MAXIMUM NORM A POSTERIORI ERROR ESTIMATE FOR A 3D SINGULARLY PERTURBED SEMILINEAR REACTION-DIFFUSION PROBLEM

NARESH M. CHADHA AND NATALIA KOPTEVA

Abstract. A singularly perturbed semilinear reaction-diffusion problem in the unit cube, is discretized on arbitrary nonuniform tensor-product meshes. We establish a second-order maximum norm a posteriori error estimate that holds true uniformly in the small diffusion parameter. No mesh aspect ratio condition is imposed. This result is obtained by combining (i) sharp bounds on the Green's function of the continuous differential operator in the Sobolev $W^{1,1}$ and $W^{2,1}$ norms and (ii) a special representation of the residual in terms of an arbitrary current mesh and the current computed solution. Numerical results on a priori chosen meshes are presented that support our theoretical estimate.

1. Introduction

Solutions of singularly perturbed differential equations typically exhibit sharp boundary and interior layers, which are narrow regions where solutions change rapidly. As shown, e.g., in [24, 23, 8, 16], by the error analysis of model problems (for which layer locations and widths are known a priori), to obtain reliable numerical approximations of layer solutions, it suffices to use relatively small numbers of mesh nodes that are independent of layer width(s) and singular perturbation parameter(s); this is attained by anisotropic mesh refinement in layer regions. Thus optimal meshes are fine in layer regions, standard outside, and include extremely thin mesh cells, i.e. have extremely high mesh aspect ratios (typically $O(\varepsilon^{-1})$, where ε is the layer width).

In contrast, a posteriori error estimates, which, ideally, are needed for reliable automated mesh adaptation, are typically obtained under the shape-regularity condition (equivalent to bounded-mesh-aspect-ratio condition), see, e.g., [1, 3, 25]; thus they are not suitable for constructing efficient layer-adapted meshes.

The aim of the present paper is to establish an a posteriori error estimate for one singularly perturbed problem under *no mesh aspect ratio condition.* Note that our error estimate will be in the maximum norm, which is sufficiently strong to capture layers and hence seems most appropriate for singularly perturbed problems. (The few known a posteriori error estimates for anisotropic meshes are in a weaker energy norm; see, e.g., [19, 20].) We follow the recent paper [17] and extend its

Date: October 31, 2008 and, in revised form, Xxxx xx, xxxx.

1991 *Mathematics Subject Classification.* Primary 65N06, 65N15, 65N50.

Key words and phrases. Semilinear reaction-diffusion, singular perturbation, a posteriori error estimate, maximum norm, no mesh aspect ratio condition, finite differences, layer-adapted mesh.

This publication has emanated from research conducted with the financial support of Science Foundation Ireland under the Research Frontiers Programme 2008; Grant 08/RFP/MTH1536.

two-dimensional analysis to a more intricate three-dimensional case. In particular, we now deal with the three-dimensional Green's function. We also refer the reader to the related one-dimensional papers [14, 15, 18, 22].

We make no attempt to suggest or analyze any particular adaptive mesh generation algorithm. But we note that many successful algorithms are based on interpolation error estimates such as presented in [10, 9, 7], roughly speaking, the criterion on the generated mesh being a small interpolation error. Thus the generated, possibly, anisotropic mesh is supposed to be (quasi-)uniform under the metric induced by the positive definite Hessian matrix of the solution (or its scaled majorant); see, e.g., [6, 11, 13, 27]. It should be noted that such algorithms are not completely theoretically justified. E.g., the relation of the actual error of a numerical method to the interpolation error under no mesh aspect ratio condition is still to be established for many problems, in particular, in the maximum norm. Furthermore, linear interpolation error bounds involve the Hessian matrix of the unknown exact solution, which is replaced in the adaptive algorithm by its computed-solution analogue. To theoretically justify this replacement, one still needs to establish Hessian-matrix recovery formulas under no mesh aspect ratio condition, which are not available in the literature.

An alternative theoretical justification, to which this paper aims to contribute, might be given by a posteriori error estimates that hold true under no mesh aspect ratio condition and directly relate the actual error to a certain discrete linear-interpolation-error-bound analogue, which involves the local mesh sizes and certain computed-solution approximations of the second-order derivatives. Indeed, roughly speaking, our a posteriori error estimate (1.3), (1.4) below is of this type, i.e. might be viewed as a discrete analogue of the linear interpolation error estimates.

We focus on the following singularly perturbed semilinear reaction-diffusion problem posed in the unit cube:

$$\begin{aligned} Tu := -\varepsilon^2 \triangle u + b(x,u) &= 0, \qquad x = (x_1, x_2, x_3) \in \Omega = (0,1)^3, \\ u(x) &= 0, \qquad x \in \partial\Omega. \end{aligned} \tag{1.1}$$

Here ε is a small positive parameter, $\triangle = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$ is the standard Laplace operator, the function b is sufficiently smooth and satisfies

$$0 < \beta < b_u(x,u) \le \bar\beta \qquad \text{for all } \ (x,u) \in [0,1]^3 \times \mathbb{R}. \tag{1.2}$$

Under condition (1.2), problem (1.1) has a unique solution, which exhibits sharp boundary layers of width $O(\varepsilon|\ln\varepsilon|)$ along the boundary $\partial\Omega$.

We discretize (1.1) using the standard second-order seven-point difference scheme—see (2.2) for details—on an arbitrary tensor-product mesh $\{x_{ij\,l}\}$ in $[0,1]^3$, where $x_{ij\,l} = (x_1^{[i]}, x_2^{[j]}, x_3^{[l]})$ with $0 = x_s^{[0]} < x_s^{[1]} < \cdots < x_s^{[N_s]} = 1$ for $s = 1,2,3$, while $h_i = x_1^{[i]} - x_1^{[i-1]}$, $\tau_j = x_2^{[j]} - x_2^{[j-1]}$ and $k_l = x_3^{[l]} - x_3^{[l-1]}$ are the local mesh sizes.

This is an idealized situation in the a posteriori mesh construction context since an irregular mesh, rather than a tensor-product mesh, seems more suitable for a practical a posteriori mesh construction algorithm. Therefore the error estimate, which we obtain, might seem more interesting from a theoretical point of view. In particular, it shows that the bounded-mesh-aspect-ratio condition/minimal-angle condition is not essential in the a posteriori error estimation. Furthermore, if tensor-product meshes are used at least in crucial layer regions, where the mesh adaptation is most needed, one might conjecture that in such regions, local analogues of our a posteriori error estimate would apply.

Our main result is the following maximum norm *a posteriori error estimate*, in which the error is understood as the difference between the exact solution and the trilinear interpolant of the computed solution:

$$\|U^{\mathrm{I}}-u\|_\infty \le C_0\Big[\max_{\substack{i=1,\dots,N_1\\ j=0,\dots,N_2\\ l=0,\dots,N_3}}\{h_i^2 M_{1,ij\,l}\}+\max_{\substack{i=0,\dots,N_1\\ j=1,\dots,N_2\\ l=0,\dots,N_3}}\{\tau_j^2 M_{2,ij\,l}\}+\max_{\substack{i=0,\dots,N_1\\ j=0,\dots,N_2\\ l=1,\dots,N_3}}\{k_l^2 M_{3,ij\,l}\}\Big], \tag{1.3}$$

—see Theorem 2.2—where, roughly speaking,

$$M_{s,ij\,l} \approx |D_s^2 U_{ij\,l}| \ln\big(2+\varepsilon/\kappa\big)+1, \qquad s=1,2,3, \tag{1.4}$$

with $\kappa = \min\{\min_i\{h_i\}, \min_j\{\tau_j\}, \min_l\{k_l\}\}$. Here U^{I} is the trilinear interpolant of the computed solution U (the finite difference computed solution is originally defined at the mesh nodes only; hence to measure the error in the entire domain, one first has to interpolate the computed solution there). The quantities $D_s^2 U_{ij\,l}$ for $s=1,2,3$ are the standard discrete approximations of $\partial^2 u/\partial x_s^2$ defined in (2.3). In (1.4), a few terms are skipped, for which the one-dimensional analysis [15] and the numerical results of [17] and §6 suggest that they are less important; see Theorem 2.2 for the precise definitions of $M_{s,ij\,l}$.

