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A robust grid equidistribution method for a one-dimensional singularly perturbed semilinear reaction-diffusion problem*

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The numerical solution of a singularly perturbed semilinear reaction-diffusion two-point boundary value problem is addressed. The method considered is adaptive movement of a fixed number $(N+1)$ of mesh points by equidistribution of a monitor function that uses discrete second-order derivatives. We extend the analysis by Kopteva & Stynes (2001) to a new equation and a more intricate monitor function. It is proved that there exists a solution to the fully discrete equidistribution problem, i.e. a mesh exists that equidistributes the discrete monitor function computed from the discrete solution on this mesh. Furthermore, in the case when the boundary value problem is linear, it is shown that after $O(|\ln\varepsilon|/\ln N)$ iterations of the algorithm, the piecewise linear interpolant of the computed solution achieves second-order accuracy in the maximum norm, uniformly in the diffusion coefficient ε^2. Numerical experiments are presented that support our theoretical results.

Keywords: grid equidistribution, reaction-diffusion, singular perturbation, adaptive mesh, finite differences, maximum norm.

1. Introduction

Solutions of singularly perturbed differential equations frequently exhibit sharp boundary and interior layers, which are narrow regions where solutions change rapidly. To obtain reliable numerical approximations of layer solutions in an efficient way, one has to use locally refined meshes that are fine in layer regions and standard outside. If the location(s) and width(s) of the layers are known a priori, one can invoke this knowledge to construct suitable layer-adapted meshes. Otherwise, which is often the case in applications, adaptive algorithms are needed that start from an initial unsophisticated mesh and then, using intermediate computed solutions, automatically adapt the mesh and ultimately detect accurate locations and widths of the layers. Although adaptive algorithms have been successfully applied to many problems, their convergence properties are not entirely understood.

The aim of this paper is to present a convergence analysis of an adaptive algorithm applied to one simple singularly perturbed problem. We follow Kopteva & Stynes (2001), where a similar algorithm, which equidistributed the arc-length of the computed solution, was applied to a singularly perturbed convection-diffusion equation and the number of iterations was estimated needed to achieve first-order accuracy. Now we extend the analysis in Kopteva & Stynes (2001) to a new equation and a more intricate monitor function that involves discrete second-order derivatives and whose equidistribution yields second-order-accurate computed solutions. Compared to the arc-length monitor function considered in Kopteva & Stynes (2001), our monitor function yields higher-order accuracy, but its equidistribution does not allow a simple geometric interpretation and hence its analysis is more complicated.

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We shall address the singularly perturbed semilinear reaction-diffusion problem

$$Tu = -\varepsilon^2 u''(x) + b(x,u) = 0 \quad \text{for } x \in (0,1), \qquad u(0) = u(1) = 0, \tag{1.1}$$

where $\varepsilon \in (0,1]$ is a small positive parameter, the function b is sufficiently smooth and

$$0 < \gamma_0^2 \leqslant b_u(x,u) \leqslant \bar{\gamma}^2 \qquad \text{for all } (x,u) \in [0,1] \times \mathbb{R}. \tag{1.2}$$

Under condition (1.2), problem (1.1) has a unique solution, which exhibits sharp boundary layers of width $O(\varepsilon|\ln\varepsilon|)$ at $x=0$ and $x=1$.

For our numerical method we consider arbitrary meshes $\{x_i\}_{i=0}^N$ with $0 = x_0 < x_1 < \ldots < x_N = 1$, where N is a positive integer, and the local mesh size $h_i := x_i - x_{i-1}$ for $i = 1,\ldots,N$. The computed solution $\{u^N\}_{i=0}^N$, associated with the mesh $\{x_i\}_{i=0}^N$, is required to satisfy the standard finite difference discretization of (1.1):

$$T^N u_i^N := -\varepsilon^2 \delta^2 u_i^N + b(x_i, u_i^N) = 0 \quad \text{for} \quad i = 1,\ldots,N-1, \qquad u_0^N = u_N^N = 0, \tag{1.3a}$$

where

$$\delta^2 u_i^N := \frac{2}{h_i + h_{i+1}} \Big(\frac{u_{i+1}^N - u_i^N}{h_{i+1}} - \frac{u_i^N - u_{i-1}^N}{h_i} \Big), \qquad i = 1,\ldots,N-1. \tag{1.3b}$$

Note that there exists a unique solution $\{u_i^N\}$ to the discrete problem (1.3); furthermore, a linearization of this problem satisfies the discrete maximum/comparison principle.

We shall invoke the a posteriori error analysis by Kopteva (2007). In particular, for the solution u of (1.1) and the standard piecewise linear interpolant $u^N(x)$ of the discrete solution $\{u_i^N\}$ of (1.3) on an arbitrary mesh $\{x_i\}$, Kopteva (2007) gives the following *a posteriori error estimate*:

$$\max_{0\leqslant x\leqslant 1} |u^N(x) - u(x)| \leqslant \bar{C} \max_{1\leqslant i\leqslant N} \{\bar{M}_i^N h_i\}^2, \quad \bar{M}_i^N := \min\{|\delta^2 u_{i-1}^N|, |\delta^2 u_i^N|\}^{1/2} + (\varepsilon|\delta^3 u_i^N|)^{1/2} + 1, \tag{1.4a}$$

which also uses the standard third-order discrete derivatives $\delta^3 u_i^N := (\delta^2 u_i^N - \delta^2 u_{i-1}^N)/h_i$ (combine Theorem 3.3, Remark 3.4 and Lemma 3.5 in Kopteva (2007)). Here $\bar{M}_0^N$ and $\bar{M}_N^N$ involve $\delta^2 u_i^N$ for $i = 0, N$, respectively, which are not given in (1.3b), but instead defined by

$$\delta^2 u_0^N := \varepsilon^{-2} b(0, u_0^N), \qquad \delta^2 u_N^N := \varepsilon^{-2} b(1, u_N^N), \tag{1.4b}$$

in agreement with the discrete equation $-\varepsilon^2 \delta^2 u_i^N + b(x_i, u_i^N) = 0$ from (1.3a) formally extended to $i = 0, N$. It is crucial that the error constant $\bar{C}$ in (1.4) is independent of ε and the mesh. We also refer the reader to a two-dimensional version of the a posteriori error estimate (1.4) recently presented in Kopteva (2007b).

Furthermore, it is shown in (Kopteva, 2007, Lemma 3.5) that on an arbitrary mesh we have

$$\max_{1\leqslant i\leqslant N} \{\bar{M}_i^N h_i\} \leqslant C \max_{1\leqslant i\leqslant N} \{\bar{\bar{M}}_i^N h_i\}, \qquad \bar{\bar{M}}_i^N := \max\{|\delta^2 u_{i-1}^N|, |\delta^2 u_i^N|\}^{1/2} + 1. \tag{1.5}$$

This implies that the discrete analogue of $\varepsilon|u'''|$ in the a posteriori error estimate (1.4a) is insignificant.

Given the a posteriori error estimate (1.4) and relation (1.5), we face the problem of finding the mesh $\{x_i\}$ and the computed solution $\{u_i^N\}$ on this mesh such that $\max_i \bar{M}_i^N h_i \leqslant CN^{-1}$ or $\max_i \bar{\bar{M}}_i^N h_i \leqslant CN^{-1}$; then, by (1.4), (1.5), this computed solution $\{u_i^N\}$ is guaranteed to be second-order accurate ε-uniformly.

We shall address this problem is in the framework of monitor-function equidistribution. A mesh $\{x_i\}$ is said to equidistribute a monitor function $M(x) > 0$ if

$$\int_{x_{i-1}}^{x_i} M(x)\,\mathrm{d}x = \frac{1}{N}\int_0^1 M(x)\,\mathrm{d}x \qquad \text{for} \quad i = 1,\ldots,N. \tag{1.6}$$

The a posteriori error estimate (1.4) and relation (1.5) imply a piecewise constant monitor function $M(x) := M_i^N$ for $x \in (x_{i-1}, x_i)$, where $M_i^N = \bar{M}_i^N$ or $M_i^N = \bar{\bar{M}}_i^N$, which yields the following formulation.

Equidistribution Problem. Find $\{(x_i, u_i^N)\}$, with the $\{u_i^N\}$ computed from the $\{x_i\}$ by means of (1.3), such that

$$M_i^N h_i = \frac{1}{N}\sum_{j=1}^{N} M_j^N h_j \quad \text{for } i = 1,2,\ldots,N. \tag{1.7}$$

Note that here both $\{x_i\}$ and $\{u_i^N\}$ are a priori unknown. Consequently, even if (1.1) is linear, the equidistribution problem, which requires the simultaneous solution of (1.3) and (1.7), is nonlinear. Following the analysis by Kopteva & Stynes (2001) of arc-length equidistribution for a convection-diffusion equation, we pose the following fundamental questions:

1. Does our equidistribution problem have a solution?

2. Is there any algorithm to solve our equidistribution problem that can be proved to yield an accurate computed solution?

The aim of the present paper is to provide answers to these important questions:

1. We establish existence of a solution to the equidistribution problem (1.7). Furthermore, our existence theorem applies to a more general equidistribution problem that uses an abstract discrete problem and an abstract monitor function (not necessarily discrete problem (1.3) and the monitor function $\bar{\bar{M}}_i^N$ from (1.5) or M_i^N from (1.8) below), which are required to satisfy a certain basic condition; see Theorem 3.1 for details. Furthermore, the abstract discrete problem might be a discretization of a continuous problem other than our problem (1.1).

2. We consider a simple algorithm, originally due to de Boor (1974), that equidistributes the discrete monitor function

   $$M_i^N := \min\{|\delta^2 u_{i-1}^N|, |\delta^2 u_i^N|\}^{1/2} + 1, \qquad \text{where } \delta^2 u_0^N := \delta^2 u_1^N,\ \delta^2 u_N^N := \delta^2 u_{N-1}^N. \tag{1.8}$$

   We choose this modification of $\bar{\bar{M}}_i^N$ for our algorithm, since using $\bar{M}_i^N$ from (1.4) or $\bar{\bar{M}}_i^N$ from (1.5) instead of this M_i^N might result in the so-called mesh starvation when the mesh nodes are temporarily drawn away from certain regions; see Figure 2 in §6 and Remarks 2.3 and 4.5.

   Following Kopteva & Stynes (2001), de Boor's original algorithm is modified by using a less stringent stopping criterion that, while preserving the accuracy of the final computed solution, requires far fewer iterations.

   We answer question 2 in the case when (1.1) is linear and under further mild assumptions. Then it is stated in Theorem 5.1 that our algorithm is guaranteed to yield a second-order accurate computed solution and this is achieved after $O(|\ln\varepsilon|/\ln N)$ iterations.

   Note that we expect *any other efficient algorithm* based on a similar monitor function to enjoy similar properties and require a similar number of iterations.

Monitor functions have been used by many authors to drive adaptive algorithms in solving differential equations; see, e.g., Huang & Sloan (1994), Cao *et al.* (1999), Linß (2001), Tang & Tang (2003), Kopteva *et al.* (2005), Huang (2005a), Huang (2005b), Mackenzie & Mekwi (2007). In particular, the monitor function M_i^N used in (1.7) is a discrete analogue of the continuous monitor function $M(x) := |u''|^{1/2} + 1$, whose equidistribution (1.6) for a linear analogue of (1.1) was investigated by Beckett & Mackenzie (2001). Note also that the same continuous monitor function $M(x)$ is a one-dimensional version of the monitor function that was suggested by Chen *et al.* (2007) based on the piecewise-linear-interpolation error analysis in the L_p norm in the case of $p = \infty$.

