq H_z$ and $h_T \simeq h_z$, then $\|v\|_{1\,;S}$ is estimated using the above argument for nodes of type (2.3). Otherwise, $S \in \mathring{\gamma}_z$ and S is an edge of some isotropic $T' \subset \omega_z$ with $h_{T'} \simeq H_{T'} \simeq |S|$ so, by (3.5), $\|v\|_{1\,;S} \lesssim |\!|\!|v|\!|\!|_{T'}$. Combining these two observations, one gets the desired assertion (3.3).

Finally, let z be an isotropic node (2.5), and note that then $\mathring{\gamma}_z = \gamma_z$ as $h_z \simeq H_z$. Again, if $S \in \gamma_z$ is an edge of some isotropic $T' \subset \omega_z$ with $h_{T'} \simeq H_{T'} \simeq |S|$, then, by (3.5), $\|v\|_{1\,;S} \lesssim |\!|\!|v|\!|\!|_{T'}$. Otherwise, S is an edge of some anisotropic T'' and either $|S| \simeq h_{T''}$ or $|S| \simeq H_{T''} \simeq H_z$. If $|S| \simeq h_{T''}$, then an application of (3.5) yields $\|v\|_{1\,;S} \lesssim |\!|\!|v|\!|\!|_{T''}$. If $|S| \simeq H_z$, one can construct an isotropic triangle $\tilde{T} \subset \omega_z$ (which is not in $\mathcal{T}$) using S as one of its edges; now another application of (3.5) yields $\|v\|_{1\,;S} \lesssim |\!|\!|v|\!|\!|_{\tilde{T}} \lesssim |\!|\!|v|\!|\!|_{\omega_z}$. Combining these three observations with (3.1) implies (3.3) for any isotropic node z.

To prove the remaining bound (3.4), note that for any segment $\bar{S}_z \subset \omega_z$ such that $|\bar{S}_z| \simeq H_z$, one can construct a triangle $\tilde{T} \subset \omega_z$ (which is not in $\mathcal{T}$) using $\bar{S}_z$ as one of its edges, such that $H_{\tilde{T}} = |\bar{S}_z| \simeq H_z$ and $h_{\tilde{T}} \simeq h_z$. Now an application of (3.5) yields $\frac{h_z}{H_z}\|v\|_{1\,;\bar{S}_z} \lesssim |\!|\!|v|\!|\!|_{\tilde{T}} \lesssim |\!|\!|v|\!|\!|_{\omega_z}$ so we have obtained the final desired bound (3.4). □

4. Bounds for the Green's function. To represent the error pointwise, we employ the Green's function for a standard linearization of $Lu_h - Lu$. Most results in this section are quoted from [6].

Remark 4.1. Only to simplify the presentation, we additionally assume that f is differentiable in u, and $f_u(x,y;u) \leqslant \bar{C}_f$ for all x, y, u. In fact, all our results can be obtained without these additional assumptions by an application of [6, Lemma 5].

There exists a Green's function $G(x',y';x,y) : \Omega \times \Omega \to \mathbb{R}$ such that for a solution u of (1.1) and any $v \in \mathring{W}^1_q(\Omega) \subset C(\bar{\Omega})$ with $q > 2$,

$$(v-u)(x',y') = \varepsilon^2 \langle \nabla v, \nabla G(x',y';\cdot)\rangle + \langle f(\cdot;v), G(x',y';\cdot)\rangle. \tag{4.1}$$

For each $(x',y') \in \Omega$, this function G, satisfies

$$\begin{array}{rcll} -\varepsilon^2\,\Delta_{(x,y)}G + p(x,y)\,G &=& \delta(x',y';x,y), & (x,y) \in \Omega, \\ G(x',y';x,y) &=& 0, & (x,y) \in \partial\Omega. \end{array} \tag{4.2}$$

Here the coefficient $p = \int_0^1 f_u(\cdot, u + [v-u]s)\,ds$ is obtained using the standard linearization $f(x,y;v) - f(x,y;u) = p(x,y)[v-u]$, and, in view of Remark 4.1, satisfies $C_f \leqslant p \leqslant \bar{C}_f$, while δ is the 2-dimensional Dirac δ-distribution.

We require the following bounds from [6].

Lemma 4.2. *Let G be from (4.2) with $0 \leqslant C_f \leqslant p \leqslant \bar{C}_f$ and $C_f + \varepsilon^2 \geqslant 1$. Then for any $(x', y') \in \Omega$,*

$$\|G(x', y'; \cdot)\|_{1;\Omega} + \varepsilon |G(x', y'; \cdot)|_{1,1;\Omega} \lesssim 1. \tag{4.3}$$

Also, for the ball $B(x', y'; \rho)$ of radius ρ centered at (x', y'), with $\ell_\rho := \ln(2 + \varepsilon\rho^{-1})$,

$$\|G(x', y'; \cdot)\|_{1\,;B(x',y';\rho)\cap\Omega} \lesssim \varepsilon^{-2}\rho^2\,\ell_\rho, \tag{4.4a}$$
$$|G(x', y'; \cdot)|_{1,1\,;B(x',y';\rho)\cap\Omega} \lesssim \varepsilon^{-2}\rho, \tag{4.4b}$$
$$|G(x', y'; \cdot)|_{2,1\,;\Omega\backslash B(x',y';\rho)} \lesssim \varepsilon^{-2}\ell_\rho. \tag{4.4c}$$

Proof. The desired bounds are given in[6, Theorem 1] for $p = C_f$. An inspection of the proof shows that it also applies to the case $C_f \leqslant p \leqslant \bar{C}_f$. □

5. Error representation via the Green's function. A calculation using (4.1) with $v = u_h$ and (1.2) implies that, $\forall\, v_h \in S_h$,

$$(u_h - u)(x', y') = \varepsilon^2\langle\nabla u_h, \nabla(G - v_h)\rangle + \langle f_h^I, G - v_h\rangle + \langle f_h - f_h^I, G\rangle, \tag{5.1}$$

where, with slight abuse of notation, $G = G(x', y'; \cdot)$. Here $\langle f_h - f_h^I, G\rangle =: \mathcal{E}_{\text{quad}}$ is the quadrature error, for which (4.3) yields

$$|\mathcal{E}_{\text{quad}}| \lesssim \|f_h - f_h^I\|_{\infty\,;\Omega}\,. \tag{5.2}$$

Next, let ϕ_z be the standard linear hat function corresponding to $z \in \mathcal{N}$, and $v_h := G_h + \sum_{z\in\mathcal{N}} \bar{g}_z\phi_z \in S_h$, where $G_h \in S_h$ is some interpolant of G, while $\bar{g}_z$ is a certain average of $G - G_h$ near z (to be specified later), but $\bar{g}_z = 0$ for $z \in \partial\Omega$ (so that $v_h \in S_h$). Now, using $g := G - G_h$, one gets $G - v_h = g - \sum_{z\in\mathcal{N}} \bar{g}_z\phi_z = \sum_{z\in\mathcal{N}}(g - \bar{g}_z)\phi_z$. Combining this with (5.1) gives a standard error representation

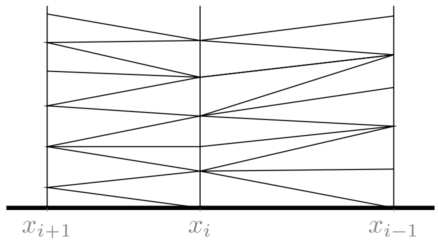
$$\begin{aligned}(u_h - u)(x', y') &= \sum_{z\in\mathcal{N}} \varepsilon^2 \int_{\gamma_z} (g - \bar{g}_z)\phi_z [\![\nabla u_h]\!] \cdot \nu + \sum_{z\in\mathcal{N}} \int_{\omega_z} f_h^I\,(g - \bar{g}_z)\phi_z + \mathcal{E}_{\text{quad}} \\ &=: I + II + \mathcal{E}_{\text{quad}}\,, \end{aligned} \tag{5.3}$$

which holds for any $G_h \in S_h$ and any $\{\bar{g}_z\}_{z\in\mathcal{N}}$ such that $\bar{g}_z = 0$ whenever $z \in \partial\Omega$.

In (5.3), $[\![\nabla u_h]\!]$ is the standard jump in the gradient of u_h across an interior edge. To be more precise, we adapt the notational convention that the unit normal ν to any edge in γ_z takes the clockwise direction about z, while $[\![w]\!]$, for any w, is the jump in w across any edge in γ_z evaluated in the anticlockwise direction about z. So $[\![\nabla u_h]\!] \cdot \nu =: J_z$ is the jump in the anticlockwise direction about z, of the derivative of u_h in the clockwise normal direction ν. Clearly, $|[\![\nabla u_h]\!]| = |J_z|$, i.e. J_z is a signed version of $|[\![\nabla u_h]\!]|$. Occasionally, when computing $[\![\nabla u_h]\!]$ across the boundary edges, we will adapt the convention that $u_h = 0$ in $\mathbb{R}^2\backslash\Omega$.

6. Error analysis on a partially structured anisotropic mesh. To illustrate our ideas, we first present a simplified version of our analysis on a simpler, partially structured anisotropic mesh in a rectangular domain $\Omega = (0, 1)^2$. Throughout this section, we assume that the triangulation satisfies the following conditions.

A1. Let $\{x_i\}_{i=0}^n$ be an arbitrary mesh in the x direction on the interval $(0, 1)$. Then, let each $T \in \mathcal{T}$, for some i,
(i) have the shortest edge on the line $x = x_i$;
(ii) have a vertex on the line $x = x_{i+1}$ or $x = x_{i-1}$ (see Figure 6.1, left).



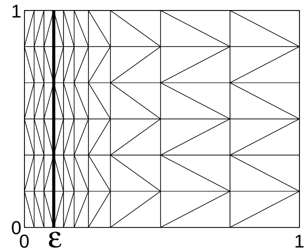


FIG. 6.1. *Partially structured anisotropic mesh (left); triangulation used in Section 6.4 (right).*

A2. Let $\mathcal{N} = \mathcal{N}_{\rm ani}$, i.e. each $z \in \mathcal{N}$ be an anisotropic node in the sense of (2.3).

A3. Let the triangulation satisfy the *Global Coordinate-System* condition in the sense that $|\sin\angle(\hat S_z, \mathrm{i}_x)| \lesssim \frac{h_z}{H_z}$, which, in view of (2.2), is equivalent to $|\sin\angle(S, \mathrm{i}_x)| \lesssim \frac{h_z}{|S|}$ for all $S \in \mathcal{S}_z$, where i_x is the unit vector in the x direction.

The above conditions essentially imply that all mesh elements are anisotropic and aligned in the x-direction. Note that A3 also implies that if $x_z = x_i$, then

$$\omega_z \subseteq \omega_z^* := (x_{i-1}, x_{i+1}) \times (y_z^-, y_z^+), \quad y_z^+ - y_z^- \simeq h_z, \quad \operatorname{diam}\omega_z \simeq \operatorname{diam}\omega_z^* \simeq H_z\,,$$

where (y_z^-, y_z^+) is the range of y within ω_z, while $x_{-1} := x_0$ and $x_{n+1} := x_n$.

A4. With some $J \lesssim 1$, let $\omega_z^* \subset \omega_z^{(J)}$ for all $z \in \mathcal{N}$, with the notation $\omega_z^{(0)} := \omega_z$ and $\omega_z^{(j+1)}$ for the patch of elements in/touching $\omega_z^{(j)}$.

Remark 6.1. Note that if a non-obtuse triangulation satisfies A1 and A2, it also satisfies A3 and A4 with $J = 1$.

6.1. Choice of $\bar g_z$. Main results. The choice of $\bar g_z$ in (5.3) is related to the orientation of anisotropic elements, and is crucial in our analysis. We let $\bar g_z = 0$ for $z \in \partial\Omega$, and, otherwise, for $x_z = x_i$ with some $1 \leqslant i \leqslant n-1$, we let

$$\int_{x_{i-1}}^{x_{i+1}} (g(x, y_z) - \bar g_z)\,\varphi_i(x)\,dx = 0. \tag{6.1}$$

Here $\varphi_i(x)$ is the standard one-dimensional hat function associated with the mesh $\{x_i\}$ (i.e. it has support on (x_{i-1}, x_{i+1}), equals 1 at $x = x_i$, and is linear on (x_{i-1}, x_i) and (x_i, x_{i+1})).

Remark 6.2. For $x_z = x_i$, let $\bar S_z \subset \omega_z^*$ be the interval joining (x_{i-1}, y_z) and (x_{i+1}, y_z). Then the definition (6.1) *of $\bar g_z$ is identical to*

$$\int_{\bar S_z} (g - \bar g_z)\,\varphi_i = 0 \qquad \text{if } x_z = x_i, \quad 1 \leqslant i \leqslant n-1. \tag{6.2}$$

Furthermore, for a non-obtuse triangulation, it is equivalent to $\int_{\bar S_z}(g - \bar g_z)\,\phi_z = 0$. By contrast, one standard choice used in the a posteriori error estimation on shape-regular meshes, which we denote by $\bar g_z'$, is $\int_{\omega_z}(g - \bar g_z')\,\phi_z = 0$ (see, e.g., [16, Lecture 5]).

THEOREM 6.3. *Let $\lambda_z = \min\{\varepsilon, H_z\}$, $g = G(x', y'; \cdot) - G_h$ with any $G_h \in S_h$, and*

$$\Theta := \varepsilon^2 \sum_{z\in\mathcal{N}} \lambda_z^{-1} |\!|\!| g |\!|\!|_{\omega_z^*}\,, \qquad \Theta' := \varepsilon^2 \sum_{z\in\mathcal{N}} \lambda_z^{-2} \|g\|_{1,\,\omega_z^*}\,. \tag{6.3}$$

Then $(u_h - u)(x', y') = I + II + \mathcal{E}_{\rm quad}$ *for any* $(x', y') \in \Omega$*, where* $\mathcal{E}_{\rm quad}$ *is bounded by* (5.2)*, and, under conditions A1–A3,*

$$|I| \lesssim \Theta \max_{z\in\mathcal{N}}\bigl\{\,\lambda_z \bigl\|J_z\bigr\|_{\infty\,;\gamma_z}\bigr\}\,, \tag{6.4}$$

$$|II| \lesssim \Theta \max_{z\in\mathcal{N}}\bigl\{\varepsilon^{-2}\,\lambda_z\,H_z\|f_h^I\|_{\infty\,;\omega_z}\bigr\}\,, \tag{6.5}$$

$$|II| \lesssim \Theta' \max_{z\in\mathcal{N}}\bigl\{\varepsilon^{-2}\,\lambda_z^2\,\|f_h^I\|_{\infty\,;\omega_z}\bigr\}\,. \tag{6.6}$$

Under the additional assumption A4, for II we have an alternative bound

$$|II| \lesssim \max_{z\in\mathcal{N}\backslash\mathcal{N}^*_{\partial\Omega}}\bigl\{\,(\Theta+\Theta')\,\varepsilon^{-2}\lambda_z\,\min\{\varepsilon,h_z\}\|f_h^I\|_{\infty\,;\omega_z} + \Theta'\,\varepsilon^{-2}\lambda_z^2\,\mathrm{osc}(f_h^I;\omega_z)\,\bigr\} + \Theta' \max_{z\in\mathcal{N}^*_{\partial\Omega}}\bigl\{\varepsilon^{-2}\,\lambda_z^2\,\|f_h^I\|_{\infty\,;\omega_z}\bigr\}\,, \tag{6.7}$$

where $\mathcal{N}^*_{\partial\Omega} = \{z\in\mathcal{N} : x_z = 0$ *or* $x_z = 1\}$ *(in agreement with* (2.6)*).*

6.2. Jump Residual. Proof of (6.4).

Proof of (6.4). Split I of (5.3) using $[\![\nabla u_h]\!]\cdot\nu = [\![\partial_x u_h]\!]\nu_x + [\![\partial_y u_h]\!]\nu_y$ as

$$\begin{aligned} I = I' + I'' + I''' := \sum_{z\in\mathcal{N}}\varepsilon^2\int_{\gamma_z}(g-\bar g_z)\phi_z[\![\partial_x u_h]\!]\,\nu_x \\ + \sum_{z\in\mathcal{N}}\varepsilon^2\int_{\gamma_z}[g - g(x,y_z)]\,\phi_z[\![\partial_y u_h]\!]\,\nu_y \\ + \sum_{z\in\mathcal{N}}\varepsilon^2\int_{\gamma_z}[g(x,y_z)-\bar g_z]\,\phi_z[\![\partial_y u_h]\!]\,\nu_y\,. \end{aligned} \tag{6.8}$$

In view of (6.3), to get the desired assertion (6.4), it suffices to show that

$$|I'| + |I''| \lesssim \sum_{z\in\mathcal{N}}\varepsilon^2|\!|\!|g|\!|\!|_{\omega_z^*}\,\bigl\|J_z\bigr\|_{\infty\,;\gamma_z}, \qquad I''' = 0. \tag{6.9}$$

Here the bound for I'' is obtained using $|\phi_z[\![\partial_y u_h]\!]| \leqslant |J_z|$ and $|\nu_y| \leqslant 1$, and also $\int_{\gamma_z}|g(x,y) - g(x,y_z)| \lesssim \|\partial_y g\|_{1\,;\omega_z^*} \lesssim |\!|\!|g|\!|\!|_{\omega_z^*}$.