The error constant C_0 in (1.3) is independent of ε, the mesh, and aspect ratios of its elements, although this constant is not specified. In a posteriori error estimation, much attention focuses on specifying the error constants. Note that for singularly perturbed problems, the error constant might blow up as ε becomes small, and hence the existence of an ε-uniform error constant is more significant than its precise value.

The paper is organized as follows. In §2, we describe the numerical method, present our a posteriori error estimate in Theorem 2.2, and outline its proof. Next, in §3, we establish some sharp bounds on the Green's function of a linearized version of (1.1) in the Sobolev $W^{1,1}$ and $W^{2,1}$ norms. They imply certain stability properties of the differential operator T from (1.1), which are presented in §4. Then in §5, we obtain a special representation of the residual in terms of an arbitrary current mesh and the current computed solution, and therefore complete the proof of Theorem 2.2. Finally, in §6, numerical results on a priori chosen meshes are given that support our theoretical estimate.

Notation. Let $\|\cdot\|_{p\,;\tilde\Omega}$, where $1\le p\le\infty$, denote the standard $L_p(\tilde\Omega)$ norm of scalar or vector functions defined in any domain $\tilde\Omega\subset\mathbb{R}^3$. Furthermore, the standard notation $W^{k,p}(\tilde\Omega)$ is used for the Sobolev spaces with the norm $\|\cdot\|_{k,p\,;\tilde\Omega}$ defined, for any scalar function $v(x)$ in a domain $\tilde\Omega$, by

$$\|v\|_{k,p\,;\tilde\Omega} = \|v\|_{p\,;\tilde\Omega} + \textstyle\sum_{l=1}^k |v|_{l,p\,;\tilde\Omega}, \qquad k=1,2,$$
$$|v|_{1,p\,;\tilde\Omega} = \textstyle\sum_{s=1}^3 \|v_{x_s}\|_{p\,;\tilde\Omega}\,, \qquad |v|_{2,p\,;\tilde\Omega} = \textstyle\sum_{s,t=1}^3 \|v_{x_s x_t}\|_{p\,;\tilde\Omega}\,;$$

see, e.g., [12]. We shall use the notation $\|\cdot\|_p$ and $\|\cdot\|_{k,p}$ for $\|\cdot\|_{p\,;\Omega}$ and $\|\cdot\|_{k,p\,;\Omega}$ when there is no ambiguity. Sometimes the domain of interest will be an open ball $B(a\,;\rho)=B(a_1,a_2,a_3\,;\rho)=\{x:\sum_{s=1}^3(x_s-a_s)^2<\rho^2\}$ centered at a of radius ρ.

Throughout the paper we let C denote a generic positive constant that may take different values in different formulas, but is always independent of the mesh and ε. A subscripted C (e.g., C_1) denotes a positive constant that is independent of the mesh and ε and takes a fixed value. For any two quantities w_1 and w_2, the notation $w_1=O(w_2)$ means $|w_1|\le Cw_2$.

Remark 1.1. The assumption $b_u(x,u) \le \bar\beta$ in (1.2) can be omitted since it follows, for some constant $\bar\beta$, from $0 < \beta < b_u(x,u)$ and u being a unique and bounded solution of (1.1); see, e.g., [28, §12]. Note that assumption (1.2) enables us to linearize (1.1) and then invoke the Green's function in our analysis.

2. Numerical method. Main result

Let our problem (1.1) satisfy the standard compatibility conditions at the corners of the domain Ω:

$$b(x,0) = 0 \qquad \text{for } x = (x_1, x_2, x_3): \ \ x_1, x_2, x_3 \in \{0,1\}, \tag{2.1}$$

which guarantee that $u \in C^3(\bar\Omega)$.

Numerical method. We require the computed solution U to satisfy the standard seven-point finite difference discretization of problem (1.1):

$$-\varepsilon^2[D_1^2 U_{ij\,l} + D_2^2 U_{ij\,l} + D_3^2 U_{ij\,l}] + b(x_{ij\,l}, U_{ij\,l}) = 0, \tag{2.2}$$

for $i = 1, \dots N_1 - 1$, $j = 1, \dots, N_2 - 1$, $l = 1, \dots, N_3 - 1$, and set $U_{ij\,l} = 0$ on the boundary, i.e. if $i = 0, N_1$ or $j = 0, N_2$ or $l = 0, N_3$. Here, as usual, $U_{ij\,l}$ is associated with the mesh node $x_{ij\,l} = (x_1^{[i]}, x_2^{[j]}, x_3^{[l]})$, and we use the standard finite difference operators, defined for a discrete function $V_{ij\,l}$ by

$$\begin{aligned}
D_1^- V_{ij\,l} &= \frac{V_{ij\,l} - V_{i-1,j,l}}{h_i}, \qquad & D_1^2 V_{ij\,l} &= \frac{D_1^- V_{i+1,j,l} - D_1^- V_{ij\,l}}{(h_i + h_{i+1})/2},\\
D_2^- V_{ij\,l} &= \frac{V_{ij\,l} - V_{i,j-1,l}}{\tau_j}, \qquad & D_2^2 V_{ij\,l} &= \frac{D_2^- V_{i,j+1,l} - D_2^- V_{ij\,l}}{(\tau_j + \tau_{j+1})/2},\\
D_3^- V_{ij\,l} &= \frac{V_{ij\,l} - V_{i,j,l-1}}{k_l}, \qquad & D_3^2 V_{ij\,l} &= \frac{D_3^- V_{i,j,l+1} - D_3^- V_{ij\,l}}{(k_l + k_{l+1})/2}.
\end{aligned} \tag{2.3}$$

By condition (1.2), there exists a unique solution of the discrete problem (2.2) on an arbitrary mesh $\{x_{ij\,l}\}$; see, e.g., [5].

Clearly, $D_1^2 U_{ij\,l}$ is defined for $i = 1, \dots, N_1 - 1$, $j = 0, \dots, N_2$, $l = 0, \dots, N_3$. Similarly $D_2^2 U_{ij\,l}$ and $D_3^2 U_{ij\,l}$ are defined for $j = 1, \dots, N_2 - 1$, $\forall\ i, l$, and $l = 1, \dots, N_3 - 1$, $\forall\ i, j$, respectively. We now extend $D_1^2 U_{ij\,l}$ to the mesh nodes $i = 0, N_1$ as follows. First, formally extend the discrete equation (2.2) to $i = 0$ and $i = N_1$, in which, using the zero boundary conditions, set $D_s^2 U_{0,j,l} = D_s^2 U_{N_1,j,l} = 0$ for $s = 2, 3$. This yields

$$D_1^2 U_{ij\,l} := \varepsilon^{-2} b(x_{ij\,l}, 0) \quad \text{for} \quad i = 0, N_1, \quad j = 0, \dots, N_2,\ l = 0, \dots, N_3. \tag{2.4a}$$

Similarly, we extend $D_2^2 U_{ij\,l}$ to $j = 0, N_2$ and $D_3^2 U_{ij\,l}$ to $l = 0, N_3$ by

$$D_2^2 U_{ij\,l} := \varepsilon^{-2} b(x_{ij\,l}, 0) \quad \text{for} \quad j = 0, N_2, \quad i = 0, \dots, N_1,\ l = 0, \dots, N_3. \tag{2.4b}$$

$$D_3^2 U_{ij\,l} := \varepsilon^{-2} b(x_{ij\,l}, 0) \quad \text{for} \quad l = 0, N_3, \quad i = 0, \dots, N_1,\ j = 0, \dots, N_2. \tag{2.4c}$$

Note that by (2.1), the above relations (2.4) imply that $D_1^2 U_{ij\,l} = D_2^2 U_{ij\,l} = D_3^2 U_{ij\,l} = 0$ at the corners of our domain, which is consistent with the boundary condition in (1.1).