Outline. The mesh movement algorithm is described in §2. Next, in §3 we prove our existence result (Theorem 3.1). Then in §4 we establish an important property of any intermediate mesh generated by our algorithm that its maximum mesh size never exceeds CN^{-1}. The entire §5 is devoted to a detailed analysis of the algorithm's behaviour in the case of linear $b(x,u)$; this culminates in an estimate of Theorem 5.1 for the number of iterations needed to achieve second-order accuracy. Numerical results supporting our theory are presented in §6. Note that to facilitate reading this paper, we deliberately follow the notation and presentation in Kopteva & Stynes (2001), which gave a similar analysis but for a simpler monitor function. We also refer the reader to Kopteva (2007a) for a constant-coefficient case of the analysis in Kopteva & Stynes (2001).

Notation: Throughout the paper C, C', $\bar{C}$ denote generic positive constants that are independent of ε, of the mesh, and of the number of iterations taken by the algorithm of §2, and can take different values in different places. A subscripted C (e.g., C_1) is a constant that is independent of ε, of the mesh, and of the number of iterations taken by the algorithm, but whose value is fixed. When choosing N sufficiently large independently of ε and the mesh, we shall mean that $N \geqslant C$ for some sufficiently large positive constant C. As the number of mesh points $(N+1)$ is fixed throughout the algorithm, we generally do not indicate dependence on N in our notation; for example we write $u^{(k)}$ for the solution computed by the kth iteration of the algorithm and $T^{(k)}$ for the difference operator on the kth mesh.

2. The algorithm

Given an arbitrary mesh, our algorithm aims to construct a mesh that solves the Equidistribution Problem (1.7), (1.8). The number N of mesh intervals is fixed throughout.

ALGORITHM:

1. Initialize mesh: The initial mesh $\{0, 1/N, 2/N, \dots, 1\}$ is uniform.
2. For $k = 0, 1, \dots$, given the mesh $\{x_i^{(k)}\}$, compute the discrete solution $\{u_i^{(k)}\}$ satisfying

$$T^{(k)}u^{(k)} = 0 \quad \text{on } \{x_i^{(k)}\}, \qquad \text{with} \quad u_0^{(k)} = u_N^{(k)} = 0. \tag{2.1}$$

Set $h_i^{(k)} = x_i^{(k)} - x_{i-1}^{(k)}$ for each i. Let the piecewise-constant monitor function $M^{(k)}(x)$ be defined by

$$M^{(k)}(x) := M_i^{(k)} = \min\{|\delta^2 u_{i-1}^{(k)}|, |\delta^2 u_i^{(k)}|\}^{1/2} + 1 \qquad \text{for} \quad x \in (x_{i-1}^{(k)}, x_i^{(k)}), \tag{2.2a}$$

where

$$\delta^2 u_0^{(k)} := \delta^2 u_1^{(k)}, \qquad \delta^2 u_N^{(k)} := \delta^2 u_{N-1}^{(k)}. \tag{2.2b}$$

Then the total integral of the monitor function $M^{(k)}$ is

$$I^{(k)} := \int_0^1 M^{(k)}(x)\,\mathrm{d}x = \sum_{i=1}^{N} M_i^{(k)} h_i^{(k)}.$$

3. Test mesh: Let C_0 be a user-chosen constant with $C_0 > 1$ (see Remark 2.1). If

$$\max_{1\leqslant i\leqslant N}\{M_i^{(k)}h_i^{(k)}\} \leqslant C_0\frac{I^{(k)}}{N}, \tag{2.3}$$

then go to Step 5. Otherwise, continue to Step 4.

4. Generate a new mesh by equidistributing monitor function $M^{(k)}$ of current computed solution: Choose the new mesh nodes $0 = x_0^{(k+1)} < x_1^{(k+1)} < \ldots < x_N^{(k+1)} = 1$ such that

$$\int_{x_{i-1}^{(k+1)}}^{x_i^{(k+1)}} M^{(k)}(x)\,\mathrm{d}x = I^{(k)}/N, \qquad i = 0,\ldots,N. \tag{2.4}$$

(Since $\int_0^x M^{(k)}(t)\,\mathrm{d}t$ is increasing in x, the above relation clearly determines the $x_i^{(k+1)}$ uniquely.) Return to Step 2.

5. Set $\{x_0^*, x_1^*, \ldots, x_N^*\} = \{x_i^{(k)}\}$ and $\{u_i^*\} = \{u_i^{(k)}\}$ then stop.

REMARK 2.1 In (2.3) we can choose any constant C_0 that satisfies $C_0 > 1$. The larger C_0 is, the fewer iterations needed by the algorithm. If we set $C_0 = 1$, then the algorithm is attempting to compute a fixed point of Theorem 3.1, so when $C_0 \approx 1$, we expect that the computed solution lies near such a fixed point. Note that the numerical experiments presented in §6, and also in Kopteva *et al.* (2005), imply that $C_0 = 2$ produces suitable layer-adapted meshes and requires quite few iterations.

REMARK 2.2 The choice of $C_0 > 1$ in the stopping criterion (2.3) implies that instead of the equidistribution problem (1.7), (1.8), the algorithm attempts to solve the following *Quasi-Equidistribution Problem*: Find $\{(x_i, u_i^N)\}$, with the $\{u_i^N\}$ computed from the $\{x_i\}$ by means of (1.3), such that

$$M_i^N h_i \leqslant C_0 \frac{1}{N}\sum_{j=1}^{N} M_j^N h_j \quad \text{for } i = 1,2,\ldots,N. \tag{2.5}$$

REMARK 2.3 Our a posteriori error estimate (1.4) and relation (1.5) involve $\bar{M}_i^N$ and $\bar{\bar{M}}_i^N$, so one might assume that it would be a better choice for the algorithm to use $\bar{M}_i^{(k)} := \bar{M}_i^N$ or $\bar{\bar{M}}_i^{(k)} := \bar{\bar{M}}_i^N$ instead of $M_i^{(k)}$. However, using $\bar{M}_i^{(k)}$ and $\bar{\bar{M}}_i^{(k)}$ results in so-called mesh starvation when the mesh nodes are drawn away from certain regions; see Figure 2 in §6 and also Remark 4.5; while the algorithm in its current form enjoys the important property that $\max_i h_i^{(k)} \leqslant CN^{-1}$ independently of ε, N and k; see Corollary 4.3.

REMARK 2.4 Instead of the monitor function M_i^N, it might be preferable to use its scaled version

$$M_{\beta,i}^N := \beta^{-1}\min\{|\delta^2 u_{i-1}^N|, |\delta^2 u_i^N|\}^{1/2} + 1 \tag{2.6}$$

for some positive constant β. This is equivalent to stretching one of the coordinates. The algorithm and its analysis can be easily modified to accommodate this more general function. Similarly, one might define $\bar{M}_{\beta,i}^N := \beta^{-1}\big[\min\{|\delta^2 u_{i-1}^N|, |\delta^2 u_i^N|\}^{1/2} + (\varepsilon|\delta^3 u_i^N|)^{1/2}\big]/2 + 1$. Furthermore, the magnitude of u^N should not affect the suitability of the monitor function, so one natural approach is to normalize by $\beta := \max_i |u_i^N|^{1/2}$; see (Kopteva *et al.*, 2005, relation (4.7)).

3. The existence theorem

In this section we establish existence of a solution to our equidistribution problem (1.7), (1.8). Furthermore, our existence theorem applies to a general equidistribution problem that uses an abstract discrete problem (not necessarily (1.3)) and an abstract monitor function (not necessarily M_i^N), which are required to satisfy a certain basic condition; see Theorem 3.1 below.

3.1 *Abstract equidistribution problem*

On an arbitrary mesh $\{x_i\}_{i=0}^N$ consider an *abstract numerical method* in the form

$$\hat{T}^N \hat{u}_i^N = 0 \qquad \text{for} \quad i = 0, \ldots, N, \tag{3.1}$$

possibly involving a small parameter $\varepsilon \in (0,1]$. Note that (3.1) might be a discretization of a continuous problem other than (1.1).

Abstract Equidistribution Problem. Find $\{(x_i, \hat{u}_i^N)\}$, with the $\{\hat{u}_i^N\}$ computed from the $\{x_i\}$ by means of (3.1), such that

$$\int_{x_{i-1}}^{x_i} \hat{M}^N(x)\,\mathrm{d}x = \frac{1}{N}\int_0^1 \hat{M}^N(x)\,\mathrm{d}x \quad \text{for } i = 1, 2, \ldots, N, \tag{3.2}$$

where $\hat{M}^N(x) = \hat{M}^N\big(\{(x_i, \hat{u}_i^N)\}; x\big)$ is an abstract monitor function computed from $\{(x_i, \hat{u}_i^N)\}$ and scaled so that $\hat{M}^N(x) \geqslant 1$.

THEOREM 3.1 (EXISTENCE OF A SOLUTION TO THE ABSTRACT EQUIDISTRIBUTION PROBLEM) Let the abstract discrete problem (3.1) have a unique solution on each mesh $\{x_i\}$ and the associated mapping $F : \mathbb{R}^{2(N+1)} \to \mathbb{R}^{N+1}$, with $F_i(\hat{u}_0, \ldots, \hat{u}_N, x_i, \ldots, x_N) := \hat{T}^N \hat{u}_i^N$, $i = 0, \ldots, N$, be continuously differentiable and have a nonsingular Jacobian matrix $\frac{\partial(F_0,\ldots,F_N)}{\partial(\hat{u}_0^N,\ldots,\hat{u}_N^N)}$ for all $\{\hat{u}_i^N\}$ and all $\{x_i\}$ such that $x_0 < x_1 < \ldots < x_N$. Furthermore, suppose that there exists a quantity $Q(\varepsilon, N) \in (0,1)$ such that the monitor function $\hat{M}^N(x)$ computed on an arbitrary mesh $\{x_i\}$ satisfies

$$1 \leqslant \hat{M}^N(x) \leqslant \frac{1}{Q(\varepsilon, N)} \qquad \text{for} \quad x \in [0,1]; \tag{3.3}$$

while $\int_0^x \hat{M}^N(t)\,\mathrm{d}t$ is continuous in $\{\hat{u}_i^N\}$ and $\{x_i\}$. Then for each ε and each N, the abstract equidistribution problem (3.2) has a solution.

Proof. We imitate the proof of (Kopteva & Stynes, 2001, Theorem 3.1), which addresses a discrete arc-length monitor function. One can regard Steps 2 and 4 of the Algorithm in §2, in which we replace $T^{(k)}$, $u^{(k)}$ and $M^{(k)}$ with $\hat{T}^N$, $\hat{u}^N$ and $\hat{M}^N$, respectively, as a mapping $\Phi : (h_1, h_2, \ldots, h_N) \longmapsto (\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_N)$, where the h_i and $\tilde{h}_i$ are the mesh-widths before and after regridding. Note that Φ is continuous, since $\int_0^x \hat{M}^N(t)\,\mathrm{d}t$ is continuous in $\{\hat{u}_i\}$ and $\{x_i\}$, while $\hat{u}_i = \hat{u}_i(x_0, \ldots, x_N)$, $i = 0, \ldots, N$, are continuous, by the implicit function theorem applied to the system $F_i(\hat{u}_0, \ldots, \hat{u}_N, x_i, \ldots, x_N) = 0$ for $i = 0, \ldots, N$.

We claim that $\Phi : S_Q \longrightarrow S_Q$, where

$$S_Q = \Big\{(h_1, h_2, \ldots, h_N) \in \mathbb{R}^N : \ h_i \geqslant Q/N \text{ for } i = 1, \ldots, N, \quad \sum_{i=1}^N h_i = 1\Big\}$$

with $Q = Q(\varepsilon, N)$ from (3.3). Note that $0 < Q < 1$ implies that S_Q is nonempty.