The estimation of I' in (6.9) is more subtle. We again use $|\phi_z[\![\partial_x u_h]\!]| \leqslant |J_z|$ and it remains to show that

$$\int_{\gamma_z}\bigl(|g\,\nu_x| + |\bar g_z\nu_x|\bigr) \lesssim |\!|\!|g|\!|\!|_{\omega_z^*}\,. \tag{6.10}$$

Here, crucially, $|\nu_x| \lesssim \frac{h_z}{|S|}$ for any $S\in\gamma_z$ (this follows from $|\nu_x| = |\sin\angle(S,\mathrm{i}_x)|$ and A3). To be more precise, for $S\subset\mathring{\gamma}_z$ one has $|S|\simeq h_z$ so $|\nu_x|\leqslant 1$, while for $S\subset\gamma_z\backslash\mathring{\gamma}_z$ one has $|S|\simeq H_z$ so $|\nu_x|\leqslant\frac{h_z}{H_z}$. Now an application of (3.3) combined with (3.2) yields $\int_{\gamma_z}|g\,\nu_x| \lesssim |\!|\!|g|\!|\!|_{\omega_z}$. One then gets $\int_{\gamma_z}|g\,\nu_x| \lesssim |\!|\!|g|\!|\!|_{\omega_z^*}$ as $\mathrm{diam}\,\omega_z \simeq \mathrm{diam}\,\omega_z^*$. Next, note that $\int_{\gamma_z}|\bar g_z\nu_x| \lesssim h_z|\bar g_z| \lesssim \frac{h_z}{H_z}\|g\|_{1\,;\bar S_z}$ where we used (6.2), in which $\bar S_z\subset\omega_z^*$ and $|\bar S_z|\simeq H_z$. Now an application of (3.4) yields $\int_{\gamma_z}|\bar g_z\nu_x| \lesssim |\!|\!|g|\!|\!|_{\omega_z^*}$. Note also that if $x_z = x_0$ or $x_z = x_n$, then $\bar g_z = 0$, so (6.10) remains valid. Thus the bound for I' in (6.9) is established.

It remains to estimate $I''' = \sum_{z\in\mathcal{N}} I_z'''$, where

$$I_z''' := \varepsilon^2\int_{\gamma_z}[g(x,y_z)-\bar g_z]\,\phi_z[\![\partial_y u_h]\!]\,\nu_y\,. \tag{6.11}$$

First, let $z \in \mathcal{N}\backslash\partial\Omega$, i.e. $x_z = x_i$ for some $i = 1, \dots n-1$, and $y_z \neq 0, 1$. In view of the mesh structure, $\phi_z = \varphi_i(x)$ on γ_z, while $\nu_y|d\nu| = -\mathrm{sgn}(x - x_i)\,dx$, and $[g(x, y_z) - \bar{g}_z]$ is a function of x only. Hence, one gets

$$I_z''' = \varepsilon^2\Big(\sum_{S\in\gamma_z^-} [\![\partial_y u_h]\!]\Big)\int_{x_{i-1}}^{x_i} [g(x, y_z) - \bar{g}_z]\,\varphi_i(x)\,dx$$
$$- \varepsilon^2\Big(\sum_{S\in\gamma_z^+} [\![\partial_y u_h]\!]\Big)\int_{x_i}^{x_{i+1}} [g(x, y_z) - \bar{g}_z]\,\varphi_i(x)\,dx\,. \tag{6.12}$$

Here $\gamma_z^+ = \{S \in \gamma_z\,:\,\mathrm{proj}_x S = (x_i, x_{i+1})\}$ and $\gamma_z^- = \{S \in \gamma_z\,:\,\mathrm{proj}_x S = (x_{i-1}, x_i)\}$, where $\mathrm{proj}_x(\cdot)$ denotes the projection onto the x-axis. In fact, $\gamma_z = \mathring{\gamma}_z \cup \gamma_z^+ \cup \gamma_z^-$, where $\mathring{\gamma}_z = \gamma_z \cap \{x = x_i\}$ contains two short edges, for which $\nu_y = 0$, so $\int_{\mathring{\gamma}_z}$ does not appear in (6.12). Combining (6.12) with the definition (6.1) of $\bar{g}_z$ implies that

$$I_z''' = \varepsilon^2\Big(\sum_{S\in\gamma_z^-\cup\gamma_z^+} [\![\partial_y u_h]\!]\Big)\int_{x_{i-1}}^{x_i} [g(x, y_z) - \bar{g}_z]\,\varphi_i(x)\,dx\,. \tag{6.13}$$

Here $\sum_{S\in\gamma_z^-\cup\gamma_z^+}[\![\partial_y u_h]\!] = \sum_{S\in\mathcal{S}_z}[\![\partial_y u_h]\!] = 0$ (in view of $\gamma_z = \mathcal{S}_z$ and $[\![\partial_y u_h]\!]\big|_{\mathring{\gamma}_z} = 0$). Hence $I_z''' = 0$ for $z \in \mathcal{N}\backslash\partial\Omega$.

Next, if $z \in \mathcal{N} \cap \partial\Omega$ and either $y_z = 0$ or $y_z = 1$, then $\bar{g}_z = 0$ and $g(x, y_z) = 0$ (as $g = 0$ on $\partial\Omega$), so, by (6.11), again $I_z''' = 0$. Finally, consider I_z''' when $z \in \mathcal{N} \cap \partial\Omega$, but $y_z \neq 0, 1$ and $x_z = x_n$ (as the case $x_z = x_0$ is similar). Now $\bar{g}_z = 0$, but both (6.12) and (6.13) remain valid with $\gamma_z^+ = \varnothing$. Also, $\sum_{S\in\gamma_z^-}[\![\partial_y u_h]\!] = \sum_{S\in\mathcal{S}_z}[\![\partial_y u_h]\!] = 0$ (in view of $\gamma_z^- = \mathcal{S}_z\backslash\partial\Omega$ and $[\![\partial_y u_h]\!]\big|_{\mathcal{S}_z\cap\partial\Omega} = 0$). Hence we again get $I_z''' = 0$. □

Remark 6.4. An inspection of the above proof shows that it remains valid if $\{\bar{g}_z\}_{z\in\mathcal{N}}$ *defined by (6.1) are replaced by* $\{\bar{g}_z^*\}_{z\in\mathcal{N}}$ *such that* $\bar{g}_z^* = 0$ *for* $z \in \partial\Omega$, *and*

$$H_z|\bar{g}_z - \bar{g}_z^*| \lesssim |\!|\!|g|\!|\!|_{\omega_z^*}\,, \qquad h_z H_z|\bar{g}_z^*| \lesssim \|g\|_{1\,;\omega_z^*}\,. \tag{6.14}$$

Indeed, I *will include an additional component* $I^* := \sum_{z\in\mathcal{N}} \varepsilon^2 \int_{\gamma_z} (\bar{g}_z - \bar{g}_z^*)\phi_z [\![\nabla u_h]\!]\cdot\nu$, *for which one easily gets* $|I^*| \leqslant \sum_{z\in\mathcal{N}} \varepsilon^2\, H_z|\bar{g}_z - \bar{g}_z^*|\, \|J_z\|_{\infty\,;\gamma_z}$ *and then (using the first relation from* (6.14)*) a bound similar to the one for* $|I'| + |I''|$ *in* (6.9).

Note that (6.14) *is satisfied if* $\bar{g}_z^*$, *for* $z \in \mathcal{N}\backslash\partial\Omega$ *with* $x_z = x_i$, *is defined by*

$$\int_{\omega_z^*} [g(x, y) - \bar{g}_z^*]\,\varphi_i(x) = 0, \tag{6.15}$$

where recall that $\omega_z^* = (x_{i-1}, x_{i+1}) \times (y_z^-, y_z^+)$ *and* $|\omega_z^*| \simeq h_z H_z$. *Now the second relation in* (6.14) *follows immediately, while* $\int_{\omega_z^*}[g(x, y_z) - g(x, y)]\,\varphi_i(x) \simeq h_z H_z(\bar{g}_z - \bar{g}_z^*)$ *implies* $H_z|\bar{g}_z - \bar{g}_z^*| \lesssim \|\partial_y g\|_{1\,;\omega_z^*}$ *and so the first relation in* (6.14).

6.3. Interior residual. Proof of (6.5)–(6.7). Now we focus on the interior-residual component $I\!I$ of the error (5.3).

Proof of (6.5). In view of (6.3), it suffices to show that

$$|I\!I| \lesssim \sum_{z\in\mathcal{N}} H_z |\!|\!|g|\!|\!|_{\omega_z^*}\, \|f_h^I\|_{\infty\,;\omega_z}. \tag{6.16}$$

When estimating I' in Section 6.2, we used $h_z|\bar{g}_z| \lesssim \frac{h_z}{H_z}\|g\|_{1\,;\bar{S}_z} \lesssim |\!|\!|g|\!|\!|_{\omega_z^*}$ (while for $z \in \partial\Omega$, one simply has $\bar{g}_z = 0$). Hence, $\|g - \bar{g}_z\|_{1\,;\omega_z} \lesssim \|g\|_{1\,;\omega_z} + h_z H_z|\bar{g}_z| \lesssim H_z |\!|\!|g|\!|\!|_{\omega_z^*}$. Combining this with the definition of $I\!I$ in (5.3) yields (6.16) and hence (6.5). □

Proof of (6.6). This bound is obtained similarly to (6.5), only the definition of Θ' is combined with $|II| \lesssim \sum_{z\in\mathcal{N}} \|g\|_{1\,;\omega_z^*}\|f_h^I\|_{\infty\,;\omega_z}$. For the latter, by Remark 6.4, replace $\{\bar g_z\}$ in (5.3) by $\{\bar g_z^*\}$ of (6.15), which, by (6.14), yields $\|g-\bar g_z^*\|_{1\,;\omega_z} \lesssim \|g\|_{1\,;\omega_z^*}$. □

Proof of (6.7). Using Remark 6.4, we again replace $\{\bar g_z\}$ in (5.3) by $\{\bar g_z^*\}$ of (6.15). Let $\mathcal{N} = \cup_{i=0}^n \mathcal{N}_i$, where $\mathcal{N}_i := \{z : x_z = x_i\}$. Note that $\mathcal{N}_0 \cup \mathcal{N}_n = \mathcal{N}^*_{\partial\Omega}$. Now split II of (5.3) as $II = \sum_{i=1}^{n-1} II_i + II_{\rm osc} + II^*_{\partial\Omega}$ as follows:

$$\begin{aligned} II_i &:= \sum_{z\in\mathcal{N}_i} \int_{\omega_z} f_h^I(x_i,y)\,(g-\bar g_z^*)\,\phi_z\,, \\ II_{\rm osc} &:= \sum_{z\in\mathcal{N}\backslash\mathcal{N}^*_{\partial\Omega}} \int_{\omega_z} [f_h^I - f_h^I(x_z,y)]\,(g-\bar g_z^*)\,\phi_z\,, \\ II^*_{\partial\Omega} &:= \sum_{z\in\mathcal{N}^*_{\partial\Omega}} \int_{\omega_z} f_h^I\,(g-\bar g_z^*)\,\phi_z\,. \end{aligned}$$

Here, for $II_{\rm osc}$ and $II^*_{\partial\Omega}$, we immediately get a version of (6.6):

$$|II_{\rm osc}| \lesssim \Theta' \max_{z\in\mathcal{N}\backslash\mathcal{N}^*_{\partial\Omega}} \Big\{\varepsilon^{-2}\,\lambda_z^2\,\|f_h^I - f_h^I(x_z,y)\|_{\infty\,;\omega_z}\Big\}, \tag{6.17}$$

$$|II^*_{\partial\Omega}| \lesssim \Theta' \max_{z\in\mathcal{N}^*_{\partial\Omega}} \Big\{\varepsilon^{-2}\,\lambda_z^2\,\|f_h^I\|_{\infty\,;\omega_z}\Big\}. \tag{6.18}$$

So it remains to estimate II_i for $1 \leqslant i \leqslant n-1$, which can be rewritten as

$$II_i = \sum_{z\in\mathcal{N}_i} \int_0^1 f_h^I(x_i,y) \int_{x_{i-1}}^{x_{i+1}} (g-\bar g_z^*)\,\phi_z\,dx\,dy.$$

Note that $\sum_{z\in\mathcal{N}_i}\phi_z = \varphi_i(x)$, so

$$\sum_{z\in\mathcal{N}_i} \int_{x_{i-1}}^{x_{i+1}} g\,\phi_z\,dx = \int_{x_{i-1}}^{x_{i+1}} g\,\varphi_i\,dx =: \hat g_i(y) \int_{x_{i-1}}^{x_{i+1}} \varphi_i\,dx = \sum_{z\in\mathcal{N}_i} \int_{x_{i-1}}^{x_{i+1}} \hat g_i(y)\,\phi_z\,dx\,.$$

Here $\hat g_i$ is deliberately defined similarly to $\bar g_z$ in (6.1). Now

$$II_i = \sum_{z\in\mathcal{N}_i} \int_0^1 f_h^I(x_i,y) \int_{x_{i-1}}^{x_{i+1}} \big(\hat g_i(y) - \bar g_z^*\big)\,\phi_z\,dx\,dy = \sum_{z\in\mathcal{N}_i} \int_{\omega_z} f_h^I(x_i,y)\,\big(\hat g_i(y) - \bar g_z^*\big)\,\phi_z\,.$$

Next,

$$|II_i| \lesssim \sum_{z\in\mathcal{N}_i} \|\hat g_i(y) - \bar g_z^*\|_{1\,;\omega_z}\,\|f_h^I\|_{\infty\,;\omega_z^*} \lesssim \sum_{z\in\mathcal{N}_i} \min\Big\{\|g\|_{1\,;\omega_z^*}, h_z|\!|\!|g|\!|\!|_{\omega_z^*}\Big\}\,\|f_h^I\|_{\infty\,;\omega_z^*}\,. \tag{6.19}$$

Here we used $\|\hat g_i(y) - \bar g_z^*\|_{1\,;\omega_z} \lesssim \|g\|_{1\,;\omega_z^*}$ (which follows from the definition of $\hat g_i$ and (6.15)), and also $|\omega_z| \simeq h_z H_z$ combined with $h_z H_z \|\hat g_i(y) - \bar g_z^*\|_{\infty,\omega_z} \lesssim h_z |\!|\!|g|\!|\!|_{\omega_z^*}$. For the latter, fix $y' \in (y_z^-, y_z^+)$, and note that the definition of $\hat g_i$ and (6.15) yield $\int_{\omega_z^*}[g(x,y') - g(x,y)]\,\varphi_i(x) \simeq h_z H_z[\hat g_i(y') - \bar g_z^*]$, where $\omega_z^* = (x_{i-1},x_{i+1}) \times (y_z^-, y_z^+)$ and $|\omega_z^*| \simeq h_z H_z$, so indeed $H_z|\hat g_i(y') - \bar g_z^*| \lesssim \|\partial_y g\|_{1\,;\omega_z^*} \lesssim |\!|\!|g|\!|\!|_{\omega_z^*}$.