Remark 2.1. Now that $D_s^2 U_{ij\,l}$, where $s = 1, 2, 3$, are extended by (2.4) to all i, j, l, our discrete equation (2.2) holds true for all $i = 0, \dots, N_1$, $j = 0, \dots, N_2$ and $l = 0, \dots, N_3$.

Trilinear interpolation notation. Let $U^{\rm I} = U^{\rm I}(x)$ be the standard trilinear interpolant of the computed solution $U_{ij\,l}$, i.e. $U^{\rm I}$ is continuous in $\bar\Omega$, trilinear on each $(x_1^{[i-1]}, x_1^{[i]}) \times (x_2^{[j-1]}, x_2^{[j]}) \times (x_3^{[l-1]}, x_3^{[l]})$, and equal to $U_{ij\,l}$ at the mesh nodes:

$$U^{\rm I}(x_{ij\,l}) = U_{ij\,l} \quad \text{for} \quad i = 0, \ldots, N_1,\ j = 0, \ldots, N_2,\ l = 0, \ldots, N_3. \tag{2.5}$$

Similarly, we define the trilinear interpolant $v^{\rm I}(x)$ for any discrete function $v_{ij\,l}$ or any continuous function $v(x)$.

Furthermore, we shall use the standard one-dimensional linear interpolants $v^{{\rm I}_s}$ with respect to x_s for $s = 1, 2, 3$, that are defined, for any function v, as follows. For each fixed x_2, x_3 in the domain of v, we have $v^{{\rm I}_1}(x_1^{[i]}, x_2, x_3) = v(x_1^{[i]}, x_2, x_3)$, and $v^{{\rm I}_1}(x)$ is linear on each $(x_1^{[i-1]}, x_1^{[i]})$. Similarly, $v^{{\rm I}_2}(x_1, x_2^{[j]}, x_3) = v(x_1, x_2^{[j]}, x_3)$, $v^{{\rm I}_3}(x_1, x_2, x_3^{[l]}) = v(x_1, x_2, x_3^{[l]})$ and, furthermore, $v^{{\rm I}_2}$ and $v^{{\rm I}_3}$ are linear on each $(x_2^{[j-1]}, x_2^{[j]})$ and $(x_3^{[l-1]}, x_3^{[l]})$, respectively.

Note that the trilinear interpolation can be represented as a product of the three one-dimensional interpolation operators independently of the order of the interpolation steps. In particular, for the trilinear interpolant $U^{\rm I}$ of $U_{ij\,l}$ we have

$$U^{\rm I}(x) = [\,U^{{\rm I}_1}]^{{\rm I}_2{\rm I}_3} = [\,U^{{\rm I}_2}]^{{\rm I}_1{\rm I}_3} = [\,U^{{\rm I}_3}]^{{\rm I}_1{\rm I}_2}. \tag{2.6}$$

Now we state a maximum norm *a posteriori* error estimate, which is the main result of the present paper.

Theorem 2.2. *Let $u(x)$ be a solution of problem (1.1), (1.2), (2.1), $U_{ij\,l}$ a solution of discrete problem (2.2) on an arbitrary mesh $\{x_{ij\,l}\}$, and $U^{\rm I}(x)$ its trilinear interpolant (2.5). Then*

$$\|U^{\rm I} - u\|_\infty \le C_0 \Big[\max_{\substack{i=1,\ldots,N_1\\ j=0,\ldots,N_2\\ l=0,\ldots,N_3}} \big\{h_i^2 M_{1,ij\,l}\big\} + \max_{\substack{i=0,\ldots,N_1\\ j=1,\ldots,N_2\\ l=0,\ldots,N_3}} \big\{\tau_j^2 M_{2,ij\,l}\big\} + \max_{\substack{i=0,\ldots,N_1\\ j=0,\ldots,N_2\\ l=1,\ldots,N_3}} \big\{k_l^2 M_{3,ij\,l}\big\} \Big],$$

where

$$\begin{aligned}
M_{1,ij\,l} &:= \min\big\{|D_1^2 U_{i-1,j,l}|, |D_1^2 U_{ij\,l}|\big\} \ln\big(2 + \varepsilon/\kappa\big) + \varepsilon |D_1^- D_1^2 U_{ij\,l}| + |D_1^- U_{ij\,l}|^2 + 1,\\
M_{2,ij\,l} &:= \min\big\{|D_2^2 U_{i,j-1,l}|, |D_2^2 U_{ij\,l}|\big\} \ln\big(2 + \varepsilon/\kappa\big) + \varepsilon |D_2^- D_2^2 U_{ij\,l}| + |D_2^- U_{ij\,l}|^2 + 1,\\
M_{3,ij\,l} &:= \min\big\{|D_3^2 U_{i,j,l-1}|, |D_3^2 U_{ij\,l}|\big\} \ln\big(2 + \varepsilon/\kappa\big) + \varepsilon |D_3^- D_3^2 U_{ij\,l}| + |D_3^- U_{ij\,l}|^2 + 1,
\end{aligned}$$

with $\kappa := \min\{\min_i\{h_i\}, \min_j\{\tau_j\}, \min_l\{k_l\}\}$, and the constant C_0 is independent of ε and the mesh.

Proof outline. Only to simplify the presentation, throughout this proof, i.e. to the end of §5, we shall assume that $N_1 = N_2 = N_3 = N$.

By (1.1), we have

$$TU^{\rm I} - Tu = -\varepsilon^2 \left[\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2\right] U^{\rm I} + b(x, U^{\rm I}),$$

where $\partial^2 U^{\rm I}/\partial x_s^2$, for $s = 1, 2, 3$, are understood in the sense of distributions. Define an auxiliary function

$$q(x) := b(x, U^{\rm I}(x))$$

and let $q^{\rm I}$ denote its trilinear interpolant on the mesh $\{(x_{ij\,l})\}$. Hence

$$TU^{\rm I} - Tu = -\varepsilon^2 \left[\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2\right] U^{\rm I} + q^{\rm I} + \big[q - q^{\rm I}\big].$$

Noting that $q_{ij\,l} := q(x_{ij\,l}) = b(x_{ij\,l}, U_{ij\,l})$ and recalling the discrete equation (2.2) and Remark 2.1, yields $q_{ij\,l} = \varepsilon^2[D_1^2 U_{ij\,l} + D_2^2 U_{ij\,l} + D_3^2 U_{ij\,l}]$ for $i,\, j,\, l = 0, \dots, N$. Next, decompose this as $q_{ij\,l} = q_{1,ij\,l} + q_{2,ij\,l} + q_{3,ij\,l}$, where

$$q_{s,ij\,l} := \varepsilon^2 D_s^2 U_{ij\,l} \qquad \text{for} \quad s = 1,2,3, \quad i,j,l = 0,\dots,N. \tag{2.7}$$