To prove this claim, let $\{\hat{u}_i^N\}$ be the solution to (3.1) computed on the mesh $\{x_i\}$ with mesh-widths $h_1, h_2, \ldots, h_N$ and let $\{\tilde{x}_i\}$ be the mesh after regridding. Then, by (2.4), we have

$$\int_{\tilde{x}_{i-1}}^{\tilde{x}_i} \hat{M}^N(x)\,\mathrm{d}x = \frac{1}{N}\int_0^1 \hat{M}^N(x)\,\mathrm{d}x, \qquad i = 0, \ldots, N,$$

while condition (3.3) implies that

$$\int_{\tilde{x}_{i-1}}^{\tilde{x}_i} \hat{M}^N(x)\,\mathrm{d}x \leqslant \tilde{h}_i/Q \qquad \text{and} \qquad \frac{1}{N}\int_0^1 \hat{M}^N(x)\,\mathrm{d}x \geqslant 1/N.$$

Hence we have $\tilde{h}_i \geqslant Q/N$ for $i = 1, \ldots, N$, i.e. indeed, Φ maps S_Q into itself.

The nonempty set S_Q is convex and compact, and Φ is continuous. By the Brouwer fixed-point theorem—see, e.g.,Smart (1974)—Φ has a fixed point in S. That is, there exists a mesh on which the computed solution satisfies $\int_{x_{i-1}}^{x_i} \hat{M}^N(x)\,\mathrm{d}x = \int_{x_{j-1}}^{x_j} \hat{M}^N(x)\,\mathrm{d}x$ for all i and j. □

3.2 *Existence of a solution for equidistribution problem (1.7) and quasi-equidistribution problem (2.5)*

Now we shall apply the above general theorem to our particular (quasi-)equidistribution problems.

COROLLARY 3.2 For each ε and each N, the equidistribution problem (1.7), (1.8) and the quasi-equidistribution problem (2.5), (1.8) have a solution.

Proof. It suffices to show that the equidistribution problem (1.7), (1.8) has a solution. Note that, by (1.2), discrete problem (1.3) satisfies the hypotheses of Theorem 3.1. Next, we verify that the monitor function M_i^N from (1.8) applied to problem (1.3) satisfies hypothesis (3.3). Combining (1.8) with (1.3), yields $M_i^N \leqslant \varepsilon^{-1} \max_{0\leqslant i\leqslant N} \sqrt{b(x_i, u_i^N)} + 1$. Now, recall the bound $|u_i^N| \leqslant C' := \gamma_0^{-2} \max_x |b(x,0)|$ from (Kopteva, 2007, Lemma 3.1), which implies that $M_i^N \leqslant Q^{-1}$ with

$$Q := \big(\varepsilon^{-1} \max_{0\leqslant x\leqslant 1, |s|\leqslant C'} \sqrt{b(x,s)} + 1\big)^{-1}.$$

Thus condition (3.3) is indeed satisfied. Furthermore, M_i^N for each i is continuous in $\{u_i^N\}$ and $\{x_i\}$ and bounded; therefore $\int_0^x M^N(t)\,\mathrm{d}t$ is continuous in $\{u_i^N\}$ and $\{x_i\}$, where $M^N(x) := M_i^N$ for $x \in (x_{i-1}, x_i)$. Hence the assertion of the corollary follows from Theorem 3.1 applied to the equidistribution problem (1.7), (1.8). □

COROLLARY 3.3 For each ε and each N, the equidistribution problem (1.7) and the quasi-equidistribution problem (2.5), in which M_i^N is replaced by $\bar{\bar{M}}_i^N$ from (1.5), (1.4b), have a solution.

Proof. Simply imitate the proof of Corollary 3.2. □

4. Maximum mesh size for meshes $\{x_i^{(k)}\}$ generated by the algorithm

In the present section we establish an important property of any mesh generated by the algorithm that $\max_i h_i^{(k)} \leqslant CN^{-1}$ independently of k, ε and N; see Corollary 4.3 below. Not only the final mesh generated by the algorithm enjoys this property, but so do all the intermediate meshes. Thus we are guaranteed to avoid what is sometimes referred to as mesh starvation, when the mesh nodes are drawn away from regions where the solution is smooth.

First, we give two auxiliary lemmas.

LEMMA 4.1 On an arbitrary mesh $\{x_i\}$, let

$$B_i = \prod_{j=1}^{i}\big(1+\alpha h_j/\varepsilon\big)^{-2} \quad \text{and} \quad \tilde{B}_i := B_N/B_i \qquad \text{for} \quad i=0,\dots N,$$

where $\alpha := \gamma_0/\sqrt{6}$. Then for $L^N B_i := -\varepsilon^2\delta^2 B_i + p_i B_i$, where $i=1,\dots,N-1$ and $\gamma_0^2 \leqslant p_i \leqslant \bar{\gamma}^2$, we have $L^N B_i \geqslant 0$. Furthermore, $L^N \tilde{B}_i \geqslant 0$ and we also have

$$\sqrt{B_i}\,\frac{h_i}{\varepsilon} = (\sqrt{B_{i-1}} - \sqrt{B_i})/\alpha, \qquad \sqrt{\tilde{B}_{i-1}}\,\frac{h_i}{\varepsilon} = (\sqrt{\tilde{B}_i} - \sqrt{\tilde{B}_{i-1}})/\alpha, \tag{4.1}$$

and

$$\sqrt{B_i} \leqslant (1+\alpha x_i/\varepsilon)^{-1}. \tag{4.2}$$

Proof. Relations (4.1) are verified by a straightforward calculation, while (4.2) immediately follows from $\prod_{j=1}^{i}(1+\alpha h_j/\varepsilon) \geqslant 1+\sum_{j=1}^{i}(\alpha h_j/\varepsilon)$. Now it suffices to prove the bound $L^N B_i \geqslant 0$, as its analogue for $\tilde{B}_i$ is obtained similarly. Using the notation $H_i := h_i/\varepsilon$, we get

$$\frac{B_i - B_{i-1}}{H_i} = -(2\alpha + \alpha^2 H_i)B_i$$

and, similarly,

$$\frac{B_{i+1}-B_i}{H_{i+1}} = -\frac{2\alpha+\alpha^2 H_{i+1}}{(1+\alpha H_{i+1})^2}B_i = \Big[-2\alpha + \alpha^2 H_{i+1}\frac{3+2\alpha H_{i+1}}{(1+\alpha H_{i+1})^2}\Big]B_i\,,$$

while

$$L^N B_i \geqslant -\frac{2}{H_i+H_{i+1}}\Big[\frac{B_{i+1}-B_i}{H_{i+1}} - \frac{B_i - B_{i-1}}{H_i}\Big] + \gamma_0^2 B_i\,.$$

Hence

$$L^N B_i \geqslant -\frac{2\alpha^2}{H_i+H_{i+1}}\Big[H_{i+1}\,\vartheta(\alpha H_{i+1}) + H_i\Big]B_i + \gamma_0^2 B_i \geqslant [\gamma_0^2 - 6\alpha^2]B_i\,,$$

where $\vartheta(t) := (3+2t)/(1+t)^2 \leqslant 3$ for $t>0$. Now $\alpha = \gamma_0/\sqrt{6}$ implies $L^N B_i \geqslant 0$. □

LEMMA 4.2 (BOUND ON TOTAL INTEGRAL OF MONITOR FUNCTION) Let $\{u_i^N\}$ be the solution of (1.3) on an arbitrary mesh $\{x_i\}$ such that $\max_i h_i \leqslant 1/2$, and let M_i^N be the monitor function from (1.8). Then for some $C_1 \geqslant 1$ we have

$$1 \leqslant I^N = \sum_{i=1}^{N} M_i^N h_i \leqslant C_1.$$

Proof. Decompose the computed solution u^N into a smooth component w^N and a singular component v_N as follows; see (Kopteva, 2007, §6). Let $u_0(x)$ be the unique solution of the reduced problem $b(x,u_0)=0$. Next let w^N satisfy $T^N w^N = 0$ for $i=1,\dots,N-1$, subject to $w_0^N = u_0(0)$ and $w_N^N = u_0(1)$. Then, by (Kopteva, 2007, Lemma 6.1), we get

$$|\delta^2 w_i^N| \leqslant C, \qquad i=1,\dots,N-1. \tag{4.3}$$

Now if $u^N = w^N + v^N$, then for v^N we have

$$L^N v_i^N = -\varepsilon^2\delta^2 v_i^N + p_i v_i^N = 0 \quad \text{for} \quad i=1,\dots,N-1, \qquad v_0^N = -u_0(0), \quad v_N^N = -u_0(1), \tag{4.4}$$

where $p_i := \int_0^1 b_u(x_i, w_i^N + s v_i^N)\mathrm{d}s$, which, by (1.2), satisfies $\gamma_0^2 \leqslant p_i \leqslant \bar{\gamma}^2$. Hence

$$1 \leqslant M_i^N \leqslant \min\big\{|\delta^2 v_{i-1}^N|, |\delta^2 v_i^N|\big\}^{1/2} + \max_i |\delta^2 w_i^N|^{1/2} + 1 \leqslant \frac{\bar{\gamma}}{\varepsilon} \min\big\{|v_{i-1}^N|, |v_i^N|\big\}^{1/2} + C, \tag{4.5}$$

where we used (4.3) and (4.4) (and, accommodating (2.2b), assumed that $|v_1^N| \leqslant |v_0^N|$ and $|v_{N-1}^N| \leqslant |v_N^N|$ only to simplify the presentation, as the other case poses no additional difficulties in the analysis). Now, recalling Lemma 4.1 and applying the discrete comparison principle to problem (4.4), yields $|v_i^N| \leqslant |u_0(0)| B_i + |u_0(1)| \tilde{B}_i$. Therefore,

$$1 \leqslant I^N \leqslant C\Big(\sum_{i=1}^{m} \sqrt{|B_i|}\frac{h_i}{\varepsilon} + R_{m+1} + \sum_{i=m+2}^{N} \sqrt{|\tilde{B}_{i-1}|}\frac{h_i}{\varepsilon} + 1\Big), \qquad R_{m+1} := \min\big\{\sqrt{B_m}, \sqrt{\tilde{B}_{m+1}}\big\}\frac{h_{m+1}}{\varepsilon},$$

where m it the unique integer such that $B_i \geqslant \tilde{B}_i$ for $i \leqslant m$ and $B_i < \tilde{B}_i$ for $i > m$. Next, invoking (4.1), we obtain $I^N \leqslant C(1 + R_{m+1})$, and to complete the proof, it suffices to show that $R_{m+1} \leqslant C$. Note that $h_{m+1} \leqslant 1/2$ implies that either $x_m \geqslant 1/4$ or $1 - x_{m+1} \geqslant 1/4$. Consider the case of $x_m \geqslant 1/4$, as the other case is similar. Then, by (4.2), we have $R_{m+1} \leqslant \sqrt{B_m}\, h_{m+1}/\varepsilon \leqslant \alpha^{-1} h_{m+1}/x_m \leqslant 2\alpha^{-1}$. Thus indeed, for some constant C_1 we have $I^N \leqslant C_1$. □

Next we give a basic property of meshes generated by the algorithm.