Finally, combining (6.19) with that $\min\{a, h_z b\} \lesssim (\lambda_z^{-2}a + \lambda_z^{-1}b)\min\{\lambda_z^2, \lambda_z h_z\}$ (for any $a, b > 0$) and $\min\{\lambda_z^2, \lambda_z h_z\} = \lambda_z \min\{\varepsilon, h_z\}$, and then with (6.17), (6.18) and $\|f_h^I - f_h^I(x_z,y)\|_{\infty\,;\omega_z} \leqslant \mathrm{osc}(f_h^I;\omega_z^*)$, and also A4 yields the desired bound (6.7). □

TABLE 6.1
Bakhvalov mesh (see Fig. 6.1, right), $M = \frac{1}{2}N$: maximum nodal errors,estimators $\mathcal{E}$ and $\mathcal{E}_{(6.6)}$.

N	$\varepsilon = 1$	$\varepsilon = 2^{-5}$	$\varepsilon = 2^{-10}$	$\varepsilon = 2^{-15}$	$\varepsilon = 2^{-20}$	$\varepsilon = 2^{-25}$	$\varepsilon = 2^{-30}$
	Errors (odd rows) & Computational Rates (even rows)						
64	3.373e-4	3.723e-3	8.952e-3	8.973e-3	8.973e-3	8.973e-3	8.973e-3
	2.00	1.91	1.01	1.00	1.00	1.00	1.00
128	8.445e-5	9.935e-4	4.446e-3	4.484e-3	4.484e-3	4.484e-3	4.484e-3
	2.00	1.98	1.04	1.00	1.00	1.00	1.00
256	2.112e-5	2.523e-4	2.165e-3	2.236e-3	2.236e-3	2.236e-3	2.236e-3
	Estimator $\mathcal{E} = \max\{\mathcal{E}_{(6.4)}, \mathcal{E}_{(6.7)}\}$ (odd rows) & Effectivity Indices (even rows)						
64	7.353e-3	1.204e-1	1.224e-1	1.230e-1	1.302e-1	1.302e-1	1.302e-1
	21.80	32.33	13.68	14.48	14.51	14.51	14.51
128	1.885e-3	3.212e-2	6.005e-2	6.621e-2	6.646e-2	6.647e-2	6.647e-2
	22.32	32.33	13.51	14.77	14.82	14.82	14.82
256	4.771e-4	8.268e-3	3.073e-2	3.328e-2	3.354e-2	3.354e-2	3.354e-2
	22.59	32.77	14.20	14.89	15.00	15.00	15.00
	Estimator $\mathcal{E}_{(6.6)}$ (odd rows) & Effectivity Indices (even rows)						
64	6.810e-3	2.516e-1	9.403e-1	9.981e-1	9.999e-1	1.000e+0	1.000e+0
	20.19	67.59	105.04	111.23	111.44	111.45	111.45
128	1.761e-3	1.120e-1	8.858e-1	9.961e-1	9.999e-1	1.000e+0	1.000e+0
	20.86	112.72	199.26	222.15	222.98	223.01	223.01
256	4.480e-4	4.036e-2	7.901e-1	9.922e-1	9.998e-1	1.000e+0	1.000e+0
	21.21	159.97	365.01	443.82	447.17	447.27	447.28

6.4. Numerical results. Before we proceed to the analysis of more general meshes, we test the estimators of Theorem 6.3 using a simple version of (1.1) with $\Omega = (0,1)^2$ and $f = u - F(x,y)$, where F is such that the unique exact solution $u = 4y\,(1-y)\,[1 - x^2 - (e^{-x/\varepsilon} - e^{-1/\varepsilon})/(1 - e^{-x/\varepsilon})]$ (the latter exhibits a sharp boundary layer at $x = 0$). We consider one a-priori-chosen layer-adapted mesh, as on Figure 6.1 (right), which is obtained by drawing diagonals from the tensor product of the Bakhvalov grid $\{\chi(\frac{i}{N})\}_{i=1}^{N}$ in the x-direction [2] and a uniform grid $\{\frac{j}{M}\}_{j=0}^{M}$ in the y-direction. The continuous mesh-generating function $\chi(t) = t$ if $\varepsilon > \frac{1}{6}$; otherwise, $\chi(t) = 3\varepsilon \ln \frac{1}{1-2t}$ for $t \in (0, \frac{1}{2} - 3\varepsilon)$ and is linear elsewhere subject to $\chi(1) = 1$.

Table 6.1 gives the maximum nodal errors, the computational convergence rates, and the two estimators $\mathcal{E} := \max\{\mathcal{E}_{(6.4)}, \mathcal{E}_{(6.7)}\}$ and $\mathcal{E}_{(6.6)}$ with their effectivity indices (computed as the ratio of the estimator to the error). Here $\mathcal{E}_{(6.\cdot)}$ denotes the right-hand side of (6.·), in which we set $\Theta = \Theta' = 1$ (while, by Lemma 8.1 below, $\Theta + \Theta' \lesssim \ell_h$), and also replace quantities of type $\min\{1, \varepsilon^{-1}a\}$ by their smoother analogues $\frac{a}{\varepsilon+a}$, e.g., $\mathcal{E}_{(6.6)} = \max_{z\in\mathcal{N}}\{\big(\frac{H_z}{\varepsilon+H_z}\big)^2 \|f_h^I\|_{\infty\,;\omega_z}\}$. Note that we define $\mathcal{E}$ as a maximum, rather than a more standard sum $\mathcal{E}_{(6.4)} + \mathcal{E}_{(6.7)} \simeq \mathcal{E}$, as this allows a more balanced comparison to $\mathcal{E}_{(6.6)}$. Note also that the estimator $\mathcal{E}$ is of type (1.4), while $\mathcal{E}_{(6.6)}$ is a sharper version of the interior-residual term from (1.3).

The mesh is chosen so that the linear interpolation error $\|u - u^I\|_{\infty\,;\infty} \lesssim N^{-2}$; however, as $\varepsilon \to 0$, the convergence rates deteriorate from 2 to 1 (this phenomenon is noted and explained in [10]). For the considered ranges of ε and N, the aspect ratios of the mesh elements take values between 1 and 3.6e+8. Considering these variations, the estimator $\mathcal{E}$ performs reasonably well and its effectivity indices stabilize as $\varepsilon \to 0$. By contrast, the ingredient $\mathcal{E}_{(6.6)}$ of the estimator (1.3) is adequate for $\varepsilon \simeq 1$, but its effectivity deteriorates in the singularly perturbed regime.

7. General-mesh a posteriori error analysis.

7.1. First Estimator. We start with a version of the estimator (6.4), (6.5), (5.2) for a general mesh, To simplify the presentation, we make the following assumption.

$\mathcal{A}1$. If $z \in \mathcal{N}$ is a corner of Ω, then $z \in \mathcal{N}_{\rm iso}$ (i.e. all corners are isotropic nodes and none of them is in $\mathcal{N}^*_{\partial\Omega}$).

This is a reasonable assumption as typical corner singularities are isotropic. Occasionally we make a further assumption.

$\mathcal{A}2$. *Quasi-non-obtuse anisotropic elements.* Let the maximum triangle angle at any anisotropic node z be bounded by $\frac{\pi}{2}+\alpha_1\frac{h_z}{H_z}$ for some positive constant α_1.

Note that for $\mathcal{A}2$, it suffices to satisfy the following stronger condition.

$\mathcal{A}2'$. Let the maximum angle in any triangle be bounded by $\frac{\pi}{2}+\alpha_1\frac{h_T}{H_T}$.

Note also that the latter condition is always satisfied by isotropic elements, and requires only the anisotropic part of the mesh to be close to a non-obtuse triangulation.

Figure 7.1 shows an example of a mesh that satisfies all assumptions made in this section (see Theorems 7.1 and 7.7), but not $\mathcal{A}2'$.

THEOREM 7.1. *Let $\lambda_z = \min\{\varepsilon, H_z\}$, $g = G(x', y'\,;\cdot) - G_h$ with any $G_h \in S_h$, and*

$$\Theta := \varepsilon^2 \sum_{z\in\mathcal{N}} \lambda_z^{-1} \sum_{T\subset\omega_z} |||g|||_T\,, \qquad \Theta' := \varepsilon^2 \sum_{z\in\mathcal{N}} \lambda_z^{-2} \|g\|_{1,\,\omega_z}\,. \tag{7.1}$$

Then $(u_h - u)(x', y') = I + II + \mathcal{E}_{\rm quad}$ for any $(x', y') \in \Omega$, where $\mathcal{E}_{\rm quad}$ is bounded by (5.2)*, and, under conditions $\mathcal{A}1$ and $\mathcal{A}2$,*

$$|I| \lesssim \Theta\ \max_{z\in\mathcal{N}}\big\{\, \lambda_z \big\|J_z\big\|_{\infty\,;\gamma_z} \big\}\,. \tag{7.2}$$

Under condition $\mathcal{A}1$ (without assuming $\mathcal{A}2$), one has

$$|I| \lesssim \Theta\ \Big(\max_{z\in\mathcal{N}}\big\{\, \lambda_z \big\|J_z\big\|_{\infty\,;\gamma_z} \big\} + \max_{z\in\widetilde{\mathcal{N}}_{\rm ani}}\big\{\, \lambda_z H_z h_z^{-1} |\mathcal{J}_z| \big\}\Big)\,, \quad \mathcal{J}_z = \sum_{S\in\mathring{\mathcal{S}}_z} \sigma_z J_z\,, \tag{7.3}$$

where $\widetilde{\mathcal{N}}_{\rm ani} = [\mathcal{N}_{\rm ani}\backslash\partial\Omega] \cup \mathcal{N}^*_{\partial\Omega} \subset \mathcal{N}_{\rm ani}$, and we use the notation $\sigma_z\big|_S = \cos\angle(S, \hat{S}_z)$, $\mathring{\mathcal{S}}_z = \mathring{\gamma}_z$ for $z \in \mathcal{N}_{\rm ani}\backslash\partial\Omega$, and $\mathring{\mathcal{S}}_z = \mathcal{S}_z \cap \partial\Omega$ for $z \in \mathcal{N}^*_{\partial\Omega}$,

$$|II| \lesssim \Theta\ \max_{z\in\mathcal{N}}\big\{\varepsilon^{-2}\, \lambda_z\, H_z \|f_h^I\|_{\infty\,;\omega_z} \big\}\,, \tag{7.4}$$

$$|II| \lesssim \Theta'\ \max_{z\in\mathcal{N}}\big\{\varepsilon^{-2}\, \lambda_z^2\, \|f_h^I\|_{\infty\,;\omega_z} \big\}\,. \tag{7.5}$$

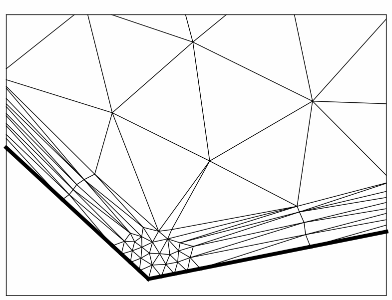


FIG. 7.1. *Example of a mesh considered in §7, which satisfies $\mathcal{A}1$, $\mathcal{A}2$, $\mathcal{A}3$ and $\mathcal{A}4$ (but not $\mathcal{A}2'$).*

Remark 7.2. Note that $\mathcal{J}_z$ *in* (7.3) *involves jumps* J_z *in the normal derivative, across edges in* $\mathring{\mathcal{S}}_z$*, evaluated in the anticlockwise direction about* z*. Here* $\mathring{\mathcal{S}}_z$ *includes exactly 2 edges of length* $\simeq h_z$*. For* $\mathring{\mathcal{S}}_z \subset \partial\Omega$*, one uses* $u_h = 0$ *in* $\mathbb{R}^2\backslash\Omega$*, while* $|\sigma_z| = |\cos\angle(\partial\Omega, \hat{S}_z)|$ *is constant on* $\mathring{\mathcal{S}}_z$*, so a calculation yields* $|\mathcal{J}_z| = |\sigma_z|\,\big|\sum_{S\in\mathring{\mathcal{S}}_z}[\![\nabla u_h]\!]\big|$.

7.2. Jump Residual. Proof of (7.2) **and** (7.3) **on a General Mesh.**

Proof of (7.2). It suffices to show that $|\mathcal{J}_z| \lesssim \frac{h_z}{H_z}\|J_z\|_{\infty\,;\gamma_z}$; then (7.2) immediately follows from (7.3), which is proved below. Indeed, for any $z \in \widetilde{\mathcal{N}}_{\rm ani}$, any edge $S \in \mathring{\mathcal{S}}_z$ is of length $\simeq h_z$, so $\mathcal{A}2$ combined with (2.2) and (2.3) implies that $\big||\angle(S,\hat{S}_z)| - \frac{\pi}{2}\big| \lesssim \frac{h_z}{H_z}$ so $|\sigma_z| = |\cos\angle(S,\hat{S}_z)| \lesssim \frac{h_z}{H_z}$. Now, if $\mathring{\mathcal{S}}_z = \mathring{\gamma}_z \subset \gamma_z$, the desired bound on $|\mathcal{J}_z|$ is straightforward. Otherwise (see Remark 7.2), $\mathring{\mathcal{S}}_z \subset \partial\Omega$, so $|\mathcal{J}_z| \lesssim \frac{h_z}{H_z}\big|\sum_{S\in\gamma_z}[\![\nabla u_h]\!]\big|$ (in view of $\mathcal{S}_z = \mathring{\mathcal{S}}_z \cup \gamma_z$ and $\sum_{S\in\mathcal{S}_z}[\![\nabla u_h]\!] = 0$). The desired bound on $|\mathcal{J}_z|$ follows. ◻

Proof of (7.3). For each fixed $z \in \mathcal{N}$, introduce the following local notation. Let the local cartesian coordinates (ξ,η) be such that $z = (0,0)$, and the unit vector i_ξ in the ξ direction lies along the longest edge $\hat{S}_z \in \mathcal{S}_z$ (see Figure 7.2 (left)).

Next, split $\mathcal{S}_z = \mathring{\mathcal{S}}_z \cup \mathcal{S}_z^+ \cup \mathcal{S}_z^-$, where $\mathring{\mathcal{S}}_z = \{S \subset \mathcal{S}_z : |S| \lesssim h_z\}$ (so $\mathring{\gamma}_z = \mathring{\mathcal{S}}_z\backslash\partial\Omega$). Here we also use $\mathcal{S}_z^+ := \{S \subset \mathcal{S}_z\backslash\mathring{\mathcal{S}}_z : S_\xi \subset \mathbb{R}_+\}$, where $S_\xi = \mathrm{proj}_\xi(S)$ denotes the projection of S onto the ξ-axis. Now, let $(\xi_z^-,\xi_z^+) \ni 0$ be the maximal interval such that $(\xi_z^-,0) \subset S_\xi$ for all $S \in \mathcal{S}_z^-$ and $(0,\xi_z^+) \subset S_\xi$ for all $S \in \mathcal{S}_z^+$. Also, let $\varphi_z(\xi)$ be the standard piecewise-linear hat-function with support on (ξ_z^-,ξ_z^+) and equal to 1 at $\xi = 0$. Note that if $\mathcal{S}_z^- = \varnothing$ (and $\mathcal{S}_z^+ = \varnothing$), then we set $\xi_z^- = 0$ (and $\xi_z^+ = 0$) and do not use φ_z for $\xi < 0$ (and $\xi > 0$).

We make a few observations on the above definitions for particular node types in the following table (for the time being, see the rows for $\mathcal{S}_z^\pm$ and $\xi_z^\pm$).