Furthermore, using analogues of (2.6) for q_s with $s = 1,2,3$, we get

$$q^{\mathrm{I}}(x) = q_1^{\mathrm{I}}(x) + q_2^{\mathrm{I}}(x) + q_3^{\mathrm{I}}(x) = [\,q_1^{\mathrm{I}_1}]^{\mathrm{I}_2\mathrm{I}_3} + [\,q_2^{\mathrm{I}_2}]^{\mathrm{I}_1\mathrm{I}_3} + [\,q_3^{\mathrm{I}_3}]^{\mathrm{I}_1\mathrm{I}_2}, \qquad x \in \bar\Omega.$$

Therefore

$$TU^{\mathrm{I}} - Tu = \big[-\varepsilon^2 \tfrac{\partial^2}{\partial x_1^2} U^{\mathrm{I}_1} + q_1^{\mathrm{I}_1}\big]^{\mathrm{I}_2\mathrm{I}_3} + \big[-\varepsilon^2 \tfrac{\partial^2}{\partial x_2^2} U^{\mathrm{I}_2} + q_2^{\mathrm{I}_2}\big]^{\mathrm{I}_1\mathrm{I}_3} + \big[-\varepsilon^2 \tfrac{\partial^2}{\partial x_3^2} U^{\mathrm{I}_3} + q_3^{\mathrm{I}_3}\big]^{\mathrm{I}_1\mathrm{I}_2} + \big[q - q^{\mathrm{I}}\big].$$

Here we used the relations

$$\tfrac{\partial^2}{\partial x_1^2} U^{\mathrm{I}} = \big[\,\tfrac{\partial^2}{\partial x_1^2} U^{\mathrm{I}_1}\big]^{\mathrm{I}_2\mathrm{I}_3}, \qquad \tfrac{\partial^2}{\partial x_2^2} U^{\mathrm{I}} = \big[\,\tfrac{\partial^2}{\partial x_2^2} U^{\mathrm{I}_2}\big]^{\mathrm{I}_1\mathrm{I}_3}, \qquad \tfrac{\partial^2}{\partial x_3^2} U^{\mathrm{I}} = \big[\,\tfrac{\partial^2}{\partial x_3^2} U^{\mathrm{I}_3}\big]^{\mathrm{I}_1\mathrm{I}_2}, \tag{2.8}$$

which follow from (2.6) since any operator $\partial^2/\partial^2 x_s$ is commutative with I_t for $t \neq s$, but not with I_s; see also Remark 2.3.

The proof is completed in §5. First, the residual $TU^{\mathrm{I}} - Tu$ is represented as

$$TU^{\mathrm{I}} - Tu = \tfrac{\partial}{\partial x_1} F_1(x) + \tfrac{\partial}{\partial x_2} F_2(x) + \tfrac{\partial}{\partial x_3} F_3(x) + \big[q - q^{\mathrm{I}}\big], \tag{2.9}$$

where F_1, F_2 and F_3 are certain functions of the current mesh and computed solution. This will enable us to estimate the error $U^{\mathrm{I}} - u$ in the maximum norm by linearizing the operator T and invoking its stability properties, which are obtained in §4 using sharp estimates of the Green's function of §3.

Remark 2.3. We understand $\partial^2 U^{\mathrm{I}}/\partial x_s^2$, for $s = 1,2,3$, in the sense of distributions. To be more precise, in (2.8) we use

$$\tfrac{\partial^2}{\partial x_1^2} U^{\mathrm{I}_1} = \textstyle\sum_{i=1}^{N-1} \big[\hbar_i\, D_1^2 U_{ij\,l}\big]\, \delta(x_1 - x_1^{[i]}),$$

$$\big[\,\tfrac{\partial^2}{\partial x_1^2} U^{\mathrm{I}_1}\big]^{\mathrm{I}_2\mathrm{I}_3} = \textstyle\sum_{i=1}^{N-1} \hbar_i\, \big[D_1^2 U_{ij\,l}\big]^{\mathrm{I}_2\mathrm{I}_3}\, \delta(x_1 - x_1^{[i]}),$$

where $\hbar_i := (h_i + h_{i+1})/2$ and $\delta(\cdot)$ is the Dirac δ-distribution.

3. Green's function

Assumption (1.2) enables us to linearize (1.1) and then invoke the Green's function in our analysis. Hence we start with a linear case of (1.1), where we set $b(x,u) := p(x)u - f(x)$ and thus arrive at

$$Lu := -\varepsilon^2 \Delta u + p(x) u = f(x) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{3.1}$$

Here $p \in L_\infty(\Omega)$ and, in accordance with (1.2), it satisfies

$$0 < \beta \le p(x) \le \bar\beta. \tag{3.2}$$

Introduce the Green's function $G(x;\xi)$ of the linear self-adjoint operator L that, for each $x = (x_1, x_2, x_3) \in \Omega$, satisfies

$$\begin{aligned} LG = -\varepsilon^2\, \Delta_\xi G + p(\xi) G &= \delta(x - \xi), \qquad && \xi \in \Omega, \\ G(x;\xi) &= 0, && \xi \in \partial\Omega, \end{aligned} \tag{3.3}$$

where $\xi = (\xi_1, \xi_2, \xi_3)$ and $\Delta_\xi = \partial^2/\partial\xi_1^2 + \partial^2/\partial\xi_2^2 + \partial^2/\partial\xi_2^2$, while $\delta(\cdot)$ is the three-dimensional Dirac δ-distribution. Then the unique solution u of problem (3.1) is

$$u(x) = \int_\Omega G(x;\xi)\, f(\xi)\, d\xi, \tag{3.4}$$

where $d\xi = d\xi_1\, d\xi_2\, d\xi_3$. Starting from (3.3) throughout the present §3, the differential operator L and an auxiliary differential operator $\bar L$ are understood as differential operators in the variable ξ; furthermore, all norms are understood as norms of functions of ξ.

Theorem 3.1. *The Green's function $G(x;\xi)$ from (3.3) satisfies*

$$|G(x;\cdot)|_{1,1\,;\Omega} \le C\,\varepsilon^{-1}. \tag{3.5a}$$

Furthermore, for any ball $B(\tilde x\,;\rho)$ of radius ρ centered at any $\tilde x \in \Omega$ we have

$$|G(x;\cdot)|_{1,1\,;B(\tilde x\,;\rho)\cap\Omega} \le C\,\varepsilon^{-2}\rho; \tag{3.5b}$$

while for the ball $B(x\,;\rho)$ of radius ρ centered at x, we have

$$|G(x;\cdot)|_{2,1\,;\Omega\setminus B(x\,;\rho)} \le C\,\varepsilon^{-2}\ln(2+\varepsilon/\rho). \tag{3.5c}$$

3.1. **Constant-coefficient case.** First, we shall establish a particular case of Theorem 3.1. Let $p := \gamma^2$, where $\gamma = \text{const} > 0$, let Ω be the positive octant space $\mathbb{R}^3_+ = \{x_1, x_2, x_3 > 0\}$. In this particular case we denote the differential operator by $\bar L$ and the Green's function by $\bar G$, and for each x we have

$$\bar L \bar G(x;\xi) := -\varepsilon^2\,\Delta_\xi \bar G + \gamma^2 \bar G = \delta(x-\xi), \qquad \xi_1, \xi_2, \xi_3 > 0. \tag{3.6}$$