COROLLARY 4.3 Let $\{x_i^{(k)}\}$ be any mesh generated by the algorithm. Then for all i we have

$$h_i^{(k)} \leqslant C_1 N^{-1}. \tag{4.6}$$

Proof. Since $C_1 \geqslant 1$, inequality (4.6) clearly holds true when $k = 0$, so assume that $k > 0$. Combining (2.4) with $1 \leqslant M^{(k-1)}$, we get

$$\int_{x_{i-1}^{(k)}}^{x_i^{(k)}} 1\,\mathrm{d}x \leqslant I^{(k-1)}/N.$$

Hence $h_i^{(k)} = x_i^{(k)} - x_{i-1}^{(k)} \leqslant I^{(k-1)}/N \leqslant C_1/N$, by Lemma 4.2. □

COROLLARY 4.4 Suppose that the algorithm reaches its stopping criterion and halts. Let the final mesh and solution generated be $\{x_i^*\}$ and $\{u_i^*\}$, and $M_i^* := M_i^{(k)}$ be the discrete monitor function on the final mesh. Then $\max_i\{M_i^* h_i^*\} \leqslant C_0 C_1 N^{-1}$.

Proof. The stopping criterion (2.3) implies that $\max_i\{M_i^* h_i^*\} \leqslant C_0 I^*/N$, where I^* is the total integral of the monitor function on the final mesh, for which, by Lemma 4.2, we have $I^* \leqslant C_1$. The desired result follows. □

REMARK 4.5 Note that the bound $I^N \leqslant C_1$ of Lemma 4.2 is essential for property (4.6) of the algorithm. Consider alternative monitor functions $\bar{M}_i^N$ from the a posteriori error estimate (1.4) or $\bar{\bar{M}}_i^N$ from (1.5). The corresponding total integrals of these monitor functions $\bar{I}^N := \sum_{i=1}^N \bar{M}_i^N h_i$ and its analogue $\bar{\bar{I}}^N$ might blow up in the sense that they are no longer bounded uniformly in ε and N. Indeed, e.g., on the uniform mesh we have $\bar{M}_1^N = O(\varepsilon^{-1})$ and hence $\bar{I}^{(k)} \geqslant \bar{M}_1^N h_1 = O(\varepsilon^{-1})N^{-1} \gg 1$ if $\varepsilon \ll N^{-1}$. (This phenomenon is also reported in (Kopteva, 2007, §5).) Hence using $\bar{M}_i^N$ or $\bar{\bar{M}}_i^N$ as a monitor function for our algorithm will result in so-called mesh starvation. If the contribution of $[0, x_1]$ to $\bar{I}^N$ significantly exceeds the contribution of $O(1)$ of the interior part of $[0,1]$, where the solution is smooth, step 4 of the algorithm will result in the mesh nodes being drawn away from the interior region; compare Figures 1 and 2 in §6. This is the reason for our choice of M_i^N instead of $\bar{M}_i^N$ or $\bar{\bar{M}}_i^N$ in the algorithm.

5. How many iterations for ε-uniform second-order accuracy?

In this section we consider the linear case of (1.1):

$$Lu := -\varepsilon^2 u'' + p(x)u = f(x), \qquad u(0) = u(1) = 0, \tag{5.1}$$

where, in agreement with (1.2), we have $0 < \gamma_0^2 \leqslant p(x) \leqslant \bar{\gamma}^2$. Now our discrete problem (1.3a) becomes

$$L^N u_i^N = -\varepsilon^2 \delta^2 u_i^N + p(x_i) u_i^N = f(x_i) \quad \text{for} \quad i = 1, \ldots, N-1, \qquad u_0^N = u_N^N = 0. \tag{5.2}$$

Similarly to the decomposition of the computed solution u^N used in the argument of Lemma 4.2, the solution u of (5.1) can be decomposed into a smooth component w and a singular component v as $u = w + v$, where $Lv = 0$ subject to the boundary conditions $v(0) = -u_0(0)$ and $v(1) = -u_0(1)$; compare with (4.4). In general, $|v(x)| \leqslant |u_0(0)|e^{-\gamma_0 x/\varepsilon} + |u_0(1)|e^{-\gamma_0(1-x)/\varepsilon}$, i.e. $v(x)$ describes the boundary layers at $x = 0$ and $x = 1$. To focus on the mesh nodes distribution in one of the two boundary layer regions, we make the simplifying assumption that

$$|u_0(0)| \geqslant C > 0, \qquad\qquad u_0(1) = 0, \tag{5.3}$$

which implies that u has a boundary layer at $x = 0$, but no boundary layer at $x = 1$. Note that if we had $u_0(0) = u_0(1) = 0$, then the solution of (5.1) would have no boundary layers and then, by (5.11) below, the computed solution would be second-order accurate on any mesh with $\max_i h_i \leqslant CN^{-1}$.

Furthermore, for simplicity we assume that

$$\varepsilon \leqslant N^{-1}, \tag{5.4}$$

which is not a restriction in practical situations. Our analysis also works if, instead of (5.4), we have $\varepsilon \leqslant \tilde{C}N^{-1}$ for some arbitrary but fixed constant $\tilde{C}$.

We also assume that N is sufficiently large independently of ε and the iteration counter k.

THEOREM 5.1 (ALGORITHM YIELDS SECOND-ORDER ACCURATE SOLUTION) Let u be the solution of (5.1) under assumptions (5.3), (5.4), and $\{u_i^{(k)}\}$ be the kth solution computed by the algorithm on the kth mesh $\{x_i^{(k)}\}$. Furthermore, let $u^{(k)}(x)$ be the standard linear interpolant of $\{u_i^{(k)}\}$. Then, if N is sufficiently large independently of ε, there exists a positive integer K, with $K \leqslant C|\ln\varepsilon|/\ln N$, such that $\max_{x\in[0,1]} |e^{(K)}(x)| \leqslant CN^{-2}$, where $e^{(k)} := u^{(k)}(x) - u(x)$ is the error in the kth solution.

The entire §5 is devoted to the proof of this theorem, which is divided into a series of intermediate results. To further help the reader, we group related material into subsections.

5.1 *Notation and preliminary material*

We introduce some notation that is used frequently later.

(i). Let t_1 and t_2 be arbitrary with $0 \leqslant t_1 < t_2 \leqslant 1$. Recall that $u_i^{(k)}$ is the computed solution on the kth mesh $\{x_i^{(k)}\}$ generated by the algorithm and $M^{(k)}(x)$ is the corresponding piecewise constant monitor function such that $M^{(k)}(x) = M_i^{(k)}$ for $x \in (x_{i-1}^{(k)}, x_i^{(k)})$; see §2, in particular (2.2). Define $I^{(k)}[t_1,t_2]$ to be the integral of the monitor function $M^{(k)}(x)$ over $[t_1,t_2]$:

$$I^{(k)}[t_1,t_2] = \int_{t_1}^{t_2} M^{(k)}(x)\,\mathrm{d}x. \tag{5.5}$$

We note that $I^{(k)}[0,1] = I^{(k)}$ in the notation of §2.

(ii). Define the parameter $\sigma = \sigma(\varepsilon, N)$ by

$$\sigma := \frac{2}{\gamma}\varepsilon \ln N \qquad \text{so that} \qquad e^{-\gamma\sigma/\varepsilon} = N^{-2}, \tag{5.6}$$

where $\gamma := \max_{x\in[0,\bar{\sigma}]} \sqrt{p(x)} \geqslant \gamma_0$ with $\bar{\sigma} := (2/\gamma_0)\varepsilon \ln N \geqslant \sigma$. This implies that

$$\gamma - C\varepsilon \ln N \leqslant \sqrt{p(x)} \leqslant \gamma \qquad \text{for} \quad x \in [0, \sigma]. \tag{5.7}$$

Note that the parameter σ is essentially the transition point between the fine and coarse meshes in Shishkin meshes; see, e.g., Miller *et al.* (1996).

(iii). Applying the decomposition of the computed solution used in the argument of Lemma 4.2, to the kth computed solution, we get $u_i^{(k)} = w_i^{(k)} + v_i^{(k)}$. Here the smooth component $w_i^{(k)}$ satisfies $L^{(k)} w_i^{(k)} = f(x_i)$ on the mesh $\{x_i^{(k)}\}$ subject to $w_0^{(k)} = u_0(0)$ and $w_N^{(k)} = u_0(1)$, while the singular component $v_i^{(k)}$ satisfies (4.4) with $p_i = p(x_i)$ and, by (5.3), we have $v_N^{(k)} = 0$. So $v_i^{(k)} = -u_0(0) z_i^{(k)}$ and $u_i^{(k)} = w_i^{(k)} - u_0(0) z_i^{(k)}$, where $\{z_i^{(k)}\}$ is the solution to the problem

$$L^{(k)} z_i^{(k)} = 0 \quad \text{on} \quad \{x_i^{(k)}\}_{i=1}^{N-1}, \qquad z_0^{(k)} = 1, \quad z_N^{(k)} = 0. \tag{5.8}$$

Note that we write $L^{(k)}$ for the difference operator L^N on the kth mesh. Finally, we observe that since the discrete operator $L^{(k)}$ satisfies the maximum principle, for each k the discrete function $z_i^{(k)}$ is decreasing in i.

(iv). For $i = 1, \ldots, N$ and $k = 0, 1, \ldots$, set

$$\lambda_i^{(k)} = \lambda_i^{(k)}(N, i, k) = \frac{C_1}{C_3 \sqrt{z_i^{(k)}} N}\ , \tag{5.9}$$

where C_1 was defined in Lemma 4.2 and C_3 is defined below in Lemma 5.3. Many subsequent inequalities involve $\lambda_i^{(k)}$. It may help the reader to note that when $\lambda_i^{(k)}$ appears in our analysis, it usually turns out that $\lambda_i^{(k)} \leqslant C$ for some C.

Now we present an auxiliary a priori error estimate, which will be used below in the proof of Lemma 5.14.

LEMMA 5.2 (A PRIORI ERROR ESTIMATE) Let u be a solution of (5.1) under assumptions (5.3), (5.4), and $\{u_i^N\}$ be a solution of (5.2) on an arbitrary mesh $\{x_i\}$ such that $h_i \leqslant CN^{-1}$ for some C. Furthermore, let $u^N(x)$ be the standard linear interpolant of $\{u_i^N\}$. Then

$$|u^N(x) - u(x)| \leqslant C \max_{1\leqslant i\leqslant N} \left[\min\left\{\frac{h_i}{\varepsilon}, 1\right\} e^{-(\gamma/2)x_{i-1}/\varepsilon} + N^{-1}\right]^2 \qquad \text{for} \quad x \in [0,1]. \tag{5.10}$$

Proof. First, note that for the solution u of (1.1) or its linear case (5.1) and the standard piecewise linear interpolant $u^N(x)$ of the discrete solution $\{u_i^N\}$ of (1.3) on an arbitrary mesh $\{x_i\}$, we have the following *a priori error estimate*:

$$\max_i \left|u_i^N - u(x_i)\right| \leqslant C\Big[\max_{1\leqslant i\leqslant N}\int_{x_{i-1}}^{x_i}\mathrm{d}s\int_{x_{i-1}}^{s}\varepsilon|u'''(x)|\mathrm{d}x + \max_{1\leqslant i\leqslant N-1}\int_{x_{i-1}}^{x_{i+1}}\mathrm{d}s\int_{x_{i-1}}^{s}\left|\tfrac{\mathrm{d}^2}{\mathrm{d}x^2}b(x,u(x))\right|\mathrm{d}x\Big]. \quad (5.11)$$

This is a version of (Kopteva, 2007, Lemma 3.2) and (Kopteva *et al.*, 2005, Theorem 3.1).