	$z \in \mathcal{N}_{\rm ani}\backslash\mathcal{N}^*_{\partial\Omega}$	$z \in \mathcal{N}^*_{\partial\Omega} \subset \mathcal{N}_{\rm ani}$	$z \in \mathcal{N}_{\rm s.ani}$	$z \in \mathcal{N}_{\rm iso}$
$\mathcal{S}_z^\pm$ $\xi_z^\pm$	$\mathcal{S}_z^-,\mathcal{S}_z^+ \neq \varnothing$ $\lvert\xi_z^-\rvert \simeq \xi_z^+ \simeq H_z$	$\mathcal{S}_z^- = \varnothing,\ \ \mathcal{S}_z^+ \neq \varnothing$ $\xi_z^- = 0,\ \ \xi_z^+ \simeq H_z$		$\mathcal{S}_z^- = \mathcal{S}_z^+ = \varnothing$ $\xi_z^- = \xi_z^+ = 0$
$\bar{g}_z$	(7.6) ⇒ (7.7)	$\bar{g}_z = 0$	(7.6) ⇒ $(7.7)\big\rvert_{\bar{S}_z^- = \varnothing}$	$\bar{g}_z = 0$
I_z'''	(7.12) $\forall\ z \in \mathcal{N}_{\rm ani}$		$I_z''' = 0$	

Next, for $\xi \in [\xi_z^-,\xi_z^+]$ define a continuous function $\bar{\eta}_z(\xi)$ with the following properties: (i) $\bar{\eta}_z(0) = 0$; (ii) $(\xi,\bar{\eta}_z(\xi)) \in \omega_z$ for all $\xi \in (\xi_z^-,\xi_z^+)$; (iii) $\bar{\eta}_z(\xi)$ is linear on $[\xi_z^-,0]$ and $[0,\xi_z^+]$. (Note that one may choose $\bar{\eta}_z(\xi)$ so that $\{(\xi,\bar{\eta}_z(\xi)) : \xi \in (\xi_z^-,0)\}$ lies on any edge in $\mathcal{S}_z^-$, while $\{(\xi,\bar{\eta}_z(\xi)) : \xi \in (0,\xi_z^+)\}$ lies on any edge in $\mathcal{S}_z^+$; see Figure 7.2 (left).)

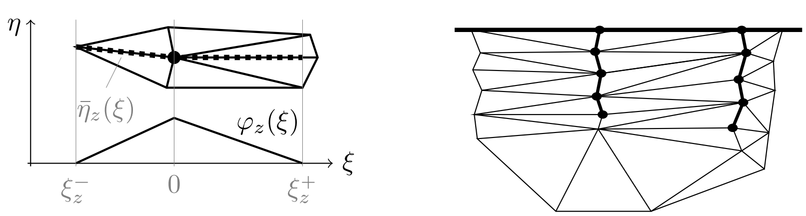


FIG. 7.2. *Local notation (left); an anisotropic path and a semi-anisotropic path highlighted (right).*

We are now prepared to specify $\bar g_z$ (see also the row for $\bar g_z$ in the above table). We let $\bar g_z := 0$ if $z \in \mathcal{N}_{\rm iso}$ is an isotropic node or $z \in \partial\Omega$, and, otherwise, let

$$\int_{\xi_z^-}^{\xi_z^+} \big[g(\xi, \bar\eta_z(\xi)) - \bar g_z\big]\, \varphi_z(\xi)\, d\xi = 0. \tag{7.6}$$

Also, let $\bar S_z^- := \{(\xi, \bar\eta_z(\xi)) : \xi \in (\xi_z^-, 0)\}$ and $\bar S_z^+ := \{(\xi, \bar\eta_z(\xi)) : \xi \in (0, \xi_z^+)\}$, i.e. $\bar S_z^\pm$ is the interval joining $(0,0)$ and $(\xi_z^\pm, \bar\eta_z(\xi_z^\pm))$. So using (7.6) and then (3.4), (3.1) yields

$$h_z|\bar g_z| \lesssim \frac{h_z}{H_z}\|g\|_{1\,;\bar S_z^- \cup \bar S_z^+} \lesssim |\!|\!| g |\!|\!|_{\omega_z} \lesssim \sum_{T\subset\omega_z} |\!|\!| g |\!|\!|_T\,. \tag{7.7}$$

To ensure that for $z \in [\mathcal{N}_{\rm ani}\backslash\mathcal{N}^*_{\partial\Omega}] \cap \partial\Omega$ and $z \in \mathcal{N}_{\rm s.ani} \cap \partial\Omega$, both (7.6) and (7.7) agree with the definition $\bar g_z = 0$, we choose $\bar\eta_z$ for these nodes such that $\{(\xi, \bar\eta_z(\xi)) : \xi \in (\xi_z^-, \xi_z^+)\}$ lies on the boundary (i.e. $\bar S_z^\pm \subset \mathcal{S}_z \cap \partial\Omega$).

Now we proceed to the estimation of I of (5.3) and split it as $I = \sum_{z\in\mathcal{N}} I_z$, and then I_z, similarly to (6.8), as

$$\begin{aligned} I_z := \varepsilon^2 \int_{\gamma_z} (g - \bar g_z)\phi_z [\![\nabla u_h]\!] \cdot \nu &= I_z' + I_z'' + I_z''' + I_z'''' \\ &:= \varepsilon^2 \int_{\mathring\gamma_z} (g - \bar g_z)\phi_z [\![\nabla u_h]\!] \cdot \nu + \varepsilon^2 \int_{\gamma_z\backslash\mathring\gamma_z} (g - \bar g_z)\phi_z [\![\partial_\xi u_h]\!]\, \nu_\xi \\ &\quad + \varepsilon^2 \int_{\gamma_z\backslash\mathring\gamma_z} [g - g(\xi, \bar\eta_z(\xi))]\ \varphi_z [\![\partial_\eta u_h]\!]\, \nu_\eta \\ &\quad + \varepsilon^2 \int_{\gamma_z\backslash\mathring\gamma_z} [g(\xi, \bar\eta_z(\xi)) - \bar g_z]\ \varphi_z [\![\partial_\eta u_h]\!]\, \nu_\eta \\ &\quad + \varepsilon^2 \int_{\gamma_z\backslash\mathring\gamma_z} (g - \bar g_z)\{\phi_z - \varphi_z\} [\![\partial_\eta u_h]\!]\, \nu_\eta\,, \end{aligned} \tag{7.8}$$

where, with slight abuse of notation, $g = g(\xi, \eta)$.

To get the desired assertion (7.3), first, we show that

$$|I_z'| + |I_z''| + |I_z''''| \lesssim \Big(\varepsilon^2 \sum_{T\subset\omega_z} |\!|\!| g |\!|\!|_T\Big)\, \big\| J_z \big\|_{\infty\,;\gamma_z} \qquad \text{for } \ z \in \mathcal{N}. \tag{7.9}$$

Here the bound for I_z'' is obtained using $|[\![\partial_\eta u_h]\!]| \leqslant |J_z|$ and $|\nu_\eta| \leqslant 1$, and also $\int_{\gamma_z\backslash\mathring\gamma_z} |g(\xi,\eta) - g(\xi, \bar\eta_z(\xi))|\varphi_z \lesssim \|\partial_\eta g\|_{1\,;\omega_z} \lesssim |\!|\!| g |\!|\!|_{\omega_z}$ combined with (3.1).

The estimation of I_z' in (7.9) is more subtle. We use $|\phi_z[\![\nabla u_h]\!]| + |\phi_z[\![\partial_\xi u_h]\!]| \leqslant |J_z|$ and $|\nu| \leqslant 1$, so it remains to show that

$$\int_{\mathring\gamma_z} \big(|g| + |\bar g_z|\big) + \int_{\gamma_z\backslash\mathring\gamma_z} \big(|g\, \nu_\xi| + |\bar g_z \nu_\xi|\big) \lesssim \sum_{T\subset\omega_z} |\!|\!| g |\!|\!|_T\,. \tag{7.10}$$

Here, crucially, $|\nu_\xi| \lesssim \frac{h_z}{|S|}$ for any $S \in \gamma_z\backslash\mathring\gamma_z$ (this follows from $|\nu_\xi| = |\sin\angle(S, \mathrm{i}_\xi)|$ and (2.2)). To be more precise, for $S \subset \gamma_z\backslash\mathring\gamma_z$ and $z \in \mathcal{N}_{\rm ani} \cup \mathcal{N}_{\rm s.ani}$ one has $|S| \simeq H_z$ so $|\nu_\xi| \leqslant \frac{h_z}{H_z}$ (while for $z \in \mathcal{N}_{\rm iso}$ we do not need to bound $|\nu_\xi|$ as $\gamma_z\backslash\mathring\gamma_z = \varnothing$). Now an application of (3.3) yields the bounds for $|g|$ and $|g\,\nu_\xi|$ announced in (7.10). Next, unless $\bar g_z = 0$, one has $\int_{\mathring\gamma_z} |\bar g_z| + \int_{\gamma_z\backslash\mathring\gamma_z} |\bar g_z \nu_\xi| \lesssim h_z|\bar g_z|$. Combining this with (7.7) immediately yields the remaining bounds for $|\bar g_z|$ and $|\bar g_z\, \nu_\xi|$ in (7.10). Finally note that the latter bounds are trivial if $\bar g_z = 0$. Thus we have established (7.10) and so the bound for I_z' in (7.9).

The bound for I_z'''' in (7.9) is obtained by combining the argument for I_z' with $|\nu_\eta| \leqslant 1$ and a crucial observation that $0 \leqslant \phi_z - \varphi_z \lesssim \frac{h_z}{H_z}$ on any $S \in \gamma_z \backslash \mathring{\gamma}_z \subset \mathcal{S}_z^+ \cup \mathcal{S}_z^-$. We now show the latter for $S \in \mathcal{S}_z^+$ (as the case of $S \in \mathcal{S}_z^-$ is similar). Consider any triangle $T \subset \omega_z$ with two edges S', $S'' \subset \mathcal{S}_z^+$, i.e. $|S'| \simeq |S''| \simeq H_z$. Then, by (2.3) or (2.4), $h_T \simeq h_z$, so, by the maximum angle condition, $||S'| - |S''|| \lesssim h_z$ and so $||S_\xi'| - |S_\xi''|| \lesssim h_z$. As a similar property holds for any two edges in $\mathcal{S}_z^+$, so $|S_\xi \backslash (0, \xi_z^+)| \lesssim h_z\ \forall S \in \mathcal{S}_z^+$. This implies $0 \leqslant \phi_z - \varphi_z \lesssim \frac{h_z}{H_z}$ on any $S \in \mathcal{S}_z^+$. Thus the final bound for I_z'''' in (7.9) is established.

It remains to estimate I_z''' in (7.8), in which $[g(\xi, \bar\eta_z(\xi)) - \bar g_z]\, \varphi_z$ is a function of ξ, while $\nu_\eta |d\nu| = -\mathrm{sgn}(\xi)\, d\xi$. Hence, one gets

$$I_z''' = \varepsilon^2 \Big(\sum_{S \in \mathcal{S}_z^- \backslash \partial\Omega} [\![\partial_\eta u_h]\!] \Big) \int_{\xi_z^-}^0 [g(\xi, \bar\eta_z(\xi)) - \bar g_z]\, \varphi_z(\xi)\, d\xi$$

$$- \varepsilon^2 \Big(\sum_{S \in \mathcal{S}_z^+ \backslash \partial\Omega} [\![\partial_\eta u_h]\!] \Big) \int_0^{\xi_z^+} [g(\xi, \bar\eta_z(\xi)) - \bar g_z]\, \varphi_z(\xi)\, d\xi\,. \tag{7.11}$$

Consider various node types separately (see also the table above).

First, for $z \in \mathcal{N}_{\rm iso}$ one has $\xi^\pm = 0$ and $\mathcal{S}_z^\pm = \varnothing$ so $I_z''' = 0$. Next, recall that for $z \in \mathcal{N}_{\rm s.ani}$, one has $\xi_z^- = 0$ and $\mathcal{S}_z^- = \varnothing$ so combining (7.11) with (7.6) immediately yields $I_z''' = 0$. Now, for $z \in [\mathcal{N}_{\rm ani} \backslash \mathcal{N}^*_{\partial\Omega}] \cap \partial\Omega$, recall that $\bar g_z = 0$, while $\bar\eta_z$ was chosen so that each $(\xi, \bar\eta_z(\xi)) \in \partial\Omega$ so $g(\xi, \bar\eta_z(\xi)) = 0$; hence again $I_z''' = 0$.

For the remaining nodes $z \in \widetilde{\mathcal{N}}_{\rm ani} = [\mathcal{N}_{\rm ani} \backslash \partial\Omega] \cup \mathcal{N}^*_{\partial\Omega}$, we claim that

$$|I_z'''| \lesssim \Big(\varepsilon^2 \sum_{T \subset \omega_z} |\!|\!| g |\!|\!|_T \Big)\ H_z h_z^{-1} |\mathcal{J}_z| \qquad \text{for}\ \ z \in \widetilde{\mathcal{N}}_{\rm ani}. \tag{7.12}$$

Indeed, consider $z \in \mathcal{N}_{\rm ani} \backslash \partial\Omega$. Then $\mathcal{S}_z^\pm \backslash \partial\Omega = \mathcal{S}_z^\pm$ and $\mathring{\mathcal{S}}_z = \mathring{\gamma}_z$, so combining (7.11) with the definition (7.6) of $\bar g_z$ and also $\sum_{S \in \mathcal{S}_z^- \cup \mathcal{S}_z^+} [\![\partial_\eta u_h]\!] = -\sum_{S \in \mathring{\mathcal{S}}_z} [\![\partial_\eta u_h]\!]$ yields

$$I_z''' = \varepsilon^2 \Big(\sum_{S \in \mathring{\mathcal{S}}_z} [\![\partial_\eta u_h]\!] \Big) \int_0^{\xi_z^+} [g(\xi, \bar\eta_z(\xi)) - \bar g_z]\, \varphi_z(\xi)\, d\xi. \tag{7.13}$$

As, by (7.6), (7.7), $\int_0^{\xi_z^+} |g(\xi, \bar\eta_z(\xi)) - \bar g_z| \varphi_z\, d\xi \lesssim \|g\|_{\bar S_z^+} \lesssim H_z h_z^{-1} \sum_{T \subset \omega_z} |\!|\!| g |\!|\!|_T$, so

$$|I_z'''| \lesssim \Big(\varepsilon^2 \sum_{T \subset \omega_z} |\!|\!| g |\!|\!|_T \Big)\ H_z h_z^{-1} \Big| \sum_{S \in \mathring{\mathcal{S}}_z} [\![\partial_\eta u_h]\!] \Big|.$$

In view of $[\![\partial_\eta u_h]\!] = -J_z \cos\angle(S, \mathrm{i}_\xi)$, where $\cos\angle(S, \mathrm{i}_\xi) = \cos\angle(S, \hat S_z) = \sigma_z\big|_S$, one gets (7.12) for $z \in \mathcal{N}_{\rm ani} \backslash \partial\Omega$. For $z \in \mathcal{N}^*_{\partial\Omega} \subset \mathcal{N}_{\rm ani}$, the bound (7.12) is obtained using a similar argument, only now (7.11) with $\mathcal{S}_z^- = \varnothing$, $\xi_z^- = 0$, and $\mathcal{S}_z^+ \backslash \partial\Omega = \mathcal{S}_z^+$ implies (7.13) (in which $\bar g_z = 0$); the latter again leads to (7.12), where now $\mathring{\mathcal{S}}_z = \mathcal{S}_z \cap \partial\Omega$. Thus (7.12) is established for all $z \in \widetilde{\mathcal{N}}_{\rm ani}$.

Combining (7.8), (7.9), (7.12) with $I_z''' = 0$ for $z \in \mathcal{N} \backslash \widetilde{\mathcal{N}}_{\rm ani}$ yields the desired bound (7.3). □

Remark 7.3. An inspection of the proof of (7.3) *shows that Theorem 7.1 remains valid if the local cartesian coordinates* (ξ, η) *satisfy* $|\angle(\hat S_z, \mathrm{i}_\xi)| \lesssim \frac{h_z}{H_z}$ *(rather than* $\angle(\hat S_z, \mathrm{i}_\xi) = 0$*), as then* $\cos\angle(S, \mathrm{i}_\xi) = \sigma_z\big|_S + O(\frac{h_z}{H_z})$ *again yields* (7.3)*. Also, the requirement that* $z = (0, 0)$ *in the coordinates* (ξ, η) *can be dropped.*

Remark 7.4. *An inspection of the above proof shows that it remains valid if $\{\bar g_z\}_{z\in\mathcal N}$ defined by* (7.6) *are replaced by $\{\bar g_z^*\}_{z\in\mathcal N}$ such that $\bar g_z^* = \bar g_z = 0$ for $z \in \mathcal N_{\rm iso} \cup \partial\Omega$, and a version of* (6.14) *holds true with $\omega_z^* := \omega_z$. Indeed, each I_z will include an additional component $I_z^* := \varepsilon^2 \int_{\gamma_z} (\bar g_z - \bar g_z^*)\phi_z [\![\nabla u_h]\!] \cdot \nu$, for which, imitating an argument from Remark 6.4, one gets $|I_z^*| \leqslant \varepsilon^2 |\!|\!| g |\!|\!|_{\omega_z} \, \|J_z\|_{\infty\,;\gamma_z}$. Combining the latter with* (3.1) *we conclude that $|I_z^*|$ is bounded by the right-hand side of* (7.9).