The fundamental solution for the operator $-\Delta_\xi + \nu^2$ in $\mathbb{R}^3$ is $e^{-\nu r}/(4\pi r)$; see, e.g., [26, §8.3]. This readily provides the fundamental solution for our differential operator $\bar L$, which is

$$g(x;\xi) := \frac{1}{4\pi\varepsilon^2}\frac{e^{-\gamma r/\varepsilon}}{r}, \quad r := \sqrt{(\xi_1-x_1)^2 + (\xi_2-x_2)^2 + (\xi_3-x_3)^2}. \tag{3.7}$$

Therefore the Green's function for $\bar L$ over the octant can be obtained by the method of images and involves eight terms of the type $\pm g(\pm x_1, \pm x_2, \pm x_3; \xi)$; to be more precise we have

$$\bar G(x;\xi) = \sum_{\sigma_1,\sigma_2,\sigma_3=-1,1} (\sigma_1\sigma_2\sigma_3)\; g\big(x^{[\sigma_1,\sigma_2,\sigma_3]};\xi\big), \quad x^{[\sigma_1,\sigma_2,\sigma_3]} := (\sigma_1 x_1, \sigma_2 x_2, \sigma_3 x_3). \tag{3.8}$$

Lemma 3.2. (i) *For $\bar G(x;\xi)$ of (3.8), estimates (3.5) of Theorem 3.1 hold true, in which G is replaced by $\bar G$.*
(ii) *Furthermore, we have*

$$\|\bar G(x;\cdot)\|_{2\,;\Omega} \le C\,\varepsilon^{-3/2}, \qquad \|\bar G(x;\cdot)\|_{2\,;\Omega\setminus B(x\,;\rho)} \le C\,\varepsilon^{-3/2} e^{-\gamma\rho/\varepsilon}. \tag{3.9}$$

Proof. It suffices to prove estimates (3.5) and (3.9) with $\bar G$ replaced by the term $g(x;\xi)$ of the representation (3.8) of $\bar G$, as the estimates for the other seven terms are similar.

Let the stretching transformation from $\xi = (\xi_1, \xi_2, \xi_3)$ to the new coordinates $\hat\xi = (\hat\xi_1, \hat\xi_2, \hat\xi_3) := (\xi - x)/\varepsilon$ map any domain $\Omega' \subset \mathbb{R}^3$ into $\hat\Omega'$. Furthermore, consider a scaled version $\hat g(\hat\xi)$ of $g(x;\xi)$ from (3.7) defined by

$$\hat g(\hat\xi) := \varepsilon^3 g(x;\xi) = \frac{1}{4\pi}\frac{e^{-\gamma\hat r}}{\hat r}, \qquad \text{where} \quad \hat r := \sqrt{\hat\xi_1^2 + \hat\xi_2^2 + \hat\xi_3^2}\,, \tag{3.10}$$

so that $g\, d\xi = \hat g\, d\hat\xi$, where $d\hat\xi = d\hat\xi_1\, d\hat\xi_2\, d\hat\xi_2 = \varepsilon^{-3} d\xi$. Therefore for any domain Ω' we have

$$|g(x;\cdot)|_{k,1\,;\Omega'} = \varepsilon^{-k}\,|\hat g|_{k,1\,;\hat\Omega'}\,, \qquad \|g(x;\cdot)\|_{2\,;\Omega'} = \varepsilon^{-3/2}\,\|\hat g\|_{2\,;\hat\Omega'}\,. \tag{3.11}$$

Now we shall establish parts (i) and (ii) of our lemma.

(i) A calculation using the standard differentiation formulas

$$\frac{\partial \hat g}{\partial \hat\xi_i}=\frac{\partial \hat g}{\partial \hat r}\cdot\frac{\partial \hat r}{\partial \hat\xi_i},\qquad \frac{\partial^2 \hat g}{\partial \hat\xi_i\,\partial\hat\xi_j}=\frac{\partial^2 \hat g}{\partial \hat r^2}\cdot\frac{\partial \hat r}{\partial \hat\xi_i}\cdot\frac{\partial \hat r}{\partial \hat\xi_j}+\frac{\partial \hat g}{\partial \hat r}\cdot\frac{\partial^2 \hat r}{\partial \hat\xi_i\,\partial\hat\xi_j},$$

where

$$\frac{\partial \hat g}{\partial \hat r}=-\frac{1}{4\pi}\frac{\gamma\hat r+1}{\hat r^2}e^{-\gamma\hat r},\qquad \frac{\partial^2 \hat g}{\partial \hat r^2}=\frac{1}{4\pi}\frac{\gamma^2\hat r^2+2\gamma\hat r+2}{\hat r^3}e^{-\gamma\hat r}$$

and also $|\partial\hat r/\partial\hat\xi_i|=|\hat\xi_i/\hat r|\le 1$ and $|\partial^2\hat r/(\partial\hat\xi_i\,\partial\hat\xi_j)|\le 1/\hat r$, yields

$$\hat r^2\Big|\frac{\partial \hat g}{\partial \hat\xi_i}\Big|\le C(\gamma\hat r+1)e^{-\gamma\hat r},\qquad \hat r^2\Big|\frac{\partial^2 \hat g}{\partial \hat\xi_i\,\partial\hat\xi_j}\Big|\le C\,\frac{\hat r^2+1}{\hat r}e^{-\gamma\hat r}.$$

Combining this with the first relation in (3.11), we obtain the required analogues of (3.5) for g as follows. First, note that

$$|g(x;\cdot)|_{1,1\,;\Omega}=\varepsilon^{-1}|\hat g|_{1,1\,;\hat\Omega}\le C\varepsilon^{-1}\int_0^\infty(\gamma\hat r+1)e^{-\gamma\hat r}\,d\hat r\le C\varepsilon^{-1}.$$

Similarly, we obtain

$$|g(x;\cdot)|_{1,1\,;B(\tilde x\,;\rho)}\le|g(x;\cdot)|_{1,1\,;B(x\,;\rho)}\le C\varepsilon^{-1}\int_0^{\rho/\varepsilon}(\gamma\hat r+1)e^{-\gamma\hat r}\,d\hat r\le C\varepsilon^{-1}(\rho/\varepsilon);$$

here replacing the integral over $B(\tilde x\,;\rho)$ by the integral over $B(x\,;\rho)$ yields an upper bound, since $(\gamma\hat r+1)e^{-\gamma\hat r}$ is a positive decreasing function. Finally, we get

$$|g(x;\cdot)|_{2,1\,;\Omega\setminus B(x\,;\rho)}\le C\varepsilon^{-2}\int_{\rho/\varepsilon}^\infty\frac{\hat r^2+1}{\hat r}e^{-\gamma\hat r}\,d\hat r\le\varepsilon^{-2}\ln(2+\varepsilon/\rho).$$

(ii) A calculation using (3.10) shows that

$$\|\hat g\|^2_{2\,;\hat\Omega}\le C\int_0^\infty e^{-2\gamma\hat r}\,d\hat r\le C,\qquad \|\hat g\|^2_{2\,;\hat\Omega\setminus\hat B(x\,;\rho)}\le C\int_{\rho/\varepsilon}^\infty e^{-2\gamma\hat r}\,d\hat r\le Ce^{-2\gamma\rho/\varepsilon}.$$

Combining these with the second relation in (3.11), we immediately get the required analogues of (3.9) for g. □

Remark 3.3. An inspection of the proof of Lemma 3.2, in which we used the explicit representation (3.8),(3.7) of the Green's function in the constant-coefficient case, shows that the estimates of the Green's function in Theorem 3.1 are sharp.

3.2. **General case. Proof of Theorem 3.1.** Fix $x=(x_1,x_2,x_3)\in\Omega$; without loss of generality, we shall consider only the case of $x\in[0,\,1/2]^3$, as the other cases are similar.