Now, imitating the argument of (Kopteva *et al.*, 2005, Corollary 3.1), in particular, representing u as in (Kopteva *et al.*, 2005, (3.6), (3.8)), and then invoking $f(1) = 0$, which follows from (5.3), we get

$$\left|\frac{d^m u}{dx^m}\right| \leqslant C\big(\varepsilon^{-m}e^{-\sqrt{p(0)}x/\varepsilon} + \varepsilon^{2-m} + 1\big), \qquad m = 0,\ldots,3.$$

Next, combining this with (5.11) and a standard linear-interpolation error bound, yields (5.10) with γ replaced by $\sqrt{p(0)}$. Hence, to establish (5.10), it suffices to show that

$$e^{-\sqrt{p(0)}x/\varepsilon} \leqslant 2[e^{-\gamma x/\varepsilon} + N^{-2}]. \quad (5.12)$$

For $x \leqslant \sigma$, by (5.7), (5.4), we get $0 \leqslant [\gamma - \sqrt{p(0)}]x/\varepsilon \leqslant C\sigma^2/\varepsilon \leqslant CN^{-1}\ln^2 N$, which, for sufficiently large N, implies

$$e^{-\sqrt{p(0)}x/\varepsilon} = e^{-\gamma x/\varepsilon}e^{[\gamma-\sqrt{p(0)}]x/\varepsilon} \leqslant 2e^{-\gamma x/\varepsilon}, \qquad x \leqslant \sigma, \quad (5.13)$$

and therefore (5.12). Furthermore, for $x > \sigma$, combining $e^{-\sqrt{p(0)}x/\varepsilon} \leqslant e^{-\sqrt{p(0)}\sigma/\varepsilon}$ with (5.13) and (5.6), again yields (5.12). Thus we established (5.12) for all $x \in [0,1]$. □

5.2 *Lower bounds on the discrete monitor function*

LEMMA 5.3 For $i = 1,2,\ldots,N$ and all k, there exist positive constants C_2 and C_3 such that

$$\sqrt{z_i^{(k)}} \geqslant C_2\varepsilon \quad \text{for some } i \qquad \Rightarrow \qquad \min_{j=1,\ldots,i} M_j^{(k)} \geqslant \frac{C_3\sqrt{z_i^{(k)}}}{\varepsilon}.$$

Proof. Since $u_i^{(k)} = w_i^{(k)} - u_0(0)z_i^{(k)}$ (see §5.1(iii) for details), combining (2.2) with (4.3), we obtain

$$M_i^{(k)} \geqslant |u_0(0)|^{1/2}\min\big\{|\delta^2 z_{i-1}^{(k)}|, |\delta^2 z_i^{(k)}|\big\}^{1/2} - \max_i|\delta^2 w_i^N|^{1/2} + 1 \geqslant |u_0(0)|^{1/2}\frac{\gamma_0}{\varepsilon}\sqrt{z_i^{(k)}} - C'$$

(see (4.5) for a similar upper bound on $M_i^{(k)}$). Here we also used $\varepsilon^2\delta z_i^{(k)} \geqslant \gamma_0^2 z_i^{(k)} \geqslant 0$, which follows from (5.8), where $z_i^{(k)}$ is decreasing in i. Now for $j \leqslant i$ we have

$$M_j^{(k)} \geqslant \big(\gamma_0|u_0(0)|^{1/2} - \frac{C'}{C_2}\big)\frac{\sqrt{z_i^{(k)}}}{\varepsilon}.$$

Choosing C_2 sufficiently large, we get the assertion of the lemma. □

The above lemma shows that, to get sharp lower bounds on our discrete monitor function $M_i^{(k)}$, we need sharp lower bounds on $z_i^{(k)}$. The next three lemmas provide such bounds in various situations.

LEMMA 5.4 Let the sub-mesh $\{x_i^{(k)}\}_{i=0}^{N'}$ be uniform for some integer $N'>1$. Then

$$z_i^{(k)} \geqslant \mathscr{B}_i - \mathscr{B}_{N'} \quad \text{for} \quad i=0,\ldots,N', \qquad \text{where} \qquad \mathscr{B}_i := \big(1+\bar{\gamma}h_1^{(k)}/\varepsilon\big)^{-2i}.$$

Proof. Set $H := h_1^{(k)}/\varepsilon$. Now imitating the argument of Lemma 4.1, for $i=1,\ldots,N'-1$ we get

$$L^{(k)}\mathscr{B}_i \leqslant -\frac{\bar{\gamma}^2}{H}\big[H\,\vartheta(\bar{\gamma}H)+H\big]\mathscr{B}_i + \bar{\gamma}^2\mathscr{B}_i \leqslant [\bar{\gamma}^2-\bar{\gamma}^2]\mathscr{B}_i = 0$$

(where $\vartheta(t)=(3+2t)/(1+t)^2>0$ for all $t\geqslant 0$), which implies $L^{(k)}[\mathscr{B}_i-\mathscr{B}_{N'}]\leqslant 0$. Furthermore, $\mathscr{B}_i-\mathscr{B}_{N'}\leqslant 1$ for $i=0$ and $\mathscr{B}_i-\mathscr{B}_{N'}=0$ for $i=N'$. Since $L^{(k)}z_i^{(k)}=0$, while $z_0^{(k)}=1$ and $z_{N'}^{(k)}>0$, applying the discrete maximum/comparison principle, we get the assertion of the lemma. □

LEMMA 5.5 Let the sub-mesh $\{x_i^{(k)}\}_{i=0}^{N'}$ be uniform with $x_{N'}^{(k)}\geqslant\sigma-C\varepsilon$. Then for $i=0,\ldots,N$ we have $z_i^{(k)}\geqslant e^{-\gamma x_i^{(k)}/\varepsilon}-CN^{-2}$.

Proof. Set $H := h_1^{(k)}/\varepsilon$. Then, by (5.7), for $i<N'$ we get

$$L^{(k)}e^{-\gamma x_i^{(k)}/\varepsilon} \leqslant \big[-\frac{e^{\gamma H}-1}{H^2}+\frac{1-e^{-\gamma H}}{H^2}+\gamma^2\big]e^{-\gamma x_i^{(k)}/\varepsilon} = \big[-\big(\eta(\gamma H/2)\big)^2+1\big]\gamma^2 e^{-\gamma x_i/\varepsilon}\leqslant 0,$$

where $\eta(t):=(\sinh t)/t\geqslant 1$ for all t. Next consider $i\geqslant N'$ and note that $\varepsilon^2\delta^2 e^{-\gamma x_i^{(k)}/\varepsilon}=\gamma^2 e^{-\gamma\xi_i/\varepsilon}>0$ for some $\xi_i\in[x_{i-1}^{(k)},x_{i+1}^{(k)}]$. Combining this with (5.6), we obtain $L^{(k)}e^{-\gamma x_i^{(k)}/\varepsilon}\leqslant\bar{\gamma}^2 e^{-\gamma x_i^{(k)}/\varepsilon}\leqslant CN^{-2}$ for $i\geqslant N'$. Thus there exists C such that $L^{(k)}[e^{-\gamma x_i^{(k)}/\varepsilon}-CN^{-2}]\leqslant 0$ for $i=1,\ldots,N-1$. Now, applying the discrete maximum/comparison principle yields the desired bound. □

LEMMA 5.6 Suppose that for every i such that $x_{i-1}^{(k)}\leqslant\sigma-C'\varepsilon$ we have $h_i^{(k)}\leqslant C''\varepsilon N^{-1/2}$ for some constants C' and C''. Then $z_i^{(k)}\geqslant e^{-\gamma x_i^{(k)}/\varepsilon}/2-CN^{-2}$ for $i=0,\ldots,N$.

Proof. Set $\rho := 1+\bar{C}N^{-1/2}$ for some $\bar{C}$ that will be chosen below, and $B(x):=e^{-\rho\gamma x/\varepsilon}$. Then

$$B(x)\geqslant e^{-\gamma x/\varepsilon}\exp(-\bar{C}N^{-1/2}\gamma\sigma/\varepsilon)\geqslant e^{-\gamma x/\varepsilon}/2 \qquad \text{for} \quad x\leqslant\sigma, \tag{5.14}$$

since, by (5.6), (5.4), for sufficiently large N we have $\bar{C}N^{-1/2}\gamma\sigma/\varepsilon\leqslant CN^{-1/2}\ln N\leqslant\ln 2$. Next, note that $\varepsilon^2\delta^2 B(x_i^{(k)})=\varepsilon^2 B''(\xi_i)$ for some $\xi_i\in[x_{i-1}^{(k)},x_{i+1}^{(k)}]$ implies $\varepsilon^2\delta^2 B(x_i^{(k)})\geqslant\rho^2\gamma^2 e^{-\rho\gamma x_{i+1}^{(k)}/\varepsilon}$. Therefore, for $x_i^{(k)}\leqslant\sigma-C'\varepsilon$, recalling that $h_{i+1}^{(k)}\leqslant C''\varepsilon N^{-1/2}$, we get

$$L^{(k)}B(x_i^{(k)})\leqslant\big[\,1-\rho^2\,e^{-h_{i+1}^{(k)}\gamma/\varepsilon}\,\big]\,\gamma^2\,e^{-\rho\gamma x_i^{(k)}/\varepsilon}\leqslant 0,$$

provided that $\bar{C}$ is sufficiently large so that $\rho\geqslant\exp(C''\gamma N^{-1/2}/2)$. Furthermore, for $x_i^{(k)}>\sigma-C'\varepsilon$, by (5.6), we get $B(x_i^{(k)})\leqslant Ce^{-\rho\gamma\sigma/\varepsilon}\leqslant Ce^{-\gamma\sigma/\varepsilon}=CN^{-2}$, which implies $L^N B(x_i^{(k)})\leqslant\bar{\gamma}^2 B(x_i^{(k)})\leqslant CN^{-2}$; see the proof of Lemma 5.5 for a similar argument. Thus there exists C such that $L^{(k)}[B(x_i^{(k)})-CN^{-2}]\leqslant 0$ for $i=1,\ldots,N-1$. Now, applying the discrete comparison principle yields $z_i^{(k)}\geqslant B(x_i^{(k)})-CN^{-2}$ for $i=0,\ldots,N$. Combining this with (5.14) implies the assertion of the lemma for $x_i^{(k)}\leqslant\sigma$. For $x_i^{(k)}>\sigma$, one again gets the assertion of the lemma with some $C\geqslant 1/2$ since $z_i^{(k)}\geqslant 0$, while, by (5.6), we have $e^{-\gamma x_i^{(k)}/\varepsilon}/2-(1/2)N^{-2}<0$. □

5.3 *Subdivision of* $h_1^{(k)}$

The next lemma will show that if the first mesh interval is too coarse, then the next iteration of the algorithm will subdivide it $O(N)$ times.

LEMMA 5.7 Let C_4 be an arbitrary positive constant, $C_5 := 2(C_4^{-1}+\bar{\gamma})C_1/C_3$, and N' be the greatest integer such that $N' \leqslant N/C_5$. Suppose that for some k, the mesh $\{x_i^{(k)}\}$ generated by the algorithm satisfies the condition of Lemma 5.4. Then

$$h_1^{(k)} \geqslant C_4\varepsilon \qquad \Rightarrow \qquad h_1^{(k+1)} \leqslant C_5 h_1^{(k)}/N.$$

Furthermore, the mesh $\{x_i^{(k+1)}\}$ also satisfies the condition of Lemma 5.4 with the same N'.