Furthermore, a more careful inspection shows that the above proof remains valid even if (6.14) *uses $\omega_z^* := \omega_z \cup \omega_{z^+}$ or $\omega_z^* := \omega_z \cup \omega_{z^+} \cup \omega_{z^-}$ as long as $z^\pm$ is connected to z by an edge from $\mathcal S_z^\pm$ and $H_{z^\pm} \simeq H_z$ (so $\lambda_{z^\pm} \simeq \lambda_z$).*

Remark 7.5. *Suppose that $z \in \mathcal N_{\rm s.ani} \backslash \partial\Omega$ and $\mathcal S_z^+$ contains at least 2 edges, or $z \in \mathcal N_{\rm ani} \backslash \partial\Omega$ and each of $\mathcal S_z^\pm$ contains at least 2 edges. Then there is a sufficiently small constant $\theta > 0$ such that for each such z, one can choose $\bar\eta_z(\xi)$ in* (7.6) *so that the domain $\tilde\omega_z := \{(\xi, \bar\eta_z(\xi) + t) : \xi \in (\xi_z^-, \xi_z^+), |t| < \theta h_z\}$ is a subset of ω_z. Now,* (6.14) *will be satisfied with $\omega_z^* := \omega_z$ if $\bar g_z^*$ is defined by*

$$\int_{\tilde\omega_z} [g(\xi,\eta) - \bar g_z^*]\, \varphi_z(\xi) = 0. \tag{7.14}$$

The second relation in (6.14) *follows as $|\tilde\omega_z| = 2\theta h_z(\xi_z^+ - \xi_z^-) \simeq h_z H_z$ and $\tilde\omega_z \subset \omega_z$ so, indeed, $h_z H_z |\bar g_z^*| \lesssim \|g\|_{1\,;\omega_z}$. Next, $\int_{\tilde\omega_z} [g(\xi, \bar\eta_z(\xi)) - g(\xi,\eta)]\varphi_z \simeq h_z H_z(\bar g_z - \bar g_z^*)$ implies $H_z|\bar g_z - \bar g_z^*| \lesssim \|\partial_\eta g\|_{1\,;\omega_z} \leqslant |\!|\!| g |\!|\!|_{\omega_z}$, i.e. one gets the first relation in* (6.14).

More generally, if $z \in \mathcal N \backslash [\mathcal N_{\rm iso} \cup \partial\Omega] = [\mathcal N_{\rm ani} \cup \mathcal N_{\rm s.ani}] \backslash \partial\Omega$ and each of $\mathcal S_z^\pm$ contains exactly one edge, then the above is true with $\tilde\omega_z \subset \omega_z^* := \omega_z \cup \omega_{z^+} \cup \omega_{z^-}$. Otherwise, if $\mathcal S_z^+$ contains exactly one edge, the above is true with $\tilde\omega_z \subset \omega_z^* := \omega_z \cup \omega_{z^+}$. In both cases we use the notation and the final conclusion of Remark 7.4 combined with the observation that (2.3)–(2.5) *imply $H_{z^\pm} \simeq H_z$.*

*Remark 7.6 (*Higher-order elements*). A version of the jump residual bound* (7.3) *for higher-order elements requires the notation $\mathcal J_z^\pm(\xi) := \sum_{S \in \mathcal S_z^\pm \backslash \partial\Omega} [\![\partial_\eta u_h]\!]$. In contrast to the linear elements, $\mathcal J_z^\pm(\xi)$ are not constant so remain inside the corresponding integrals in* (7.11). *So one gets* (7.3) *with $|\mathcal J_z|$ replaced by $|\mathcal J_z^-(0) + \mathcal J_z^+(0)| + {\rm osc}(\mathcal J_z^\pm; \omega_z)$, and $\widetilde{\mathcal N}_{\rm ani}$ by $\widetilde{\mathcal N}_{\rm ani} \cup \mathcal N_{\rm s.ani}$. (For $z \in \mathcal N_{\rm s.ani}$, one can use a simpler ${\rm osc}(\mathcal J_z^+; \omega_z)$ in place of $|\mathcal J_z|$.) Note also that with the interior residual f_h^I corrected to a more general $-\varepsilon^2 \triangle u_h + f_h^I$, all results of §7 for the interior residual apply to higher-order elements.*

7.3. Interior Residual. Proof of (7.4) **and** (7.5)**.**

Proof of (7.4). The proof of this bound closely follows the proof of (6.5) in §6.3. The main difference is that, unless $\bar g_z = 0$, one uses $h_z|\bar g_z| \lesssim |\!|\!| g |\!|\!|_{\omega_z}$ from (7.7), so now $\|g - \bar g_z\|_{1\,;\omega_z} \lesssim H_z |\!|\!| g |\!|\!|_{\omega_z}$. Combining this with (3.1) leads to (7.4). □

Proof of (7.5). This bound is also obtained similarly to (6.5), only the definition of Θ' is combined with $|I\!I| \lesssim \sum_{z\in\mathcal N} \|g\|_{1\,;\omega_z^*} \|f_h^I\|_{\infty\,;\omega_z}$, where $\omega_z^* = \omega_z$, or $\omega_z^* = \omega_z \cup \omega_{z^+}$ or $\omega_z^* = \omega_z \cup \omega_{z^+} \cup \omega_{z^-}$ with $H_{z^\pm} \simeq H_z$ and $\lambda_{z^\pm} \simeq \lambda_z$. For the latter, using Remarks 7.4 and 7.5, replace $\{\bar g_z\}$ in (5.3) by $\{\bar g_z^*\}$ of (7.14), which, by (6.14), yields $\|g - \bar g_z^*\|_{1\,;\omega_z} \lesssim \|g\|_{1\,;\omega_z^*}$. □

7.4. Further Mesh Assumptions. Second Estimator. To get a version of the estimator (6.4), (6.7), (5.2) for a more general mesh, we make further assumptions on our triangulation. It is essential in this part of the analysis that we look at sequences of short edges that connect anisotropic nodes. This concept is implemented with the help of the following definition.

Definition. A *(Semi-)Anisotropic Path*, or simply a *Path*, is an ordered sequence $\{z_j\}_{j=1}^k$ of nodes in $\mathcal{N}_{\rm ani}\backslash\mathcal{N}^*_{\partial\Omega}$ (or $\mathcal{N}_{\rm s.ani}$), for some $k>1$ (which may differ for different paths and for which no upper-bound assumption is made), such that each z_j, $j=1,\ldots,k-1$, is connected to z_{j+1} by an edge of length $\simeq h_{z_j}\simeq h_{z_{j+1}}$, and each of the start and end nodes z_l, $l=1,k$, either lies on $\partial\Omega$, or is connected by an edge of length $\simeq h_{z_l}$ to an isotropic node of patch diameter $\simeq H_{z_l}$. (E.g., if $z_1\notin\partial\Omega$, then is it connected to some node $z_0\in\mathcal{N}_{\rm iso}$ by an edge of length $\simeq h_{z_1}$ and $H_{z_0}\simeq H_{z_1}$.)

Let $\mathcal{N}_i\subset\mathcal{N}_{\rm ani}\backslash\mathcal{N}^*_{\partial\Omega}$ be an anisotropic path for $i=1,\ldots,n_{\rm ani}$, and $\mathcal{N}_i\subset\mathcal{N}_{\rm s.ani}$, be a semi-anisotropic path for $i=n_{\rm ani}+1,\ \ldots,n_{\rm ani}+n_{\rm s.ani}$ (with no upper bound assumption made on $n_{\rm ani}$ or $n_{\rm s.ani}$). Furthermore, let $\mathcal{N}_{\rm paths}:=\cup_{i=1}^{n_{\rm ani}+n_{\rm s.ani}}\mathcal{N}_i$. Note that $\mathcal{N}\backslash\mathcal{N}_{\rm paths}$ may include (semi-)anisotropic nodes that do not belong to any path.

$\mathcal{A}3$. *Path Coordinate-System condition.* For each (semi-)anisotropic path $\mathcal{N}_i$, $i=1,\ldots,n_{\rm ani}+n_{\rm s.ani}$, let there exist a cartesian coordinate system $(\xi,\eta)=(\xi_i,\eta_i)$ such that $|\sin(\angle(S,{\rm i}_\xi))|\lesssim\frac{h_z}{|S|}$ for any $S\subset\mathcal{S}_z$ of any node $z\in\mathcal{N}_i$ (while, if $\mathcal{N}_i$ is semi-anisotropic a stronger condition $|\angle(S,{\rm i}_\xi)|\lesssim\frac{h_z}{|S|}$ is satisfied).

Note that assumption $\mathcal{A}3$ implies that for any $z\in\mathcal{N}_i$, $i=1,\ldots,n_{\rm ani}+n_{\rm s.ani}$, with the notation $\Omega_i:=\cup_{z\in\mathcal{N}_i}\omega_z$, one has

$$\omega_z\subseteq\omega_z^*:=\Omega_i\cap\bigl\{\eta_i\in(\eta_z^-,\eta_z^+)\bigr\},\quad \eta_z^+-\eta_z^-\simeq h_z,\quad {\rm diam}\,\omega_z\simeq{\rm diam}\,\omega_z^*\simeq H_z\,,\tag{7.15}$$

where (η_z^-,η_z^+) is the range of η in ω_z.

$\mathcal{A}4$. We also assume that $\omega_z^*\subset\omega_z^{(J)}$, where $J\lesssim 1$ (with $\omega_z^{(J)}$ defined in Section 6).

Theorem 7.7. *Under conditions of Theorem 7.1 and also $\mathcal{A}3$ and $\mathcal{A}4$ (but without assuming $\mathcal{A}2$), one has $(u_h-u)(x',y')=I+II+\mathcal{E}_{\rm quad}$ for any $(x',y')\in\Omega$, where I is bounded by* (7.3), *$\mathcal{E}_{\rm quad}$ is bounded by* (5.2), *and*

$$|II|\lesssim\max_{z\in\mathcal{N}_{\rm paths}}\bigl\{\,(\Theta+\Theta')\,\varepsilon^{-2}\lambda_z\,\min\{\varepsilon,h_z\}\|f_h^I\|_{\infty\,;\omega_z}+\Theta'\,\varepsilon^{-2}\lambda_z^2\,{\rm osc}(f_h^I;\omega_z)\,\bigr\}$$
$$+\,\Theta'\max_{z\in\mathcal{N}\backslash\mathcal{N}_{\rm paths}}\bigl\{\varepsilon^{-2}\,\lambda_z^2\,\|f_h^I\|_{\infty\,;\omega_z}\bigr\}\,.\tag{7.16}$$

Under the additional condition $\mathcal{A}2$, the error component I is also bounded by (7.2).

Sections 7.5 and 7.6 below are devoted to the proof of this theorem.

*Remark 7.8 (*Curvilinear Layers*). Condition $\mathcal{A}3$ applies to a particular path (for example, the two paths on Figure 7.2 (right) may have different coordinate systems (ξ_i,η_i)). Hence, $\mathcal{A}3$ does not prohibit anisotropic mesh elements to be aligned along a curvilinear layer. However, the local coordinate condition* (2.2) *restricts such alignments to the case $\varrho H_z^2\lesssim h_z$ (where ϱ is the layer curvature).*

On the other hand, this restriction agrees with the linear interpolation error. To illustrate the latter, $u=e^{-(1-|x|)/\varepsilon}$ in $\Omega=\{|x|<1\}$ exhibits a circular boundary layer. If the boundary nodes form an equidistant mesh of diameter H_z on $\partial\Omega$, then ${\rm dist}(\partial\Omega,\partial\Omega_h)\simeq H_z^2$ *(where $\Omega_h:=\cup_{T\in\mathcal{T}}T$), so $u-u^I\simeq\min\{\varepsilon^{-1}H_z^2,\,1\}$ on $\partial\Omega_h$, while on the short edges of length $\simeq h_z$ originating on $\partial\Omega$, one has $|u-u^I|\lesssim\min\{\varepsilon^{-1}h_z,\,1\}$.*

7.5. Choice of $\bar g_z$ for the second estimator. Jump residual. To get the sharper estimator (7.16) for the interior residual, we need to tweak the definition (7.6) of $\bar g_z$ for each path $\mathcal{N}_i=\{z_j\}_{j=1}^k$ (where $k=k(i)$) as follows. For each $z_j\in\mathcal{N}_i$, let the local cartesian coordinates (ξ,η) described in Section 7.2 and used in (7.6) coincide with (ξ_i,η_i). (Recall that originally (ξ,η) were chosen independently for each z, now $(\xi,\eta)=(\xi_i,\eta_i)$ remain unchanged for all nodes of each path).

In view of Remark 7.3 and $\mathcal{A}3$, all conclusions of Theorem 7.1, with the exception of (7.5), remain valid under the above choice of local coordinates (ξ,η).

Remark 7.9. For all bounds in Theorem 7.1, including (7.5)*, to hold true, it suffices to replace* $\{\bar g_z\}$ *in* (5.3) *by* $\{\bar g_z^*\}$ *that satisfies* (6.14)*, where* ω_z^* *is from* (7.15) *and satisfies* $\mathcal{A}4$ *for* $z\in\mathcal{N}_{\rm paths}$*, and* ω_z^* *is from Remark 7.4 for* $z\in\mathcal{N}\backslash\mathcal{N}_{\rm paths}$*. Hence, for* $z\in\mathcal{N}\backslash\mathcal{N}_{\rm paths}$ *let* $\{\bar g_z^*\}$ *be as in the proof of* (7.5)*, i.e. as in Remark 7.5 (in particular,* $\bar g_z^*=\bar g_z=0$ *for* $z\in\mathcal{N}_{\rm iso}\cup\partial\Omega$*, while, otherwise, use* (7.14)*). For* $z\in\mathcal{N}_{\rm paths}$*, such* $\bar g_z^*$ *will be specified in* §*7.6 below.*

7.6. Interior Residual. Proof of (7.16)**.**

Proof. First, replace $\{\bar g_z\}$ in (5.3) by $\{\bar g_z^*\}$ from Remark 7.9 (such $\bar g_z^*$ for $z\in\mathcal{N}_{\rm paths}$ will be specified below). Now, recall that $\mathcal{N}=\bigl(\cup_{i=1}^{n_{\rm ani}+n_{\rm s.ani}}\mathcal{N}_i\bigr)\cup\bigl(\mathcal{N}\backslash\mathcal{N}_{\rm paths}\bigr)$. So split $I\!I$ of (5.3) as $I\!I=\sum_i I\!I_i+I\!I_{\rm osc}+I\!I_{\backslash\rm paths}$ as follows:

$$\begin{aligned}
I\!I_i &:= \sum_{z\in\mathcal{N}_i}\int_{\omega_z} f_h^I(X_z,Y_z)\,(g-\bar g_z^*)\,\phi_z\,,\\
I\!I_{\rm osc} &:= \sum_{z\in\mathcal{N}_{\rm paths}}\int_{\omega_z} [f_h^I-f_h^I(X_z,Y_z)]\,(g-\bar g_z^*)\,\phi_z\,,\\
I\!I_{\backslash\rm paths} &:= \sum_{z\in\mathcal{N}\backslash\mathcal{N}_{\rm paths}}\int_{\omega_z} f_h^I\,(g-\bar g_z^*)\,\phi_z\,.
\end{aligned}$$

Here all integrals are in the original variables (x,y), and we use some $(X_z,Y_z)=(X_z(x,y),Y_z(x,y))\in\omega_z^*$ assigned to each $(x,y)\in\omega_z$ for $z\in\mathcal{N}_{\rm paths}$. Hence, for $I\!I_{\rm osc}$ and $I\!I_{\backslash\rm paths}$, we immediately get versions of (7.5) (using Remark 7.9):

$$|I\!I_{\rm osc}|\lesssim\Theta'\max_{z\in\mathcal{N}_{\rm paths}}\Bigl\{\varepsilon^{-2}\,\lambda_z^2\,{\rm osc}(f_h^I\,;\omega_z^*)\Bigr\},\tag{7.17}$$

$$|I\!I_{\backslash\rm paths}|\lesssim\Theta'\max_{z\in\mathcal{N}\backslash\mathcal{N}_{\rm paths}}\Bigl\{\varepsilon^{-2}\,\lambda_z^2\,\|f_h^I\|_{\infty\,;\omega_z}\Bigr\}.\tag{7.18}$$

To complete the proof, it now suffices to show that for $i=1,\ldots,n_{\rm ani}$

$$\begin{aligned}
|I\!I_i|\lesssim{}&\Bigl(\sum_{z\in\mathcal{N}_i}\lambda_z^{-1}|\!|\!|g|\!|\!|_{\omega_z^*}+\sum_{z\in\mathcal{N}_i}\lambda_z^{-2}\|g\|_{1\,;\omega_z^*}\Bigr)\max_{z\in\mathcal{N}_i}\Bigl\{\min\{\lambda_z^2,\lambda_z h_z\}\|f_h^I\|_{\infty\,;\omega_z}\Bigr\}\\
&+\Bigl(\sum_{z\in\mathcal{N}_i}\lambda_z^{-2}\|g\|_{1\,;\omega_z^*}\Bigr)\Bigl[\max_{z\in\mathcal{N}_i}\bigl\{\lambda_z^2\,{\rm osc}(f_h^I\,;\omega_z^*)\bigr\}+\max_{z\in\mathcal{N}\backslash\mathcal{N}_{\rm paths}}\bigl\{\lambda_z^2\,\|f_h^I\|_{\infty\,;\omega_z}\bigr\}\Bigr],
\end{aligned}\tag{7.19}$$

and that for $i=n_{\rm ani}+1,\ldots,n_{\rm ani}+n_{\rm s.ani}$, one has a version of (7.17) with $|\!|\!|g|\!|\!|_{\omega_z^*}$ replaced by $|\!|\!|g|\!|\!|_{\omega_z^*}+\sum_{T\subset\omega_z}|\!|\!|g|\!|\!|_T$. Indeed, then the desired assertion (7.16) follows from (7.17) and (7.18) combined with the observation that $\sum_i|I\!I_i|$ is bounded by the right-hand side in (7.16). The latter can be shown by combining (7.19) for all i with $\min\{\lambda_z^2,\lambda_z h_z\}=\lambda_z\min\{\varepsilon,h_z\}$, and then (3.1) and (7.1), and also noting that, by $\mathcal{A}4$, one can replace ω_z^* in (7.19) by ω_z.