Let the auxiliary function $\bar G$ satisfy (3.6) with the coefficient $\gamma^2:=\beta$. Since, by Lemma 3.2(i), estimates (3.5) hold true for $\bar G$, to get the desired estimates (3.5) for G, it suffices to show that

$$\varepsilon^2|(\bar G-G)(x;\cdot)|_{2,1\,;B(x\,;\varepsilon)\cap\Omega}+\varepsilon|(\bar G-G)(x;\cdot)|_{1,1\,;B(x\,;\varepsilon)\cap\Omega}\le C,\tag{3.12a}$$

$$\varepsilon|(\bar G-G)(x;\cdot)|_{1,1\,;B(\tilde x\,;\rho)\cap\Omega}\le C\,\rho/\varepsilon\quad\text{for }\rho\le\varepsilon,\tag{3.12b}$$

$$\varepsilon^2|G(x;\cdot)|_{2,1\,;\Omega\setminus B(x\,;\varepsilon)}+\varepsilon|G(x;\cdot)|_{1,1\,;\Omega\setminus B(x\,;\varepsilon)}\le C.\tag{3.12c}$$

Indeed, (3.5a) follows from its analogue for $\bar G$ combined with (3.12a) and (3.12c). The next estimate (3.5b) follows from (3.5a) if $\rho>\varepsilon$, and from its analogue for $\bar G$ combined with (3.12b) otherwise. Finally, estimate (3.5c) follows from (3.12c) for $\rho \geq \varepsilon$, and is obtained combining its analogue for $\bar{G}$ with (3.12a) and (3.12c) otherwise.

Now, to complete the proof, we shall establish each of the estimates in (3.12).

(a) Note that, by (3.3) and (3.6), we get

$$L(\bar{G}-G) = [-\varepsilon^2\triangle_\xi + p(\xi)\,]\,(\bar{G}-G) = [p(\xi)-\beta]\,\bar{G}. \tag{3.13}$$

Therefore, by (3.2), we have $L(\bar{G}-G) \geq 0$. Combining this with $\bar{G}-G \geq 0$ on $\partial\Omega$ and then applying the maximum/comparison principle yields $0 \leq G \leq \bar{G}$.

Next, using the variable $\hat{\xi} = (\xi - x)/\varepsilon$ and the notation $\hat{w}(\hat{\xi}) := w(\xi)$ for any function w, rewrite (3.13) in terms of the variable $\hat{\xi}$ as $[-\triangle+\hat{p}\,](\hat{\bar{G}}-\hat{G}) = [\hat{p}-\beta]\hat{\bar{G}}$. This implies that $|\triangle(\hat{\bar{G}}-\hat{G})| \leq C|\hat{\bar{G}}|$ and therefore

$$\|\hat{\bar{G}}-\hat{G}\|_{2,2\,;\hat{B}(x\,;\varepsilon)\cap\hat{\Omega}} \leq C_2\|\hat{\bar{G}}\|_{2\,;\hat{B}(x\,;2\varepsilon)\cap\hat{\Omega}}\,; \tag{3.14}$$

see [21, Chapter 3, (8.6), p. 171]; here the constant C_2 is independent of ε since $\mathrm{dist}(\partial\hat{B}(x\,;\varepsilon), \partial\hat{B}(x\,;2\varepsilon)) = 1$. Note that to obtain estimate (3.14), it is crucial that $\bar{G}-G = 0$ for $\xi \in \partial\Omega \cap B(x\,;2\varepsilon)$ (this readily holds true for $\varepsilon < 1/4$ since $x \in [0,\, 1/2]^3$; otherwise, we get estimates (3.12) with $\varepsilon \in [1/4,\, 1]$ replaced by $\varepsilon' := \varepsilon/5 < 1/4$, which still imply (3.5)).

Rewriting (3.14) in terms of the original variable ξ, we have

$$\varepsilon^2|(\bar{G}-G)(x;\cdot)|_{2,2\,;B(x\,;\varepsilon)\cap\Omega} + \varepsilon|(\bar{G}-G)(x;\cdot)|_{1,2\,;B(x\,;\varepsilon)\cap\Omega} \leq C\|\bar{G}(x;\cdot)\|_{2\,;B(x\,;2\varepsilon)\cap\Omega},$$

where $\|\bar{G}(x;\cdot)\|_{2\,;B(x\,;2\varepsilon)\cap\Omega} \leq C\,\varepsilon^{-3/2}$, by the first estimate in (3.9). Combining this with $|\bar{G}-G|_{k,1\,;B(x\,;\varepsilon)\cap\Omega} \leq C\varepsilon^{3/2}|\bar{G}-G|_{k,2\,;B(x\,;\varepsilon)\cap\Omega}$, for $k = 1, 2$, yields (3.12a).

(b) Let x be an arbitrary point in Ω and $\tilde{x} \in [0,\, 1/2]^3$ (as the other cases are similar). To show (3.12b), imitate the argument used to prove (3.12a) for $B(\tilde{x}\,;\rho)$ instead of $B(x\,;\varepsilon)$ invoking $|\bar{G}-G|_{k,1\,;B(\tilde{x}\,;\rho)} \leq C\rho^{3/2}|\bar{G}-G|_{k,2\,;B(\tilde{x}\,;\rho)}$ and $\sqrt{\rho/\varepsilon} \leq 1$.

(c) Let $\rho_j := 2^j\varepsilon$ and divide the domain $\Omega\backslash B(x\,;\varepsilon)$ into the non-overlapping subdomains $\mathcal{D}_j := \{\xi \in \Omega \,:\, \rho_j < r < \rho_{j+1}\}$ where $j = 0, 1, \ldots$. Furthermore, $\mathcal{D}_j \subset \mathcal{D}'_j := \mathcal{D}_{j-1} \cup \bar{\mathcal{D}}_j \cup \mathcal{D}_{j+1}$, so that $\mathrm{dist}(\partial\mathcal{D}'_j\backslash\partial\Omega, \partial\mathcal{D}_j\backslash\partial\Omega) \geq \varepsilon/2$.

Let the stretching transformation from ξ to $\hat{\xi} = (\xi - x)/\varepsilon$ map $\mathcal{D}_j$ into $\hat{\mathcal{D}}_j$. Rewriting the equation from (3.3) for $\xi \neq x$ in terms of the stretched variable $\hat{\xi}$ as $-\Delta\hat{G} + \hat{p}\,\hat{G} = 0$ yields

$$\|\hat{G}\|_{2,2\,;\hat{\mathcal{D}}_j} \leq C_2\|\hat{G}\|_{2\,;\hat{\mathcal{D}}'_j}\,; \tag{3.15}$$

see [21, Chapter 3, (8.6), p. 171]; here the constant C_2 is independent of ε since $\mathrm{dist}(\partial\hat{\mathcal{D}}_j\backslash\partial\hat{\Omega}, \partial\hat{\mathcal{D}}'_j\backslash\partial\hat{\Omega}) \geq 1/2$. Note that to obtain (3.15), it is crucial that $\hat{G} = 0$ on $\partial\hat{\mathcal{D}}'_j \cap \partial\hat{\Omega}$.