Proof. By Lemma 5.4, setting $t := (1+\bar{\gamma}h_1^{(k)}/\varepsilon)^{-2}$, we get

$$z_1^{(k)} \geqslant t - t^{N'} = t(1-t^{N'-1}) \geqslant t/4,$$

where $1-t^{N'-1} \geqslant 1/4$ follows for sufficiently large N from $t \leqslant (1+\bar{\gamma}C_4)^{-2}$. Thus we have

$$\sqrt{z_1^{(k)}} \geqslant \frac{1}{2(1+\bar{\gamma}h_1^{(k)}/\varepsilon)} \geqslant \frac{\varepsilon}{2h_1^{(k)}(C_4^{-1}+\bar{\gamma})}. \tag{5.15}$$

By (4.6), this implies $\sqrt{z_1^{(k)}} \geqslant \varepsilon/[2C_1N^{-1}(C_4^{-1}+\bar{\gamma})] \geqslant C_2\varepsilon$ for sufficiently large N. Now, applying Lemma 5.3, we get $M_1^{(k)} \geqslant C_3\sqrt{z_1^{(k)}}/\varepsilon$. Combining this with (5.5), (5.15) and Lemma 4.2, yields

$$I^{(k)}[0,x_1^{(k)}] = M_1^{(k)}h_1^{(k)} \geqslant \frac{C_3\sqrt{z_1^{(k)}}}{\varepsilon}h_1^{(k)} \geqslant \frac{C_3}{2(C_4^{-1}+\bar{\gamma})} \geqslant \frac{C_3/C_1}{2(C_4^{-1}+\bar{\gamma})}I^{(k)} = I^{(k)}/C_5$$

and furthermore,

$$I^{(k)}[0,x_1^{(k)}] \geqslant I^{(k)}/N.$$

Hence, by the construction of the algorithm, we have $x_1^{(k+1)} < x_1^{(k)}$ and

$$I^{(k)}[0,x_1^{(k+1)}] = I^{(k)}/N, \qquad\qquad h_1^{(k+1)} = h_1^{(k)}\frac{I^{(k)}[0,x_1^{(k+1)}]}{I^{(k)}[0,x_1^{(k)}]}.$$

The second relation here follows from $I^{(k)}[0,x_1^{(k+1)}] = M_1^{(k)}h_1^{(k+1)}$ and $I^{(k)}[0,x_1^{(k)}] = M_1^{(k)}h_1^{(k)}$. Finally, a calculation yields the desired bound for $h_1^{(k+1)}$:

$$h_1^{(k+1)} = h_1^{(k)}\frac{I^{(k)}/N}{I^{(k)}[0,x_1^{(k)}]} \leqslant h_1^{(k)}(C_5/N).$$

Similarly, combining $I^{(k)}[0,x_1^{(k)}] \geqslant I^{(k)}/C_5$ with $I^{(k)}[x_{j-1}^{(k+1)},x_j^{(k+1)}] = I^{(k)}/N$, for $j=1,\dots,N'$ we get $x_j^{(k+1)} \in [0,x_1^{(k)}]$ and $h_j^{(k+1)} = h_1^{(k+1)}$, i.e. the sub-mesh $\{x_j^{(k+1)}\}_{j=0}^{N'}$ is uniform and thus the next mesh $\{x_j^{(k+1)}\}_{j=0}^{N}$ generated by the algorithm also satisfies the condition of Lemma 5.4 with the same value of N'. □

REMARK 5.8 To be more precise, under the conditions of Lemma 5.7, there exists an integer $N'' \geqslant N'$ such that

$$h_j^{(k+1)} = h_1^{(k+1)}, \qquad j = 1, \ldots, N''; \qquad\qquad x_{N''}^{(k+1)} > h_1^{(k)} - h_1^{(k+1)}.$$

COROLLARY 5.9 Let C_4 be an arbitrary positive constant. Then there exists a non-negative integer $K' = K'(\varepsilon, N, C_4)$ such that we have

$$h_1^{(k)} \geqslant C_4\varepsilon \quad \text{and} \quad h_1^{(k+1)} \leqslant C_5 h_1^{(k)}/N \quad \text{for} \quad k = 0, 1, \ldots, K'-1; \qquad h_1^{(K')} < C_4\varepsilon,$$

(where C_5 is defined in Lemma 5.7). Furthermore,

$$K' \leqslant \frac{-\ln(C_4C_5\varepsilon)}{\ln(N/C_5)} \leqslant C\frac{|\ln\varepsilon|}{\ln N}.$$

Proof. Note that for $k = 0$ the mesh $\{x_i^{(k)}\}$ is uniform and hence satisfies the condition of Lemma 5.4 with N' from Lemma 5.7. Now imitate the proof of (Kopteva & Stynes, 2001, Corollary 5.3). □

5.4 *Properties of mesh intervals generated by the algorithm*

This subsection contains two crucial corollaries that shed light on the lengths of certain mesh intervals generated inside the boundary layer. It may help the reader if we point out that a subscript i usually corresponds to the mesh $\{x_i^{(k)}\}$, while a subscript j usually corresponds to the mesh $\{x_j^{(k+1)}\}$.

LEMMA 5.10 Suppose that for some i and k we have $\sqrt{z_i^{(k)}} \geqslant C_2\varepsilon$.
(i) For any $[t_1, t_2] \subseteq [0, x_i^{(k)}]$ such that $I^{(k)}[t_1, t_2] = I^{(k)}/N$, we have

$$|t_2 - t_1| \leqslant \varepsilon\lambda_i^{(k)}. \tag{5.16}$$

(ii) If

$$x_i^{(k)} \geqslant \varepsilon\lambda_i^{(k)}, \tag{5.17}$$

then

$$I^{(k)}[0, x_i^{(k)}] \geqslant I^{(k)}/N. \tag{5.18}$$

Proof. First note that, by (5.5) and Lemma 5.3, for $[t_1, t_2] \subseteq [0, x_i^{(k)}]$ we get

$$I^{(k)}[t_1, t_2] = \int_{t_1}^{t_2} M^{(k)}(x)\,\mathrm{d}x \geqslant |t_2 - t_1|\frac{C_3\sqrt{z_i^{(k)}}}{\varepsilon}. \tag{5.19}$$

(i) Suppose that $[t_1, t_2] \subseteq [0, x_i^{(k)}]$ and $I^{(k)}[t_1, t_2] = I^{(k)}/N$. Then, combining inequality (5.19), Lemma 4.2 and (5.9), immediately yields

$$|t_2 - t_1| \leqslant \frac{\varepsilon I^{(k)}[t_1, t_2]}{C_3\sqrt{z_i^{(k)}}} = \frac{\varepsilon I^{(k)}}{C_3\sqrt{z_i^{(k)}}N} \leqslant \varepsilon\lambda_i^{(k)}.$$

(ii) Now suppose instead that (5.17) holds true. Setting $[t_1,t_2]=[0,x_i^{(k)}]$ in (5.19) and then using (5.17), (5.9) and Lemma 4.2, we obtain

$$I^{(k)}[0,x_i^{(k)}] \geqslant x_i^{(k)} \frac{C_3\sqrt{z_i^{(k)}}}{\varepsilon} \geqslant C_3\lambda_i^{(k)}\sqrt{z_i^{(k)}} = C_1/N \geqslant I^{(k)}/N.$$

□

This lemma has the following two corollaries.

COROLLARY 5.11 Suppose that for some i and k, the inequalities $\sqrt{z_i^{(k)}} \geqslant C_2\varepsilon$ and $x_i^{(k)} \geqslant \varepsilon\lambda_i^{(k)}$ are satisfied. On the next mesh $\{x_j^{(k+1)}\}$ generated by the algorithm,

$$x_{j-1}^{(k+1)} \leqslant x_i^{(k)} - \varepsilon\lambda_i^{(k)} \quad \text{for some} \quad j \geqslant 1 \qquad \Rightarrow \qquad h_j^{(k+1)} \leqslant \varepsilon\lambda_i^{(k)}.$$

Proof. We imitate the proof of Corollary 5.1 in Kopteva & Stynes (2001). First, (5.18) implies that there is a unique point $x^* \in [0,x_i^{(k)})$ such that $I^{(k)}[x^*,x_i^{(k)}] = I^{(k)}/N$. Hence (5.16) yields $x_i^{(k)} - x^* \leqslant \varepsilon\lambda_i^{(k)}$. But then $x_{j-1}^{(k+1)} \leqslant x_i^{(k)} - \varepsilon\lambda_i^{(k)}$ implies $x_{j-1}^{(k+1)} \leqslant x^*$, so $I^{(k)}[x_{j-1}^{(k+1)},\, x_i^{(k)}] \geqslant I^{(k)}/N$. By the construction of the algorithm, $I^{(k)}[x_{j-1}^{(k+1)},\, x_j^{(k+1)}] = I^{(k)}/N$. Consequently $x_j^{(k+1)} \leqslant x_i^{(k)}$. Now take $[t_1,t_2] = [x_{j-1}^{(k+1)},\, x_j^{(k+1)}]$ in (5.16) to get $h_j^{(k+1)} = x_j^{(k+1)} - x_{j-1}^{(k+1)} \leqslant \varepsilon\lambda_i^{(k)}$. □

COROLLARY 5.12 Suppose that for some i and k, the inequality $\sqrt{z_i^{(k)}} \geqslant C_2\varepsilon$ is satisfied. On the next mesh $\{x_j^{(k+1)}\}$ generated by the algorithm,

$$x_j^{(k+1)} \leqslant x_i^{(k)} \quad \text{for some} \quad j \geqslant 1 \qquad \Rightarrow \qquad h_j^{(k+1)} \leqslant \varepsilon\lambda_i^{(k)}.$$

Proof. We imitate the proof of Corollary 5.2 in Kopteva & Stynes (2001). By the construction of the algorithm, $I^{(k)}[x_{j-1}^{(k+1)},\, x_j^{(k+1)}] = I^{(k)}/N$, while $x_j^{(k+1)} \leqslant x_i^{(k)}$ implies $[x_{j-1}^{(k+1)},\, x_j^{(k+1)}] \subseteq [0,x_i^{(k)}]$. Thus we can take $[t_1,t_2] = [x_{j-1}^{(k+1)},\, x_j^{(k+1)}]$ in (5.16) to get $h_j^{(k+1)} = x_j^{(k+1)} - x_{j-1}^{(k+1)} \leqslant \varepsilon\lambda_i^{(k)}$. □

5.5 *A sufficient condition for second-order accuracy*

CONDITION 5.13 For every i such that $x_{i-1}^{(k)} \leqslant \sigma - C_6\varepsilon$, let

$$h_i^{(k)} \leqslant C_7\varepsilon. \tag{5.20}$$

LEMMA 5.14 (SUFFICIENT CONDITION FOR ACCURACY) Suppose that for some $k \geqslant 0$ the mesh $\{x_i^{(k)}\}$ satisfies Condition 5.13, where C_6 and C_7 are arbitrary constants such that

$$C_6 \geqslant C_7 \geqslant C_1/(C_2C_3), \qquad e^{\gamma(C_6-C_7)} \geqslant 2(C_2^2 + C_8), \tag{5.21}$$

for some constant C_8. Furthermore,let

$$z_i^{(k)} \geqslant e^{-\gamma x_i^{(k)}/\varepsilon}/2 - C_8N^{-2}, \qquad i = 0,\dots,N. \tag{5.22}$$

(i) Then the mesh $\{x_j^{(k+1)}\}$ also satisfies Condition 5.13 with C_6 replaced by C_6+C_7.
(ii) Furthermore, there exists C such that

$$|e^{(k+1)}(x)| = |u^{(k+1)}(x) - u(x)| \leqslant CN^{-2}, \qquad x \in [0,1], \tag{5.23}$$

where $e^{(k+1)}(x) := u^{(k+1)}(x) - u(x)$ is the error in the computed solution on the $(k+1)$st mesh.
(iii) In fact, for any positive integer m, the meshes $\{x_i^{(k+m)}\}$ also satisfy Condition 5.13 with C_6 replaced by $C_6'(m) \geqslant C_6 + mC_7$, while the discrete functions $\{z_i^{(k+m)}\}$ satisfy (5.22) with C_8 replaced by some $C_8'(m)$, and we have $|e^{(k+m)}(x)| \leqslant CN^{-2}$ for $x \in [0,1]$ and $m = 1,2,\ldots$, where $C = C(m)$.