We now proceed to establishing (7.19). First, consider $I\!I_i$ for some $i=1,\ldots,n_{\rm ani}$, which corresponds to an anisotropic path $\mathcal{N}_i=\{z_j\}_{j=1}^k$ (with $k=k(i)$). Recall that $\Omega_i:=\cup_{z\in\mathcal{N}_i}\omega_z$, and let $\Omega_i^*:=\Omega_i\backslash(\omega_{z_1}^*\cup\omega_{z_k}^*)$. Within Ω_i^*, we shall use the local cartesian coordinates $(\xi,\eta)=(\xi_i,\eta_i)$, as described in Section 7.5. Note that, by $\mathcal{A}3$ and the maximum angle condition, the polygonal curve joining consecutive nodes of the path $\mathcal{N}_i$ can be described in these coordinates by some function $\xi=\kappa_i(\eta)$, while the two disjoint polygonal curves forming $\partial\Omega_i\backslash[\partial\omega_{z_1}\cup\partial\omega_{z_k}]$ can be described by some functions $\xi=\kappa_i^{\pm}(\eta)$. Furthermore, Ω_i^* is a curvilinear rectangle bounded by the curves $\xi=\kappa^{\pm}(\eta)$ and the lines $\eta=\eta_{z_1}^+$ and $\eta=\eta_{z_k}^-$ (assuming, without loss of generality, that η increases as we move along the path $\mathcal{N}_i$ from the start node z_1 to the end node z_k). Note also that $|\frac{d}{d\eta}\kappa_i|+|\frac{d}{d\eta}\kappa_i^{\pm}|\lesssim 1$, so

$$\begin{aligned}&\pm[\kappa_i^{\pm}(\eta)-\kappa_i(\eta)]\simeq H_z\qquad \text{for }\ \eta\in(\eta_z^-,\eta_z^+)\cup(\eta_{z_1}^+,\eta_{z_k}^-),\\&\operatorname{osc}(\kappa(\eta)\,;\omega_z^*)+\operatorname{osc}(\kappa^{\pm}(\eta)\,;\omega_z^*)\lesssim h_z,\end{aligned}\tag{7.20}$$

where we also used (7.15) and $\mathcal{A}4$.

As $\omega_z=(\omega_z\cap\omega_{z_1}^*)\cup(\omega_z\cap\omega_{z_k}^*)\cup(\omega_z\cap\Omega_i^*)$, split $II_i=II_i^{(1)}+II_i^{(k)}+II_i^*$, where $II_i^{(1)}$ and $II_i^{(k)}$ respectively correspond to $\omega_z\cap\omega_{z_1}^*$ and $\omega_z\cap\omega_{z_k}^*$, while II_i^* corresponds to $\omega_z\cap\Omega_i^*$. Furthermore, in each $\omega_z\cap\omega_{z_l}^*$ choose $(X_z(x,y),Y_z(x,y))=(x,y)$ so

$$II_i^{(l)}=\sum_{z\in\mathcal{N}_i}\ \int_{\omega_z\cap\omega_{z_l}^*}f_h^I\,(g-\bar g_z^*)\,\phi_z,\qquad \text{for }\ l=1,k.\tag{7.21}$$

Next, we rewrite II_i^* in the coordinates (ξ,η). Now, $(X_z(x,y),Y_z(x,y))$ becomes some $(\Psi_z(\xi,\eta),\Upsilon_z(\xi,\eta))$, and then in each $\omega_z\cap\Omega_i^*$ we choose $(\Psi_z,\Upsilon_z)=(\kappa_i(\eta),\eta)\in\omega_z^*$. So, with slight abuse of notation, we arrive at

$$II_i^*=\sum_{z\in\mathcal{N}_i}\int_{\eta_{z_1}^+}^{\eta_{z_k}^-}f_h^I(\kappa_i(\eta),\eta)\int_{\kappa_i^-(\eta)}^{\kappa_i^+(\eta)}(g-\bar g_z^*)\,\phi_z\,d\xi\,d\eta,\tag{7.22}$$

where $g=g(\xi,\eta)$ and $\phi_z=\phi_z(\xi,\eta)$.

Without loss of generality, let $z_k\in\partial\Omega$, and z_1 be connected to some $z_0\in\mathcal{N}_{\rm iso}$ by and edge of length $\simeq h_{z_1}$ with $H_{z_0}\simeq H_{z_1}$. Set $\mathcal{N}_i^{(l)}:=\{z\in\mathcal{N}_i:\omega_z\cap\omega_{z_l}^*\neq\varnothing\}$ for $l=1,k$, and then let $\omega_z^{**}:=\omega_z^*\cup\omega_{z_k}$ for $z\in\mathcal{N}_i^{(k)}$, and $\omega_z^{**}:=\omega_z^*\cup\omega_{z_1}$ for $z\in\mathcal{N}_i^{(1)}\backslash\mathcal{N}_i^{(k)}$. As $\mathcal{A}4$ implies that ω_z^{**} satisfies (7.15) and a version $\omega_z^{**}\subset\omega_z^{(2J)}$ of $\mathcal{A}4$, so the redefinition $\omega_z^*:=\omega_z^{**}$ for $z\in\mathcal{N}_i^{(1)}\cup\mathcal{N}_i^{(k)}$ will not affect any of the above arguments. Now, we are prepared to <u>define $\bar g_z^*$</u> for $z\in\mathcal{N}_i$. Set $\bar g_z^*=0$ for $z\in\mathcal{N}_i^{(k)}$ (which we show to satisfy (6.14) when dealing with $II_i^{(k)}$ below). Otherwise, for $z\in\mathcal{N}_i\backslash\mathcal{N}_i^{(k)}$, use the definition (7.14) of $\bar g_z^*$, in which $\tilde\omega_z:=(\xi_z^-,\xi_z^+)\times(\eta_z^-,\eta_z^+)$ for $z\in\mathcal{N}_i\backslash[\mathcal{N}_i^{(1)}\cup\mathcal{N}_i^{(k)}]$, and $\tilde\omega_z:=\tilde\omega_{z_1}$ for $z\in\mathcal{N}_i^{(1)}\backslash\mathcal{N}_i^{(k)}$, while $\tilde\omega_{z_1}$ is chosen as in Remark 7.5. (To simplify the presentation, assume that $\tilde\omega_{z_1}\subset\omega_{z_1}$; the general case is addressed in Remark 7.10 below.) For $z\in\mathcal{N}_i\backslash\mathcal{N}_i^{(k)}$, the second relation in (6.14) follows from $\tilde\omega_z\subset\omega_z^*$, while for the first relation we imitate the related argument from Remark 7.5. (Note, that, strictly speaking, $\tilde\omega_z\subset\omega_z^*$ is valid only if the support of $\varphi_z(\xi)$ satisfies $(\xi_z^-,\xi_z^+)\subset(\kappa_i^-(\eta),\kappa_i^+(\eta))$ for the range of η in ω_z. In view of (7.20), to satisfy this condition (if this is not the case), the support interval width reduction of $\lesssim h_z$ will suffice, while all other arguments will remain unchanged.)

Now consider $II_i^{(1)}$ and $II_i^{(k)}$ from (7.21) separately. For $II_i^{(1)}$, recalling the second relation in (6.14), one easily gets

$$|II_i^{(1)}|\lesssim\Big(\sum_{z\in\mathcal{N}_i^{(1)}}\lambda_z^{-2}\|g\|_{\omega_z^*}\Big)\max_{z\in\mathcal{N}_i}\Big\{\lambda_z^2\,\|f_h^I\|_{\infty\,;\omega_z\cap\omega_{z_1}^*}\Big\}.\tag{7.23}$$

Note that $\|f_h^I\|_{\infty\,;\omega^*_{z_1}} \leqslant \|f_h^I\|_{\infty\,;\omega_{z_0}} + \operatorname{osc}(f_h^I\,;\omega^*_{z_1})$, while $\mathcal{A}4$ yields $H_z \simeq H_{z_1} \simeq H_{z_0}$ so $\lambda_z \simeq \lambda_{z_1} \simeq \lambda_{z_0}$ within $\omega^*_{z_1}$, so

$$\max_{z\in\mathcal{N}_i}\Big\{\lambda_z^2\,\|f_h^I\|_{\infty\,;\omega_z\cap\omega^*_{z_1}}\Big\} \lesssim \lambda_{z_0}^2\,\|f_h^I\|_{\infty\,;\omega_{z_0}} + \lambda_{z_1}^2\,\operatorname{osc}(f_h^I\,;\omega^*_{z_1}). \tag{7.24}$$

Combining (7.24) with (7.23) and then recalling that $z_0 \in \mathcal{N}_{\rm iso} \subset \mathcal{N}\backslash\mathcal{N}_{\rm paths}$, while $z_1 \in \mathcal{N}_i$, one concludes that $|\mathit{II}_i^{(1)}|$ is bounded by the right-hand side of (7.19).

To get a similar bound of type (7.19) for $\mathit{II}_i^{(k)}$, note that (7.21) yields

$$|\mathit{II}_i^{(k)}| \lesssim \Big[\sum_{z\in\mathcal{N}_i^{(k)}}(\lambda_z^{-2}+\lambda_z^{-1}h_z^{-1})\|g-\bar g_z^*\|_{1\,;\omega_z\cap\omega^*_{z_k}}\Big]\,\max_{z\in\mathcal{N}_i}\Big\{\min\{\lambda_z^2,\lambda_z h_z\}\|f_h^I\|_{\infty\,;\omega_z}\Big\}, \tag{7.25}$$

where $\bar g_z^* = 0$. Next, $\mathcal{A}4$ implies that $h_z \simeq h_{z_k}$, $H_z \simeq H_{z_k}$ and $\lambda_z \simeq \lambda_{z_k}$ within $\omega^*_{z_k}$, so the $\sum$ term in (7.25) is $\lesssim (\lambda_{z_k}^{-2} + \lambda_{z_k}^{-1}h_{z_k}^{-1})\|g\|_{1\,;\omega^*_{z_k}}$. Combining this with the crucial observation (which is explained below) that

$$\|g\|_{1\,;\omega^*_{z_k}} \lesssim h_{z_k}\|\nabla g\|_{1\,;\omega^*_{z_k}} \lesssim h_{z_k}|\!|\!|g|\!|\!|_{\omega^*_{z_k}}, \tag{7.26}$$

implies that the $\sum$ term in (7.25) is $\lesssim \lambda_{z_k}^{-2}\|g\|_{1\,;\omega^*_{z_k}} + \lambda_{z_k}^{-1}|\!|\!|g|\!|\!|_{\omega^*_{z_k}}$, so one gets the desired bound of type (7.19) for $\mathit{II}_i^{(k)}$. Now note that the first relation in (7.26) is due to $z_k \in \partial\Omega$ and $|\mathcal{S}_{z_k} \cap \partial\Omega| \simeq H_{z_k}$. To be more precise, we use $g = 0$ on $\partial\Omega$ combined with that every point in $\omega^*_{z_k}$ is within a distance of $\lesssim h_{z_k}$ from $\mathcal{S}_{z_k} \cap \partial\Omega$. Note also that, in view of (7.7), a similar argument yields $H_z|\bar g_z| \lesssim \|g\|_{1\,;\bar S_z^-\cup\bar S_z^+} \lesssim \|\nabla g\|_{1\,;\omega_z^*}$ for $z \in \mathcal{N}_i^{(k)}$; the latter immediately implies that our choice $\bar g_z^* = 0$, for $z \in \mathcal{N}_i^{(k)}$, satisfies (6.14). Thus $|\mathit{II}_i^{(k)}|$ is proved to be bounded by the right-hand side of (7.19).

To complete the proof of (7.19) for II_i, it now remains to show that a similar bound is satisfied by its third component II_i^*, given by (7.22). Set $\varphi_i(\xi,\eta) := \sum_{z\in\mathcal{N}_i}\phi_z$ and note that for all $\eta \in (\eta^+_{z_1}, \eta^-_{z_k})$

$$\sum_{z\in\mathcal{N}_i}\int_{\kappa_i^-(\eta)}^{\kappa_i^+(\eta)} g\,\phi_z\,d\xi = \int_{\kappa_i^-(\eta)}^{\kappa_i^+(\eta)} g\,\varphi_i\,d\xi =: \hat g_i(\eta)\int_{\kappa_i^-(\eta)}^{\kappa_i^+(\eta)}\varphi_i\,d\xi = \sum_{z\in\mathcal{N}_i}\int_{\kappa_i^-(\eta)}^{\kappa_i^+(\eta)}\hat g_i(\eta)\,\phi_z\,d\xi. \tag{7.27}$$

Here the central relation defines $\hat g_i(\eta)$. Now, (7.22) can be rewritten as

$$\begin{aligned}\mathit{II}_i^* &= \sum_{z\in\mathcal{N}_i}\int_{\eta^+_{z_1}}^{\eta^-_{z_k}} f_h^I(\kappa_i(\eta),\eta)\int_{\kappa_i^-(\eta)}^{\kappa_i^+(\eta)}\big(\hat g_i(\eta) - \bar g_z^*\big)\,\phi_z\,d\xi\,d\eta\\ &= \sum_{z\in\mathcal{N}_i}\int_{\omega_z\cap\Omega_i^*} f_h^I(\kappa_i(\eta),\eta)\,\big(\hat g_i(\eta)-\bar g_z^*\big)\,\phi_z\,.\end{aligned}$$

So a calculation yields

$$\begin{aligned}|\mathit{II}_i^*| &\lesssim \sum_{z\in\mathcal{N}_i}\|\hat g_i(\eta)-\bar g_z^*\|_{1\,;\omega_z\cap\Omega_i^*}\,\|f_h^I\|_{\infty\,;\omega_z^*} \lesssim \sum_{z\in\mathcal{N}_i}\min\Big\{\|g\|_{1\,;\omega_z^*}, h_z|\!|\!|g|\!|\!|_{\omega_z^*}\Big\}\,\|f_h^I\|_{\infty\,;\omega_z^*}\\ &\lesssim \Big(\sum_{z\in\mathcal{N}_i}\lambda_z^{-2}\|g\|_{1\,;\omega_z^*} + \sum_{z\in\mathcal{N}_i}\lambda_z^{-1}|\!|\!|g|\!|\!|_{\omega_z^*}\Big)\,\max_{z\in\mathcal{N}_i}\Big\{\min\{\lambda_z^2,\lambda_z h_z\}\|f_h^I\|_{\infty\,;\omega_z^*}\Big\},\end{aligned} \tag{7.28}$$

where we used $\|\hat g_i(\eta)-\bar g_z^*\|_{1\,;\omega_z\cap\Omega_i^*} \lesssim \|g\|_{1\,;\omega_z^*}$ (which follows from the definition of $\hat g_i$ and the second relation in (6.14)), and also $\|\hat g_i(\eta)-\bar g_z^*\|_{1\,;\omega_z\cap\Omega_i^*} \lesssim h_z|\!|\!|g|\!|\!|_{\omega_z^*}$, for which we combine $|\omega_z| \simeq h_z H_z$ with $|\bar{g}_z - \bar{g}_z^*| \lesssim H_z^{-1} \|\!|g|\!\|_{\omega_z^*}$ from (6.14) and another crucial bound

$$\|\hat{g}_i(\eta) - \bar{g}_z\|_{\infty\,;\omega_z \cap \Omega_i^*} \lesssim H_z^{-1} \|\!|g|\!\|_{\omega_z^*}, \tag{7.29}$$

which we establish below. Now (7.28) gives the desired bound of type (7.19) for II_i^*.