Rewritten in terms of the original variable ξ, estimate (3.15) implies that

$$\varepsilon^2|G(x;\cdot)|_{2,2\,;\mathcal{D}_j} + \varepsilon|G(x;\cdot)|_{1,2\,;\mathcal{D}_j} \leq C\|G(x;\cdot)\|_{2\,;\mathcal{D}'_j} \leq C\|\bar{G}(x;\cdot)\|_{2\,;\mathcal{D}'_j}. \tag{3.16}$$

where we also used $G \leq \bar{G}$. Noting that $\mathcal{D}'_j \subset \Omega\backslash B(x\,;\rho_{j-1})$ and recalling the second estimate in (3.9), we get $\|\bar{G}(x;\cdot)\|_{2\,;\mathcal{D}'_j} \leq C\varepsilon^{-3/2}e^{-\gamma\rho_{j-1}/\varepsilon}$. Combining this with (3.16) and $|G|_{k,1\,;\mathcal{D}_j} \leq C\rho_j^{3/2}|G|_{k,2\,;\mathcal{D}_j}$, for $k = 1, 2$, we arrive at

$$\varepsilon^2|G(x;\cdot)|_{2,1\,;\mathcal{D}_j} + \varepsilon|G(x;\cdot)|_{1,1\,;\mathcal{D}_j} \leq C\rho_j^{3/2}\varepsilon^{-3/2}e^{-\gamma\rho_{j-1}/\varepsilon}.$$

Now, the required estimate (3.12c) is obtained recalling that $\Omega\backslash B(x\,;\varepsilon)=\cup_{j=0}^{\infty}\mathcal{D}_j$ and noting that we have

$$\sum_{j=0}^{\infty}\Bigl(\frac{\gamma\rho_j}{\varepsilon}\Bigr)^{3/2}e^{-\gamma\rho_{j-1}/\varepsilon}\le C\sum_{j=0}^{\infty}\frac{\gamma(\rho_j-\rho_{j-1})}{4\varepsilon}e^{-\gamma\rho_j/(4\varepsilon)}\le C\int_{\gamma/8}^{\infty}e^{-s}ds,$$

since $s^{1/2}e^{-2s}\le Ce^{-s}$ and $\rho_j=2(\rho_j-\rho_{j-1})$, and for the decreasing function e^{-s} the right Riemann sum gives a lower estimate of the corresponding integral. □

4. Stability properties of differential operators

The main result of this section is the following stability theorem for the semilinear differential operator T from (1.1), which we shall further apply to relation (2.9).

Consider the right-hand side f in the special form

$$f(x)=-\nabla\cdot\bigl[F(x)+\bar F(x)\bigr]+\bar f(x), \tag{4.1a}$$

where $F=(F_1,F_2,F_3)$ and $\bar F=(\bar F_1,\bar F_2,\bar F_3)$ are vector functions, whose components together with $\bar f$ are in $L_\infty(\Omega)$, and $\nabla\cdot F=\partial F_1/\partial x_1+\partial F_2/\partial x_2+\partial F_3/\partial x_3$. Furthermore, we assume that

$$F_1(x)=A_i(x_2,x_3)\,(x_1-x_1^{[i-1/2]})\quad\text{for } x\in(x_1^{[i-1]},x_1^{[i]})\times[0,1]\times[0,1], \tag{4.1b}$$

$$F_2(x)=B_j(x_1,x_3)\,(x_2-x_2^{[j-1/2]})\quad\text{for } x\in[0,1]\times(x_2^{[j-1]},x_2^{[j]})\times[0,1], \tag{4.1c}$$

$$F_3(x)=Q_l(x_1,x_2)\,(x_3-x_3^{[l-1/2]})\quad\text{for } x\in[0,1]\times[0,1]\times(x_3^{[l-1]},x_3^{[l]}), \tag{4.1d}$$

where $i,j,\,l=1,\ldots,N$, respectively, and the notation $x_s^{[i-1/2]}:=(x_s^{[i-1]}+x_s^{[i]})/2$ was used with $s=1,2,3$.

Theorem 4.1. *Let the function b in (1.1) satisfy (1.2). Then, for any functions $v,\,w\in W^{1,2}(\Omega)$ such that $Tv(x)-Tw(x)=f(x)$, where f is defined by (4.1), and $v=w$ on $\partial\Omega$, we have*

$$\|v-w\|_\infty\le C\varepsilon^{-2}\bigl[E_1+E_2+E_3\bigr]\,\ln\bigl(2+\varepsilon/\kappa\bigr)+C\varepsilon^{-1}\|\bar F\|_\infty+\beta^{-1}\|\bar f\|_\infty,$$

where $\kappa=\min\{\min_i\{h_i\},\min_j\{\tau_j\},\min_l\{k_l\}\}$ and

$$E_1:=\max_{i=1,\ldots,N}\bigl\{h_i^2\max_{x_2,x_3\in[0,1]}|A_i(x_2,x_3)|\bigr\},\quad E_2:=\max_{j=1,\ldots,N}\bigl\{\tau_j^2\max_{x_1,x_3\in[0,1]}|B_j(x_1,x_3)|\bigr\},$$

$$E_3:=\max_{l=1,\ldots,N}\bigl\{k_l^2\max_{x_1,x_2\in[0,1]}|Q_l(x_1,x_2)|\bigr\}.$$

The proof is deferred to §4.2.

4.1. **Linear reaction-diffusion.** First we address the linear problem (3.1), (3.2) with the right-hand side (4.1). Since the differential operator L is linear, it is convenient to establish stability of u with respect to various components of f separately.

Lemma 4.2. *There exists a unique solution $u\in L_\infty(\Omega)$ of problem (3.1), (3.2) with the right-hand side (4.1a). Furthermore, if $F=(F_1,F_2,F_3):=0$, then*

$$\|u\|_\infty\le C\varepsilon^{-1}\|\bar F\|_\infty+\beta^{-1}\|\bar f\|_\infty. \tag{4.2}$$

Proof. Since L is linear, it suffices to establish the desired estimate in the following two cases. Case A: $f=\bar f$, while $\bar F:=0$. Then estimate (4.2) is well known and follows from the maximum/comparison principle extended to functions in the Sobolev space $W^{1,2}$ [12, §8.1]. Case B: $\bar f:=0$. Now our assertion (4.2) follows from (3.4) combined with estimate (3.5a). □

Lemma 4.3. *Let $\bar F := 0$ and $\bar f := 0$ in (4.1a). Then the solution $u \in L_\infty(\Omega)$ of problem (3.1), (3.2) with the right-hand side (4.1) satisfies*

$$\|u\|_\infty \le C\varepsilon^{-2}\big[E_1 + E_2 + E_3\big]\,\ln\big(2+\varepsilon/\kappa\big),$$

where κ and E_s, for $s = 1, 2, 3$, are defined in Theorem 4.1.

Proof. It suffices to consider only the case of $f := -\partial F_1/\partial x_1$, i.e. $F_2 = F_3 := 0$, as the cases of $f := -\partial F_s/\partial x_s$, for $s = 2, 3$, are considered similarly, and our differential operator L is linear.