Proof. (i) Let n be the largest value of i for which $x_{i-1}^{(k)} \leqslant \sigma - C_6\varepsilon$, so, by (5.20), we have

$$x_{n-1}^{(k)} \leqslant \sigma - C_6\varepsilon < x_n^{(k)} \leqslant \sigma - (C_6 - C_7)\varepsilon. \tag{5.24}$$

To apply Corollary 5.11 with $i = n$, we check its hypotheses. By (5.22), (5.24) and (5.6), one has

$$\begin{aligned} z_n^{(k)} &\geqslant \exp(-\gamma x_n^{(k)}/\varepsilon)/2 - C_8N^{-2} \geqslant \exp(-\gamma[\sigma - (C_6 - C_7)\varepsilon]/\varepsilon)/2 - C_8N^{-2} \\ &\geqslant [e^{\gamma(C_6-C_7)}/2 - C_8]N^{-2}. \end{aligned} \tag{5.25}$$

Thus, by (5.21) and (5.4), we get

$$\sqrt{z_n^{(k)}} \geqslant C_2N^{-1} \geqslant C_2\varepsilon. \tag{5.26}$$

Now we verify the next hypothesis $x_n^{(k)} \geqslant \varepsilon\lambda_n^{(k)}$ of Corollary 5.11. By (5.24), (5.6), we get

$$x_n^{(k)} > \sigma - C_6\varepsilon = \varepsilon[(2/\gamma)\ln N - C_6] \geqslant C_7\varepsilon$$

for N sufficiently large. But, by (5.9), (5.26) and (5.21), we have

$$\varepsilon\lambda_n^{(k)} = \frac{C_1\varepsilon}{C_3\sqrt{z_n^{(k)}}N} \leqslant \frac{C_1\varepsilon}{C_2C_3} \leqslant C_7\varepsilon. \tag{5.27}$$

Thus the hypotheses of Corollary 5.11 with $i = n$ are satisfied, while (5.27) implies $\lambda_n^{(k)} \leqslant C_7$. Now Condition 5.13 for the mesh $\{x_j^{(k+1)}\}$ with C_6 replaced by $C_6 + C_7$ follows from Corollary 5.11. Indeed,

$$x_{j-1}^{(k+1)} \leqslant \sigma - (C_6 + C_7)\varepsilon = [\sigma - C_6\varepsilon] - C_7\varepsilon$$

implies, by (5.24) and (5.27), that

$$x_{j-1}^{(k+1)} \leqslant x_n^{(k)} - \varepsilon\lambda_n^{(k)}.$$

Hence, by Corollary 5.11, we have

$$h_j^{(k+1)} \leqslant C_7\varepsilon. \tag{5.28}$$

(ii) Consider two cases.
Case A. $x_{j-1}^{(k+1)} \leqslant \sigma - (C_6 + C_7)\varepsilon$. Then (5.28) implies

$$x_j^{(k+1)} \leqslant \sigma - C_6\varepsilon.$$

Let i be such that $x_j^{(k+1)} \in [x_{i-1}^{(k)}, x_i^{(k)}]$. Then $x_{i-1}^{(k)} \leqslant x_j^{(k+1)} \leqslant \sigma - C_6\varepsilon$, so $i \leqslant n$ by definition of n, while, by (5.20), we also have

$$h_i^{(k)} \leqslant C_7\varepsilon. \tag{5.29}$$

Since $i \leqslant n$ and $z_i^{(k)} \geqslant z_n^{(k)}$, then, by (5.26), we can apply Corollary 5.12 to obtain

$$h_j^{(k+1)} \leqslant \varepsilon\lambda_i^{(k)} = \frac{C_1\varepsilon}{C_3\sqrt{z_i^{(k)}}N},$$

which immediately yields

$$\frac{h_j^{(k+1)}}{\varepsilon}\exp\Big(-\frac{\gamma x_{j-1}^{(k+1)}}{2\varepsilon}\Big) \leqslant \frac{C_1}{C_3\sqrt{z_i^{(k)}}N}\exp\Big(-\frac{\gamma x_{j-1}^{(k+1)}}{2\varepsilon}\Big).$$

Examining (5.25) and (5.26), we note that $\exp(-\gamma x_n^{(k)}/\varepsilon)/2 - C_8N^{-2} \geqslant C_2^2N^{-2}$, which implies

$$(C_2^2 + C_8)N^{-2} \leqslant \exp(-\gamma x_n^{(k)}/\varepsilon)/2 \leqslant \exp(-\gamma x_i^{(k)}/\varepsilon)/2.$$

Combining this with (5.22), we get

$$z_i^{(k)} \geqslant \exp(-\gamma x_i/\varepsilon)/2 - C_8N^{-2} \geqslant \frac{\exp(-\gamma x_i/\varepsilon)}{2(1+C_8/C_2^2)}$$

and hence

$$\frac{h_j^{(k+1)}}{\varepsilon}\exp\Big(-\frac{\gamma x_{j-1}^{(k+1)}}{2\varepsilon}\Big) \leqslant \frac{C_1}{C_3N}\sqrt{2(1+C_8/C_2^2)}\,\exp\Big(\gamma\frac{x_i^{(k)} - x_{j-1}^{(k+1)}}{2\varepsilon}\Big). \tag{5.30}$$

Now (5.28) together with (5.29) implies that $x_i^{(k)} - x_{j-1}^{(k+1)} \leqslant 2C_7\varepsilon$. From (5.30) we therefore get

$$\frac{h_j^{(k+1)}}{\varepsilon}\exp\Big(-\gamma\frac{x_{j-1}^{(k+1)}}{2\varepsilon}\Big) \leqslant CN^{-1}. \tag{5.31}$$

Case B. $x_{j-1}^{(k+1)} > \sigma - (C_6 + C_7)\varepsilon$. By (5.6), we have

$$\begin{aligned}\exp\Big(-\gamma\frac{x_{j-1}^{(k+1)}}{2\varepsilon}\Big) &= N^{-1}\exp\Big(\gamma\frac{\sigma - x_{j-1}^{(k+1)}}{2\varepsilon}\Big) \leqslant N^{-1}e^{\gamma(C_6+C_7)}\\ &\leqslant CN^{-1}.\end{aligned} \tag{5.32}$$

Combining bounds (5.31) and (5.32) for cases A and B, for $j = 1, \ldots, N$ we get

$$\min\Big\{\frac{h_j^{(k+1)}}{\varepsilon},\, 1\Big\}\exp\Big(-\gamma\frac{x_{j-1}^{(k+1)}}{2\varepsilon}\Big) \leqslant CN^{-1}. \tag{5.33}$$

Now the desired estimate (5.23) follows from Lemma 5.2 combined with Corollary 4.3.

(iii) Note that (5.33) also implies an analogue of the error estimate (5.23) for the solution $\{z_j^{(k+1)}\}$ of problem (5.8) with k replaced by $k+1$, and the solution $z(x)$ of the equation $Lz = 0$ subject to the boundary conditions $z(0)=1$ and $z(1)=0$; thus we have $|z_j^{(k+1)} - z(x_j^{(k+1)})| \leqslant CN^{-2}$. This yields an analogue of (5.22) on the mesh $\{x_j^{(k+1)}\}$:

$$z_j^{(k+1)} \geqslant z(x_j^{(k+1)}) - CN^{-2} \geqslant e^{-\gamma x_j^{(k+1)}/\varepsilon}/2 - C_8' N^{-2}, \qquad j = 0, \ldots, N,$$

for some constant C_8'. Here we used the bound $z(x) \geqslant e^{-\gamma x/\varepsilon} - CN^{-2}$, which is obtained imitating the argument of Lemma 5.5, but for the continuous operator L.

Thus the hypotheses of the lemma (with C_6 and C_8 replaced by $C_6' \geqslant C_6 + C_7$ and C_8', respectively, such that (5.21) holds true) apply on mesh $k+1$, so by the same argument we obtain analogues of (i), (ii) and (5.22) on mesh $k+2$. Continue repeating this argument a fixed number of iterations. □

5.6 *Proof of Theorem 5.1*

To simplify the notation and presentation, it is convenient to prove the theorem with K replaced by $K+2$. From Corollary 5.9 and (5.6), there exists $K \leqslant K'$ such that

$$h_1^{(K-1)} \geqslant \sigma, \qquad h_1^{(K)} < \sigma, \qquad h_1^{(K)} \leqslant C_5 h_1^{(K-1)}/N.$$

Furthermore, by Remark 5.8 for Lemma 5.7, for some $N'' > N/C_5 - 1$ we have

$$h_i^{(K)} = h_1^{(K)}, \qquad i = 1, \ldots, N''; \qquad x_{N''}^{(K)} > h_1^{(K-1)} - h_1^{(K)}. \tag{5.34}$$

Let C_9 satisfy $\ln(1+\bar{\gamma}C_9)/C_9 = \gamma/8$; there exists such C_9 since the function $\ln(1+t)/t$ equals 1 at $t=0$ and is decaying to zero for $t>0$, while $(\gamma/8)/\bar{\gamma} < 1$. We now consider two cases.

Case A. $h_1^{(K)} \leqslant C_9\varepsilon$. Combining this with $h_1^{(K-1)} \geqslant \sigma$ and (5.34) implies $x_{N''}^{(K)} \geqslant \sigma - C_9\varepsilon$. Then, invoking Lemma 5.5, we get (5.22) for $k=K$ and some constant C_8. Furthermore, by (5.34), the mesh $\{x_i^{(K)}\}$ satisfies Condition 5.13 with arbitrary C_6 and C_7 such that $C_6 \geqslant 2C_9$ and $C_7 \geqslant C_9$. Now choosing C_6 and C_7 that satisfy (5.21), and then invoking Lemma 5.14 yields the assertion of Theorem 5.1.

Case B. $h_1^{(K)} > C_9\varepsilon$. First note that, by (5.6), the bound $h_1^{(K)} \leqslant C_5 h_1^{(K-1)}/N$ implies $h_1^{(K-1)} > (C_9/C_5)\varepsilon N > 2\sigma$. Combining this with $h_1^{(K)} < \sigma$ and (5.34), we observe that $x_{N''}^{(K)} > \sigma$. Now let $n \leqslant N''$ be such that $\sigma \in [x_{n-1}^{(K)}, x_n^{(K)}]$. Then, by (5.6), we have

$$n \leqslant \frac{\sigma}{h_1^{(K)}} + 1 \leqslant \frac{2\sigma}{h_1^{(K)}} \leqslant \frac{4\ln N}{\gamma H}, \qquad \text{where} \quad H := h_1^{(K)}/\varepsilon > C_9.$$

Next, Lemma 5.4 yields $z_n^{(K)} \geqslant t^{-n} - t^{-N''}$ with $t := (1+\bar{\gamma}H)^{-2}$, where $n \leqslant C\ln N$, while $N'' > N/C_5 - 1$. Then, imitating the argument used to obtain (5.15), for sufficiently large N we get

$$\begin{aligned} \sqrt{z_n^{(K)}} &\geqslant (1+\bar{\gamma}H)^{-n}/2 = \exp\bigl(-n\ln(1+\bar{\gamma}H)\bigr)/2 \geqslant \exp\Bigl(-\frac{4\ln N}{\gamma H}\ln(1+\bar{\gamma}H)\Bigr)/2 \\ &\geqslant N^{-1/2}/2, \end{aligned} \tag{5.35}$$

where we used $\ln(1+\bar{\gamma}H)/H \leqslant \ln(1+\bar{\gamma}C_9)/C_9 = \gamma/8$. Now, for sufficiently large N, combining (5.35) with (5.4) yields $\sqrt{z_n^{(K)}} \geqslant C_2 N^{-1} \geqslant C_2\varepsilon$ and then, by (5.9),

$$\lambda_n^{(K)} = \frac{C_1}{C_3\sqrt{z_n^{(K)}}N} \leqslant \frac{2C_1}{C_3}N^{-1/2}.$$

Hence, by Corollary 5.11 with $i = n$, on the next mesh $\{x_j^{(K+1)}\}$ generated by the algorithm we have

$$x_{j-1}^{(K+1)} \leqslant \sigma - \varepsilon \frac{2C_1}{C_3} N^{-1/2} \quad \text{for some} \quad j \geqslant 1 \qquad \Rightarrow \qquad h_j^{(K+1)} \leqslant \varepsilon \frac{2C_1}{C_3} N^{-1/2}.$$

Therefore the mesh $\{x_j^{(K+1)}\}$ satisfies Condition 5.13 for arbitrary C_6 and C_7; then choose C_6 and C_7 that satisfy hypothesis (5.21) of Lemma 5.14. Furthermore, by Lemma 5.6, the next hypothesis (5.22) is also satisfied for some C_8. Now invoking Lemma 5.14 yields the assertion of Theorem 5.1 in Case B. □

6. Numerical results

First, we refer the reader to Kopteva *et al.* (2005), where extensive numerical results were presented for the algorithm of §2 that used a similar monitor function and was applied to both linear and nonlinear test problems. Mostly for completeness we shall give further numerical results here for one simple problem.