To obtain the assertion (7.29) (which we already employed), we need to show that $|\hat{g}_i(\eta) - \bar{g}_z| \lesssim H_z^{-1} \|\!|g|\!\|_{\omega_z^*}$ for $\eta \in (\eta_z^-, \eta_z^+) \cap (\eta_{z_1}^+, \eta_{z_k}^-)$. Recalling the definitions (7.27) and (7.6) of, respectively, $\hat{g}_i$ and $\bar{g}_z$ yields, for any $\eta \in (\eta_z^-, \eta_z^+) \cap (\eta_{z_1}^+, \eta_{z_k}^-)$,

$$(\hat{g}_i(\eta) - \bar{g}_z) \int_{\kappa_i^-}^{\kappa_i^+} \varphi_i(\xi, \eta) d\xi \lesssim \int_{\kappa_i^-}^{\kappa_i^+} g[\varphi_i - \varphi_z] d\xi + \int_{\kappa_i^-}^{\kappa_i^+} [g - g(\xi, \bar{\eta}_z(\xi))] \varphi_z d\xi + \bar{g}_z \int_{\kappa_i^-}^{\kappa_i^+} [\varphi_z - \varphi_i] d\xi,$$

where $\kappa^\pm = \kappa_i^\pm(\eta)$, $g = g(\xi, \eta)$ and $[\varphi_i - \varphi_z] = [\varphi_i(\xi, \eta) - \varphi_z(\xi)]$. (Note that the application of (7.6) here is valid as the support of $\varphi_z(\xi)$ satisfies $(\xi_z^-, \xi_z^+) \subset (\kappa_i^-(\eta), \kappa_i^+(\eta))$ for the range of η under consideration; see the choice of $\bar{g}_z^*$ for $z \in \mathcal{N}_i$.)

Next, the definition of φ_i implies that it equals 1 on $\xi = \kappa_i(\eta)$ and vanishes on $\xi = \kappa_i^\pm(\eta)$; combining this with (7.20) gives $\int_{\mathbb{R}} \varphi_i(\xi, \eta) d\xi \simeq H_z$, and then

$$H_z |\hat{g}_i(\eta) - \bar{g}_z| \lesssim \vartheta_z^* \int_{\kappa_i^-}^{\kappa_i^+} |g| d\xi + \|\partial_\eta g\|_{1\,;\omega_z^*} + \vartheta_z^* H_z |\bar{g}_z|, \quad \text{where } \vartheta_z^* := \|\varphi_i - \varphi_z\|_{\infty\,;\omega_z^* \cap \Omega_i^*}.$$

Note that a version of (3.4) yields $\int_{\kappa_i^-}^{\kappa_i^+} |g| d\xi \lesssim \frac{H_z}{h_z} \|\!|g|\!\|_{\omega_z^*}$, while $\|\partial_\eta g\|_{1\,;\omega_z^*} \lesssim \|\!|g|\!\|_{\omega_z^*}$, and we also have $H_z |\bar{g}_z| \leqslant \frac{H_z}{h_z} \|\!|g|\!\|_{\omega_z}$ from (7.7), in which, by $\mathcal{A}4$, $\|\!|g|\!\|_{\omega_z} \lesssim \|\!|g|\!\|_{\omega_z^*}$. Combining the above, one concludes that $|\hat{g}_i(\eta) - \bar{g}_z| \lesssim H_z^{-1}(\vartheta_z^* \frac{H_z}{h_z} + 1) \|\!|g|\!\|_{\omega_z^*}$. So to complete the proof of (7.29), it remains to show that $\vartheta_z^* \lesssim \frac{h_z}{H_z}$, to which we now proceed.

First, $\mathcal{A}4$ implies that the number of nodes in $\mathcal{N}_z^* := \{z' : \omega_{z'} \cap \omega_z^* \cap \Omega_i^* \neq \varnothing\} \subset \mathcal{N}_i$ is $\lesssim 1$, and hence $h_z \simeq h_{z'}$ and $H_z \simeq H_{z'}$ within $\omega_z^* \cap \Omega_i^*$. Next, (7.20) implies that $|\varphi_{z'} - \varphi_z| \lesssim \frac{h_z}{H_z}$ for any $z' \in \mathcal{N}_z^*$. Hence $\vartheta_z^* \leqslant \max_{z' \in \mathcal{N}_z^*} \vartheta_{z'} + \frac{h_z}{H_z}$, where $\vartheta_z := \|\varphi_i - \varphi_z\|_{\infty\,;\omega_z \cap \Omega_i^*} \lesssim \frac{h_z}{H_z}$, so we indeed get $\vartheta_z^* \lesssim \frac{h_z}{H_z}$. The bound for ϑ_z, which we used here, is easily shown by noting that $\varphi_i - \varphi_z$ is linear on each mesh element in the variable η, so it suffices to check that $|\varphi_i - \varphi_z| \lesssim \frac{h_z}{H_z}$ on all edges of all mesh elements within $\omega_z \cap \Omega_i^*$. In particular, on the edges corresponding to $\xi = \kappa^\pm(\eta)$ one has $\varphi_i = \varphi_z = 0$ so $\varphi_i - \varphi_z = 0$. On the edges corresponding to $\xi = \kappa(\eta)$ one has $\varphi_i = 1$ and $0 \leqslant \varphi_i - \varphi_z = 1 - \varphi_z \lesssim \frac{h_z}{H_z}$, in view of (7.20). For the edges on $\gamma_z \backslash \mathring{\gamma}_z$ (of length $\simeq H_z$), one has $\varphi_i = \phi_z$, while when estimating the component I_z'''' of I (in Section 7.2), we used $0 \leqslant \phi_z - \varphi_z \lesssim \frac{h_z}{H_z}$ on $\gamma_z \backslash \mathring{\gamma}_z$; so again $0 \leqslant \varphi_i - \varphi_z \lesssim \frac{h_z}{H_z}$. The remaining edges have length $\simeq H_z$ and lie on $\partial\omega_z$, so they are on $\gamma_{z'} \backslash \mathring{\gamma}_{z'}$ for some $z' \in \mathcal{N}_i$ connected to z, so combining $0 \leqslant \varphi_i - \varphi_{z'} \lesssim \frac{h_{z'}}{H_{z'}} \simeq \frac{h_z}{H_z}$ with $|\varphi_{z'} - \varphi_z| \lesssim \frac{h_z}{H_z}$ again yields $|\varphi_z - \varphi_i| \lesssim \frac{h_z}{H_z}$. This observation completes the proof of the desired bound for ϑ_z^* and hence the proof of (7.29). Thus we have established (7.19) for $i = 1, \ldots, n_{\text{ani}}$.

The estimation of II_i for each $i = n_{\text{ani}} + 1, \ldots, n_{\text{ani}} + n_{\text{s.ani}}$, which corresponds to a semi-anisotropic path $\mathcal{N}_i$, is similar with the following modifications. First, we now choose $\kappa_i^- = \kappa_i$; then all the bounds obtained for an anisotropic path remain valid. However, the component II_i^* now corresponds to $\Omega_i^* \cap \{\xi > \kappa(\eta)\}$, so II_i involves an additional component II_i^{iso} that corresponds to $\Omega_i^{\text{iso}} := \Omega_i^* \cap \{\xi < \kappa(\eta)\}$. To be more precise, $\mathit{II}_i^{\rm iso} := \sum_{z\in\mathcal{N}_i}\int_{\omega_z\cap\Omega_i^{\rm iso}} f_h^I(X_z,Y_z)\,(g-\bar g_z)\,\phi_z$ and $\mathit{II}_i = \mathit{II}_i^{(1)}+\mathit{II}_i^{(k)}+\mathit{II}_i^{*}+\mathit{II}_i^{\rm iso}$. The notation for this new component is due to that all mesh elements in $\Omega_i^{\rm iso}$ are isotropic. Choosing $(X_z(x,y),Y_z(x,y))=(x,y)$ in $\Omega_i^{\rm iso}$, one easily gets

$$|\mathit{II}_i^{\rm iso}| \lesssim \Bigl[\sum_{z\in\mathcal{N}_i}(\lambda_z^{-2}+\lambda_z^{-1}h_z^{-1})\|g-\bar g_z^*\|_{1\,;\omega_z\cap\Omega_i^{\rm iso}}\Bigr]\max_{z\in\mathcal{N}_i}\Bigl\{\min\{\lambda_z^2,\lambda_z h_z\}\|f_h^I\|_{\infty\,;\omega_z}\Bigr\}. \quad (7.30)$$

Finally note that all mesh elements in $\omega_z\cap\Omega_i^{\rm iso}$ are shape-regular of diameter $\simeq h_z$ so $\|\bar g_z^*\|_{1\,;\omega_z\cap\Omega_i^{\rm iso}} \lesssim h_z^2|\bar g_z^*| \lesssim h_z H_z^{-1}\|g\|_{1\,;\omega_z^*} \lesssim \min\bigl\{\|g\|_{1\,;\omega_z^*}, h_z|\!|\!|g|\!|\!|_{\omega_z^*}\bigr\}$ (where we also used the the second relation from (6.14)), while $\|g\|_{1\,;\omega_z\cap\Omega_i^{\rm iso}} \lesssim \|g\|_{1\,;\omega_z^*}$ and at the same time $\|g\|_{1\,;\omega_z\cap\Omega_i^{\rm iso}} \lesssim h_z\sum_{T\cap\omega_z\cap\Omega_i^{\rm iso}\neq\varnothing} h_T^{-1}\|g\|_{1\,;T} \lesssim h_z\sum_{T\subset\omega_z}|\!|\!|g|\!|\!|_T$. Combining these three observations with (7.30), we get the desired assertion of type (7.19) for II_i, $i=n_{\rm ani}+1,\dots,n_{\rm ani}+n_{\rm s.ani}$. This completes the proof of (7.16). ∎

Remark 7.10. To simplify the presentation, it was assumed in the above proof that $\tilde\omega_{z_1}\subset\omega_{z_1}$*, where* $\tilde\omega_{z_1}$ *is chosen as in Remark 7.5. In general, using the notation of Remark 7.5, one may also have* $\tilde\omega_{z_1}\subset\widehat\omega_{z_1}$ *with* $\widehat\omega_{z_1}=\omega_{z_1}\cup\omega_{z_1^+}\cup\omega_{z_1^-}$ *or* $\widehat\omega_{z_1}=\omega_{z_1}\cup\omega_{z_1^+}$*. Then a few minor adjustments need to be made in the above argument. In particular, for* $z\in\mathcal{N}_i^{(1)}\backslash\mathcal{N}_i^{(k)}$ *one gets a version of* (6.14) *with* $\omega_z^*:=\omega_z^*\cup\widehat\omega_{z_1}$*. Hence,* (7.23) *will include* $|\!|\!|g|\!|\!|_{\omega_z^*\cup\widehat\omega_{z_1}}$ *in place of* $|\!|\!|g|\!|\!|_{\omega_z^*}$*. Consequently,* $\mathcal{N}_i$ *in* $\sum_{z\in\mathcal{N}_i}$ *in the second line of* (7.19) *will be replaced by* $\mathcal{N}_i\cup\{z_1^\pm\}$ *or* $\mathcal{N}_i\cup\{z_1^+\}$*.*

7.7. Numerical results. We tested our estimators using the same model problem as in §6.4. Mesh elements were marked using a version of the estimator of Theorem 7.7 formed as $\mathcal{E}:=\max\{\mathcal{E}_{(7.2)},\mathcal{E}_{(7.16)},\mathcal{E}_{(5.2)}\}$. Here each $\mathcal{E}_{(7.\cdot)}$ reflects the right-hand side of (7.·), in which we set $\Theta=\Theta'=1$ (while, by Lemma 8.1 below, $\Theta+\Theta'\lesssim\ell_h$). We also replace quantities of type $\min\{1,\varepsilon^{-1}a\}$ by the smoother $\frac{a}{\varepsilon+a}$. So, to be more precise, $\mathcal{E}_{(7.2)} := \max_{T\in\mathcal{T}}\bigl\{\max_{z\in\mathcal{N}}\bigl[\frac{\varepsilon H_z}{\varepsilon+H_z}\|J_z\|_{\infty\,;\gamma_z\cap\partial T}\bigr]\bigr\}$, while $\mathcal{E}'_{(7.16)} := \max_{T\in\mathcal{T}}\bigl\{\frac{h_T H_T}{(\varepsilon+h_T)(\varepsilon+H_T)}\|f_h^I\|_{\infty\,;T}\bigr\}$ and $\mathcal{E}''_{(7.16)} := \max_{T\in\mathcal{T}}\bigl\{\bigl(\frac{H_T}{\varepsilon+H_T}\bigr)^2{\rm osc}(f_h^I;T)\bigr\}$ form $\mathcal{E}_{(7.16)} := \max\{\mathcal{E}'_{(7.16)},\mathcal{E}''_{(7.16)}\}$. (Isotropic and anisotropic elements are not distinguished in view of $h_T\simeq H_T$ for isotropic elements, and also ${\rm osc}(f_h^I;T)\leqslant\|f_h^I\|_{T,\infty}$.) The term $\mathcal{E}_{(5.2)} := \max_{T\in\mathcal{T}} H_T^2$ was included as a rough replacement of the quadrature error from (5.2) (and did not play any significant role in the refinement process).

In each experiment, we started with a uniform mesh of right-angled triangles of diameter $H_T=2^{-8}$, 2^{-16}, 2^{-32}, and aspect ratio $\frac{H_T}{h_T}=2$. At each iteration, we marked for refinement the mesh elements responsible for at least 5% of the overall estimator $\mathcal{E}$, but no more than 15% of the elements. The marked elements were

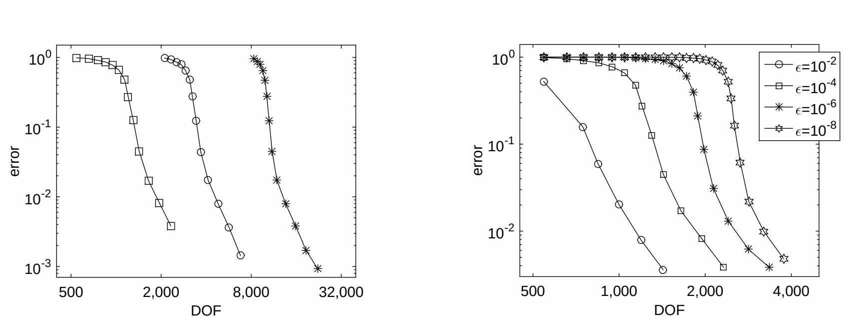


FIG. 7.3. *Maximum errors for* $\varepsilon=10^{-4}$ *and initial DOF varied (left), and* ε *varied (right).*

refined only in the x direction using a single or triple green refinement (depending on the orientation of the mesh element). Edge swapping was also employed to improve geometric properties of the mesh and/or possibly reduce $\max_{T\in\mathcal{T}}\{\mathrm{osc}(f_h^I;T)\}$. The iterations continued until $\mathrm{osc}(f_h^I;T) \leqslant 4H_T$ for all T. In Figure 7.3 we plot the observed errors $\|u-u_h\|_{\infty\,;\Omega}$ versus degrees of freedom (DOF) for fixed $\varepsilon = 10^{-4}$ (left) and ε varied (right). We observe that the mesh refinement yields a very dramatic error reduction (compared with the isotropic mesh refinement [6]). While the maximum mesh aspect ratios vary between 2 and 3.35e+7, the effectivity indices do not exceed 85 in all our experiments. Considering these variations (and the observation made by [10]), the estimator appears to perform reasonably well.

A more comprehensive numerical study of the proposed estimators (combined with a more sophisticated anisotropic mesh refinement algorithm) certainly needs to be conducted, and will be presented elsewhere.

8. Green's function interpolant. Bounds for $\Theta + \Theta'$. We have proved a number of a posteriori error estimates, but they still involve the quantities Θ and Θ'. The purpose of this section is to bound these quantities and thus complete our a posteriori error analysis. Although throughout the paper, we used slightly different definitions (6.3) and (7.1) for Θ and Θ', but whether these quantities appear in Theorem 6.3 under condition A4, or in Theorems 7.1 and 7.7, they satisfy

$$\Theta + \Theta' \lesssim \bar\Theta := \varepsilon^2 \sum_{T\in\mathcal{T}} \Bigl(\lambda_T^{-1}\|\nabla g\|_{1\,;T} + \lambda_T^{-2}\|g\|_{1\,;T}\Bigr), \qquad \lambda_T := \min\{\varepsilon, H_T\}, \tag{8.1}$$

(this is shown using (3.1)). Recall also that $g = G(x', y'\,; \cdot) - G_h$ with any $G_h \in S_h$.

To deal with $g = G - G_h$ in $\bar\Theta$, it is convenient to employ a quasi-interpolant (of Clément/Scott-Zhang type) with the property $|G - G_h|_{k,p\,;T} \lesssim H_T^{j-k}|G|_{j,p\,;\omega_T}$ for any $0 \leqslant k \leqslant j \leqslant 2$, $1 \leqslant p \leqslant \infty$; see [6] for the case of shape-regular triangulations. However, such interpolants are not readily available for general anisotropic meshes (we refer the reader to [1, Chapt. 3] for a discussion of Scott-Zhang-type interpolation on anisotropic tensor-product meshes). Because of this difficulty, we employ a less standard interpolant G_h, which gives a version of the Lagrange interpolant whenever $H_T \lesssim \varepsilon$, and vanishes whenever $H_T \gtrsim \varepsilon$; however, this construction requires additional mild assumptions on the triangulation; see Remark 8.2 below.

For any point $(x', y') \in \Omega$, let $\omega_{(x',y')} := \omega_{z'}$ or $\omega_{(x',y')} := \omega_{S'}$ or $\omega_{(x',y')} := \omega_{T'}$ if, respectively, $(x', y') = z'$ for some $z' \in \mathcal{N}$, or $(x', y') \in S'$ for some $S' \in \mathcal{S}$, or $(x', y') \in T'$ for some $T' \in \mathcal{T}$. Here the standard notation ω_z, ω_S and ω_T is used for the patch of elements touching any $z \in \mathcal{N}$, $S \in \mathcal{S}$ and $T \in \mathcal{T}$, respectively. Also, let

$$\hbar_{(x',y')} := \min_{T\subset\omega_{(x',y')}} h_T, \qquad \ell_{h,(x',y')} := \ln\bigl(2 + \varepsilon^{-1}\hbar_{(x',y')}\bigr). \tag{8.2}$$

Clearly, for any $(x', y') \in \Omega$, one has $\hbar_{(x',y')} \geqslant \underline{h}$ so $\ell_{h,(x',y')} \leqslant \ell_h = \ln(2 + \varepsilon^{-1}\underline{h})$.

The next lemma also uses the notation $\mathrm{dist}_\Omega(\cdot,\cdot)$ for the distance understood as the infimum path length measured in the interior of Ω (so $\mathrm{dist}_\Omega = \mathrm{dist}$ if Ω is convex).

Lemma 8.1. *Let $\mathcal{T}_1 := \{T : H_T \geqslant c_1\varepsilon\}$ and $\mathcal{T}_2 := \{T : H_T \leqslant c_2\varepsilon\}$ for arbitrary fixed constants $0 < c_1 < c_2$. There exists $G_h \in S_h$ such that $\bar\Theta$ from* (8.1) *satisfies*

$$\bar\Theta \lesssim \ell_{h,(x',y')}$$

if $\mathcal{T} = \mathcal{T}_1$, or $\mathcal{T} = \mathcal{T}_2$, or, otherwise, $\mathrm{dist}_\Omega(\mathcal{T}_1\backslash\mathcal{T}_2\,,\, \mathcal{T}_2\backslash\mathcal{T}_1) \gtrsim \varepsilon$.

Remark 8.2. The condition $\mathrm{dist}_\Omega(\mathcal{T}_1\backslash\mathcal{T}_2\,,\, \mathcal{T}_2\backslash\mathcal{T}_1) \gtrsim \varepsilon$ in Lemma 8.1 essentially requires that H_T does not switch too abruptly between $H_T \leqslant c_1\varepsilon$ and $H_T \geqslant c_2\varepsilon$, and, roughly speaking, the transition regions $\{c_1\varepsilon \leqslant H_T \leqslant c_2\varepsilon\}$ have an internal width $\gtrsim \varepsilon$.

Proof. Throughout this proof, (x', y') remains fixed, so we write G for $G(x', y'\,;\cdot)$, $\hbar$ for $\hbar_{(x',y')}$, and ℓ'_h for $\ell_{h,(x',y')}$. We start by showing that

$$\bar{\Theta}_1 := \varepsilon^2 \sum_{T\in\mathcal{T}_1} \Big(\lambda_T^{-1}\|\nabla G\|_{1\,;T} + \lambda_T^{-2}\|G\|_{1\,;T}\Big) \lesssim 1. \tag{8.3}$$

Indeed, $T \in \mathcal{T}_1$ implies $\lambda_T \simeq \varepsilon$, so $\bar{\Theta}_1 \lesssim \varepsilon\|\nabla G\|_{1\,;\Omega} + \|G\|_{1\,;\Omega}$. Combining this with (4.3) yields the desired bound (8.3).

Next, we construct a Lagrange-type interpolant $\tilde{G}^I$ of G such that

$$\bar{\Theta}_2 := \varepsilon^2 \sum_{T\in\mathcal{T}_2} \Big(\lambda_T^{-1}\|\nabla(G-\tilde{G}^I)\|_{1\,;T} + \lambda_T^{-2}\|G-\tilde{G}^I\|_{1\,;T}\Big) \lesssim \ell'_h. \tag{8.4}$$

Note that here $T \in \mathcal{T}_2$ implies $\lambda_T \simeq H_T$. Let $\tilde{G} := \min\{G,\, \varepsilon^{-2}c_3\,\ell'_h\}$, for some $c_3 > 0$, be a continuous approximation of G, and then $\tilde{G}^I$ be the standard piecewise-linear Lagrange interpolant of $\tilde{G}$. Assuming $C_f \simeq 1$, note the upper bound $G \leqslant G_0 := \varepsilon^{-2}(2\pi)^{-1}K_0(\sqrt{C_f}\,r/\varepsilon)$, where $r = [(x-x')^2 + (y-y')^2]^{1/2}$, and K_0 is the modified Bessel function of the second kind of order zero [6]. (Unless $C_f \geqslant \frac{1}{2}$, one has $\varepsilon^2 \geqslant \frac{1}{2}$ so a similar $G_0 := \varepsilon^{-2}(2\pi)^{-1}\ln(r^{-1}\mathrm{diam}\,\Omega)$ can be used.) Recalling that $K_0(s) \lesssim \ln(2+s^{-1})$ for $s > 0$, one concludes that there is a sufficiently large c_3 such that $\{r \geqslant \hbar\} \subseteq \{G_0 \leqslant \varepsilon^{-2}c_3\,\ell'_h\}$, and so $\Omega\backslash B(x', y'\,;\hbar) \subseteq \{G = \tilde{G}\}$ independently of ε or $\hbar$. Now let $\mathcal{T}_2^* := \{T \in \mathcal{T}_2 : T \cap B(x', y'\,;\hbar) = \varnothing\}$, and $\bar{\Theta}_2^*$ be defined as $\bar{\Theta}_2$ in (8.4) only with $\mathcal{T}_2^*$ replacing $\mathcal{T}_2$. As $G = \tilde{G}$ for any $T \in \mathcal{T}_2^*$, so $\tilde{G}^I = G^I$, so

$$\bar{\Theta}_2^* \simeq \varepsilon^2 \sum_{T\in\mathcal{T}_2^*} \Big(H_T^{-1}\|\nabla(G-G^I)\|_{1\,;T} + H_T^{-2}\|G-G^I\|_{1\,;T}\Big) \lesssim \varepsilon^2|G|_{2,1\,;\Omega\backslash B(x',y'\,;\hbar)},$$

where we used [1, Corollary 2.1]. Combining this with (4.4c) yields $\bar{\Theta}_2^* \lesssim \ell'_h$.

To deal with $\mathcal{T}_2\backslash\mathcal{T}_2^* = \{T \in \mathcal{T}_2 : T \cap B(x', y'\,;\hbar) \neq \varnothing\}$, note that (8.2) implies

$$B(x', y'\,;\hbar) \subset \omega_{(x',y')}, \qquad T \cap B(x', y'\,;\hbar) \neq \varnothing \;\Rightarrow\; T \subset B(x', y'\,;2H_T). \tag{8.5}$$

Here the first observation is straightforward, while the second observation is due to $T \cap B((x', y')\,;\hbar) \neq \varnothing$ implying that (i) T is within the distance of $\hbar$ from (x', y'), (ii) $T \subset \omega_{(x',y')}$ (also using the first observation) so $\hbar = \hbar_{(x',y')} \leqslant h_T \leqslant H_T$. Now,

$$\bar{\Theta}_2 - \bar{\Theta}_2^* \simeq \varepsilon^2 \sum_{T\in\mathcal{T}_2\backslash\mathcal{T}_2^*} \Big(H_T^{-1}\|\nabla(G-\tilde{G}^I)\|_{1\,;T} + H_T^{-2}\|G-\tilde{G}^I\|_{1\,;T}\Big).$$

For G, combining (8.5) with (4.4a), (4.4b) yields $\varepsilon^2\big(H_T^{-1}\|\nabla G\|_{1\,;T} + H_T^{-2}\|G\|_{1\,;T}\big) \lesssim \ell_\rho$, where $\rho = 2H_T$ so $\ell_\rho \leqslant \ell'_h$. A similar bound for $\tilde{G}^I$ follows from $\varepsilon^2|\tilde{G}^I| \leqslant c_3\,\ell'_h$ and $\varepsilon^2|\nabla\tilde{G}^I| \leqslant c_3\,\ell'_h h_T^{-1}$ combined with $|T| \simeq h_T H_T$. As (8.5) also implies that the number of $T \in \mathcal{T}_2\backslash\mathcal{T}_2^*$ is $\simeq 1$, so $\bar{\Theta}_2 - \bar{\Theta}_2^* \leqslant \ell'_h$. As $\bar{\Theta}_2^* \leqslant \ell'_h$, the bound (8.4) follows.

Now, in view of (8.3) and (8.4), if $\mathcal{T} = \mathcal{T}_1$ or $\mathcal{T} = \mathcal{T}_2$, the choice $G_h = 0 \in S_h$ or $G_h = \tilde{G}^I \in S_h$, respectively, yields the statement of the lemma. We proceed to the remaining case $\mathrm{dist}_\Omega(\mathcal{T}_1\backslash\mathcal{T}_2\,,\,\mathcal{T}_2\backslash\mathcal{T}_1) \gtrsim \varepsilon$ and set $G_h := (\mu^I\tilde{G})^I \in S_h$, where μ^I is the piecewise-linear Lagrange interpolant of a weight function μ that we now define. For any point $(x, y) \in \Omega$, set the weight $\mu := 0$ in $\mathcal{T}_1\backslash\mathcal{T}_2$, $\mu := 1$ in $\mathcal{T}_2\backslash\mathcal{T}_1$, $\mu := \min\big\{1\,;\,\mathrm{dist}_\Omega((x, y)\,, \mathcal{T}_1\backslash\mathcal{T}_2)/\mathrm{dist}_\Omega(\mathcal{T}_1\backslash\mathcal{T}_2\,,\,\mathcal{T}_2\backslash\mathcal{T}_1)\big\}$ for $(x, y) \in \mathcal{T}_1 \cap \mathcal{T}_2$. Note that μ is continuous, and furthermore, for its Lagrange interpolant μ^I, triangle-type inequalities yield $\|\nabla\mu^I\|_{\infty\,;\Omega} \leqslant 1/\mathrm{dist}_\Omega(\mathcal{T}_1\backslash\mathcal{T}_2\,,\,\mathcal{T}_2\backslash\mathcal{T}_1) \lesssim \varepsilon^{-1}$.

Next, split $g = G - G_h$ in (8.1) as $g = (1-\mu^I)G + [\mu^I G - (\mu^I \tilde{G})^I]$ and note that $1-\mu^I = 0$ in $\mathcal{T}\backslash\mathcal{T}_1$, while $\mu^I = 0$ in $\mathcal{T}\backslash\mathcal{T}_2$. Consequently $\bar\Theta \leqslant \bar\Theta_1^\mu + \bar\Theta_2^\mu$, where $\bar\Theta_1^\mu$ is defined as $\bar\Theta_1$ in (8.3), only with $(1-\mu^I)G$ replacing G, while $\bar\Theta_2^\mu$ is defined as $\bar\Theta_2$ in (8.4), only with $\mu^I G - (\mu^I\tilde{G})^I$ replacing $G - \tilde{G}^I$. Imitating the argument used to estimate $\bar\Theta_1$, one gets $\bar\Theta_1^\mu \lesssim \varepsilon\|\nabla\{(1-\mu^I)G\}\|_{1\,;\Omega} + \|(1-\mu^I)G\|_{1\,;\Omega} \lesssim 1$. Similarly, imitating the argument used to estimate $\bar\Theta_2$, one gets $\bar\Theta_2^\mu \lesssim \ell'_h$. In particular, for a version of $\bar\Theta_2^*$ one has $\bar\Theta_2^{\mu,*} \lesssim \varepsilon^2|\mu^I G|_{2,1\,;\Omega\backslash B(x',y'\,;\hbar)} \lesssim \varepsilon^2|G|_{2,1\,;\Omega\backslash B(x',y'\,;\hbar)} + \varepsilon|G|_{1,1\,;\Omega}$; so combining this with (4.4c) and (4.3) yields $\bar\Theta_2^{\mu,*} \lesssim \ell'_h$. Also, $\bar\Theta_2^\mu - \bar\Theta_2^{\mu,*}$ is estimated similarly to $\bar\Theta_2 - \bar\Theta_2^*$; in particular, we use $\varepsilon^2\bigl(H_T^{-1}\|\nabla(\mu^I G)\|_{1\,;T} + H_T^{-2}\|\mu^I G\|_{1\,;T}\bigr) \lesssim \ell'_h$, for which we additionally note that $\varepsilon H_T^{-1}\|G\|_{1\,;T} \lesssim \sqrt{\ell'_h}$, in view of (4.4a) and (4.3). Combining the above observations yields the desired bound $\bar\Theta \leqslant \bar\Theta_1^\mu + \bar\Theta_2^\mu \lesssim \ell'_h$. $\square$

9. Conclusion. The main results of the paper, Theorems 6.3, 7.1 and 7.7, give a number a posteriori error estimates in the maximum norm on general anisotropic meshes. To be more precise, the error is represented as $(u_h - u)(x', y') = I + II + \mathcal{E}_{\rm quad}$, where the quadrature error $\mathcal{E}_{\rm quad}$ is bounded by (5.2), while the above three theorems give bounds for the jump residual component I and the interior residual component II. The latter bounds acquire meaning when they are combined with $\Theta + \Theta' \lesssim \ell_{h,(x',y')}$ (which is given by Lemma 8.1).

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