Our test problem is the linear equation from (5.1) with the boundary conditions $u(0) = 2, u(1) = -1$, in which $p(x) = 4(1+x)^{-4}[1+\varepsilon(1+x)]$, and whose exact solution

$$u(x) = -\cos(2\pi t) + 3(e^{-t/\varepsilon} - e^{-1/\varepsilon})[1 - e^{-1/\varepsilon}]^{-1}, \qquad t := 2x/(x+1),$$

exhibits a boundary layer near $x = 0$; the right-hand side f and the boundary conditions are chosen so that (5.1) is satisfied; see Schatz & Wahlbin (1983), O'Riordan & Stynes (1986), Kopteva *et al.* (2005), Kopteva (2007).

Applying the algorithm to this test problem, instead of the monitor function M_i^N, its scaled version $M_{\beta,i}^N$ from (2.6) was used with $\beta := \max_i |u_i^N|^{1/2} = \sqrt{2}$; see Remark 2.4. (Similarly, we used the scaled version $\bar{M}_{\beta,i}^N$ of $\bar{M}_i^N$, wherever appropriate, in our computations.)

Figure 1, which should be read from bottom to top, shows the mesh after each iteration. Each of these meshes is labeled with the value of C_0, for which the stopping criterion (2.3) becomes an equation; one can deduce how many iterations of the algorithm would have taken to converge given a priori a value of C_0. We shall take $C_0 = 2$ in the computations presented in Table 1. One can see from Figure 1 that

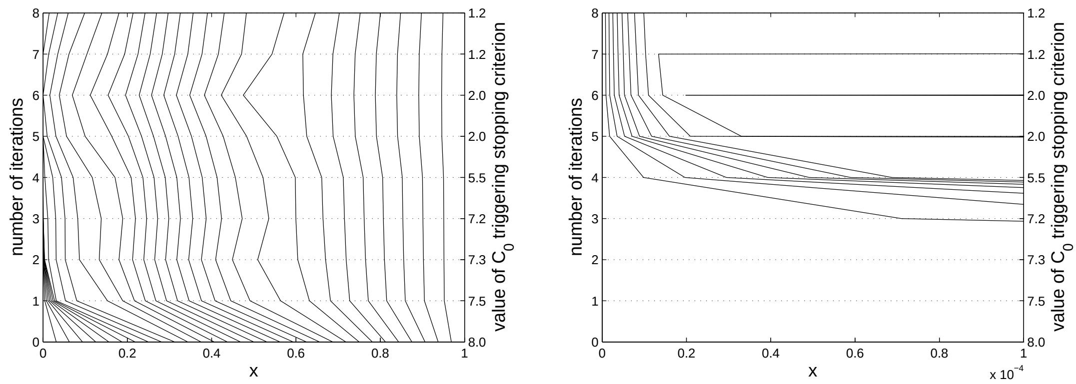

FIG. 1. Mesh movement (right-hand picture enlarges boundary layer); $\varepsilon = 10^{-5}$, $N = 32$.

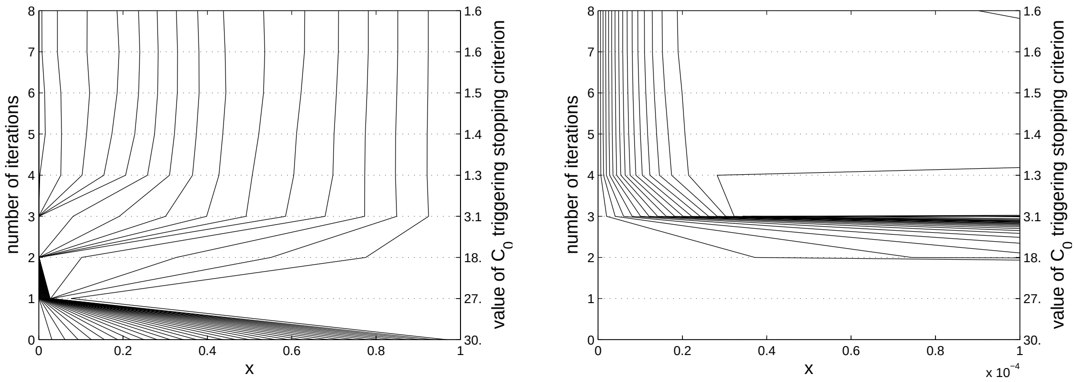

FIG. 2. Mesh movement when $M_i^{(k)}$ in (2.2) is replaced by $\bar{M}_i^N$ from (1.4); right-hand picture enlarges boundary layer; $\varepsilon = 10^{-5}$, $N = 32$. (Using $\bar{\bar{M}}_i^N$ from (1.5) instead of $M_i^{(k)}$ yields similar results.)

taking any smaller value of C_0 will increase the number of iterations without significantly changing the mesh. It should also be noted that the meshes computed by the algorithm are very close to a mesh by Bakhvalov (1969); see (Kopteva *et al.*, 2005, Figure 5.4) for more details.

Figure 2 plots the meshes generated by a modification of the algorithm, in which $M_i^{(k)}$ in (2.2) was replaced by the scaled version $\bar{M}_{\beta,i}^N$ of $\bar{M}_i^N$ from the a posteriori error estimate (1.4). We observe a temporary mesh starvation: at the first few iterations all the mesh nodes are drawn into the narrow layer region at $x = 0$; compare with Figure 1 and also see Remarks 2.3 and 4.5 for a further discussion of this undesirable phenomenon.

Numerical results for the original version of the algorithm, as described in §2, are presented in Table 1. For various values of ε and N, we computed the errors in the maximum norm, denoted $\|e\|_\infty$. Under each error is given the rate of convergence, expressed as a power of N and computed in the standard way. These rates approach the value 2 as N increases, confirming Theorem 5.1; the irregular

Table 1. Accuracy and speed of algorithm: errors in the maximum norm $\|e\|_\infty$, their rates of convergence (even rows), number of iterations K, its ratio to the predicted number of iterations $O(|\ln\varepsilon|/(\ln N))$, and C' to compare M_i^N and $\bar{M}_i^N$.

	$\varepsilon = 1$		$\varepsilon = 10^{-3}$			$\varepsilon = 10^{-8}$			$\max_{\varepsilon=10^{-m}, m=0,\dots,8} \|e\|_\infty$	$\max_{\varepsilon=10^{-m}, m=2,\dots,8} C'$
N	$\|e\|_\infty$	K	$\|e\|_\infty$	K	$\frac{K\ln N}{\|\ln\varepsilon\|}$	$\|e\|_\infty$	K	$\frac{K\ln N}{\|\ln\varepsilon\|}$		
32	1.44e-2	1	2.16e-2	3	1.51	2.83e-2	9	1.69	4.96e-2	1.36
	1.96		1.34			2.00			2.54	
64	3.71e-3	1	8.51e-3	3	1.81	7.08e-3	7	1.58	8.51e-3	1.28
	2.00		3.01			1.80			1.23	
128	9.29e-4	1	1.06e-3	2	1.40	2.03e-3	6	1.58	3.63e-3	1.28
	1.98		2.45			1.30			2.14	
256	2.35e-4	1	1.94e-4	2	1.61	8.25e-4	5	1.51	8.25e-4	1.31
	2.04		2.10			3.66			2.19	
512	5.74e-5	1	4.52e-5	2	1.81	6.54e-5	4	1.35	1.81e-4	1.26
	2.02		2.02			2.39			2.96	
1024	1.42e-5	1	1.11e-5	2	2.01	1.25e-5	4	1.51	2.33e-5	1.39

nature of the convergence is presumably due to the nonlinear nature of the algorithm. For each ε and N, the table also lists K, the number of iterations needed before the stopping criterion (2.3) is satisfied. We took $C_0 = 2$ in (2.3); values of C_0 close to 1 yielded slightly more accurate solutions but required many more iterations of the algorithm. The column $K(\ln N)/|\ln\varepsilon|$ measures the ratio between the actual number of iterations K and the predicted number of iterations $O(|\ln\varepsilon|/(\ln N))$. We observe that the algorithm requires very few iterations, even for small ε, and that our estimate $O(|\ln\varepsilon|/(\ln N))$ of the number of iterations is sharp.

Finally, note that our main results, Theorem 5.1, establishes that the algorithm achieves second-order accuracy after $O(|\ln\varepsilon|/\ln N)$ iterations. But, strictly speaking, we have not proved and therefore are not guaranteed that after the algorithm reaches its stopping criterion, our computed solution is already second-order accurate. One can still estimate the error in terms of values obtained in the computation process invoking our a posteriori error estimate (1.4). Recall that (1.4) involves the monitor function $\bar{M}_i^N$ different from M_i^N, which we used in the algorithm. However, comparing M_i^N and $\bar{M}_i^N$, intuitively one might expect that $M_i^N \approx \bar{M}_i^N$ on a suitable layer-adapted mesh, which is indeed confirmed by numerical experiments. In particular, this is illustrated by the last column of Table 1, which describes $C' = \frac{\max_i\{\bar{M}_{\beta,i}^N h_i\}}{\max_i\{M_{\beta,i}^N h_i\}}$ on the final mesh produced by the algorithm for $\varepsilon < 10^{-1}$ (for $\varepsilon = 1$ and $\varepsilon = 10^{-1}$ we observe $C' \leqslant 7.2$ and $C' \leqslant 2.9$, respectively). We see from Table 1 that the values of C' are quite close to 1.3; the influence of C' on the accuracy is clarified in the following remark.

REMARK 6.1 (ACCURACY OF SOLUTION WHEN THE ALGORITHM TERMINATES) After the algorithm reaches its stopping criterion, compute $C' = \frac{\max_i\{\bar{M}_i^* h_i^*\}}{\max_i\{M_i^* h_i^*\}}$ (where $\bar{M}_i^* := \bar{M}_i^N$ and $M_i^* := M_i^N$ on the final mesh $\{x_i^*\}$); then, by Corollary 4.4 combined with the a posteriori error bound (1.4), we get

$$\max_{0\leqslant x\leqslant 1} |u^*(x) - u(x)| \leqslant \bar{C}C'^2 C_0^2 C_1^2 N^{-2},$$

for the solution u of (1.1) and the linear interpolant $u^*(x)$ of the final computed solution $\{u_i^*\}$.

In summary, the numerical results obtained confirm our theoretical results and, furthermore, show that the monitor function M_i^N used in the algorithm is superior to the other monitor functions considered in the sense that it guarantees any intermediate mesh generated by the algorithm from mesh starvation.

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