Fix x and denote $v(\xi) := G(x;\xi)$. Then, by (3.4), we have

$$u(x) = \int_\Omega F_1(\xi)\, v_{\xi_1}(\xi)\, d\xi = \sum_{i=1}^N \int_{\Omega_i} A_i(\xi_2, \xi_3)\,(\xi_1 - x_1^{[i-1/2]})\, v_{\xi_1}(\xi)\, d\xi, \tag{4.3}$$

where $\Omega_i := (x_1^{[i-1]}, x_1^{[i]}) \times (0,1) \times (0,1)$ for $i = 1, \ldots, N$. Furthermore, for some $0 < m < N$ let $x_1 \in [x_1^{[m-1/2]}, x_1^{[m+1/2]}]$ (the cases of $x_1 \in [0, x_1^{[1/2]}]$ and $x_1 \in [x_1^{[N-1/2]}, 1]$ are similar). Now, introduce the rectangular box domain

$$\Omega' := (x_1^{[m-1]}, x_1^{[m+1]}) \times (x_2 - \tilde h_m, x_2 + \tilde h_m) \times (x_3 - \tilde h_m, x_3 + \tilde h_m),$$

where $\tilde h_m := \min\{h_m, h_{m+1}\}/2$, so that

$$B(x_1, x_2, x_3; \tilde h_m) \subset \Omega' \subset B(x_1^{[m-1/2]}, x_2, x_3; h_m) \cup B(x_1^{[m+1/2]}, x_2, x_3; h_{m+1}). \tag{4.4}$$

Clearly (4.3) can be written as $u(x) = S_1 + S_2$, where

$$S_1 = \sum_{i=1}^N \int_{\Omega_i\backslash\Omega'} A_i(\xi_2, \xi_3)\,(\xi_1 - x_1^{[i-1/2]})\, v_{\xi_1}(\xi)\, d\xi,$$

$$S_2 = \sum_{i=m}^{m+1} \int_{\Omega_i\cap\Omega'} A_i(\xi_2, \xi_3)\,(\xi_1 - x_1^{[i-1/2]})\, v_{\xi_1}(\xi)\, d\xi\,.$$

To estimate S_1, note that $v_{\xi_1\xi_1}$ is well-defined in each $\Omega_i\backslash\Omega'$ since the singularity of v occurs at $x \in \Omega'$. Furthermore, $\xi = (\xi_1, \xi_2, \xi_3) \in \Omega_i\backslash\Omega'$ implies $(t, \xi_2, \xi_3) \in \Omega_i\backslash\Omega'$ for all $t \in (x_1^{[i-1]}, x_1^{[i]})$. Therefore, we shall invoke the representation

$$v_{\xi_1}(\xi) = v_{\xi_1}(x_1^{[i-1]}, \xi_2, \xi_3) + \int_{x_1^{[i-1]}}^{\xi_1} v_{\xi_1\xi_1}(t, \xi_2, \xi_3)\, dt \qquad \text{for } \xi \in \Omega_i\backslash\Omega'.$$

Combining this with $\int_{x_1^{[i-1]}}^{x_1^{[i]}} (\xi_1 - x_1^{[i-1/2]})\, d\xi_1 = 0$ yields

$$\Big|\int_{x_1^{[i-1]}}^{x_1^{[i]}} (\xi_1 - x_1^{[i-1/2]})\, v_{\xi_1}(\xi)\, d\xi_1\Big| = \Big|\int_{x_1^{[i-1]}}^{x_1^{[i]}} d\xi_1\,(\xi_1 - x_1^{[i-1/2]}) \int_{x_1^{[i-1]}}^{\xi_1} v_{\xi_1\xi_1}(t, \xi_2, \xi_3)\, dt\Big|$$
$$\le \frac{h_i^2}{4} \int_{x_1^{[i-1]}}^{x_1^{[i]}} |v_{\xi_1\xi_1}(t, \xi_2, \xi_3)| dt = \frac{h_i^2}{4} \int_{x_1^{[i-1]}}^{x_1^{[i]}} |v_{\xi_1\xi_1}(\xi)| d\xi_1.$$

Hence we have

$$|S_1| \le \sum_{i=1}^N \frac{h_i^2}{4} \int_{\Omega_i\backslash\Omega'} |A_i(\xi_2, \xi_3)|\, |v_{\xi_1\xi_1}(\xi)|\, d\xi \le \frac{1}{4}\, E_1 \int_{\Omega\backslash\Omega'} |v_{\xi_1\xi_1}(\xi)|\, d\xi\,.$$

Finally, recalling (4.4) and estimate (3.5c) for $v_{\xi_1\xi_1} = G_{\xi_1\xi_1}$, we get the desired estimate for S_1:

$$|S_1| \le \frac{1}{4}\, E_1 \int_{\Omega\setminus B(x;\tilde h_m)} |v_{\xi_1\xi_1}(\xi)|\, d\xi \le C\varepsilon^{-2} E_1 \ln(2+\varepsilon/\tilde h_m) \le C\varepsilon^{-2} E_1 \ln(2+\varepsilon/\kappa).$$

It remains to obtain a similar estimate for S_2, for which, invoking (4.4), we have

$$\begin{aligned} S_2 &\le \sum_{i=m}^{m+1} \max_{\xi_2,\xi_3\in[0,1]} |A_i(\xi_2,\xi_3)|\; h_i \int_{B(x_1^{[i-1/2]},x_2,x_3;h_i)} |v_{\xi_1}(\xi)|\, d\xi \\ &\le \sum_{i=m}^{m+1} \max_{\xi_2,\xi_3\in[0,1]} |A_i(\xi_2,\xi_3)|\; h_i\, \frac{Ch_i}{\varepsilon^2} \;\le\; C\,\varepsilon^{-2} E_1. \end{aligned}$$

Here we also used estimate (3.5b) for $v_{\xi_1} = G_{\xi_1}$. □

4.2. **Semilinear reaction-diffusion. Proof of Theorem 4.1.** Using the standard linearization technique, we have $Tv(x) - Tw(x) = L\big[v(x) - w(x)\big]$, where the linear operator L is defined by (3.1) with $p(x) = \int_0^1 b_u(x, w(x) + t[v(x) - w(x)])\, dt$, which, by (1.2), satisfies condition (3.2). Hence the desired estimate follows from Lemmas 4.2 and 4.3. □

5. Analysis of the numerical method. Proof of Theorem 2.2

To complete the proof of our main result, Theorem 2.2, which we started in §2, we shall invoke the following lemma.

Lemma 5.1. *We have*

$$-\varepsilon^2 \tfrac{\partial^2}{\partial x_s^2} U^{\mathrm{I}_s} + q_s^{\mathrm{I}_s} = \tfrac{\partial}{\partial x_s} F_s, \qquad s = 1, 2, 3,$$

where the semi-discrete functions $F_1 = F_1(x_1, x_2^{[j]}, x_3^{[l]})$, $F_2 = F_2(x_1^{[i]}, x_2, x_3^{[l]})$ *and* $F_3 = F_3(x_1^{[i]}, x_2^{[j]}, x_3)$ *are defined by*

(5.1a) $$F_1 := q_{1,ij\,l}\,(x_1 - x_1^{[i-1/2]}) + \tfrac12 D_1^- q_{1,ij\,l}\,(x_1^{[i]} - x_1)^2, \quad x_1 \in (x_1^{[i-1]}, x_1^{[i]}),$$

for $i = 1,\dots,N$ *and* $j,\, l = 0,\dots,N$,

(5.1b) $$F_2 := q_{2,ij\,l}\,(x_2 - x_2^{[j-1/2]}) + \tfrac12 D_2^- q_{2,ij\,l}\,(x_2^{[j]} - x_2)^2, \quad x_2 \in (x_2^{[j-1]}, x_2^{[j]}),$$

for $j = 1,\dots,N$ *and* $i,\, l = 0,\dots,N$,

(5.1c) $$F_3 := q_{3,ij\,l}\,(x_3 - x_3^{[l-1/2]}) + \tfrac12 D_3^- q_{3,ij\,l}\,(x_3^{[l]} - x_3)^2, \quad x_3 \in (x_3^{[l-1]}, x_3^{[l]}),$$

for $l = 1,\dots,N$ *and* $i,\, j = 0,\dots,N$.

Proof. We closely imitate the one-dimensional argument used in the proofs of [15, Theorem 3.3] and [17, Lemma 5.1] and include this proof here for completeness only.

It suffices to obtain the desired relation for $s = 1$, as the other cases are similar. To simplify the presentation, within this proof, fix $x_2^{[j]}$, $x_3^{[l]}$, and therefore use the notation $U^{\mathrm{I}_1}(x_1) := U^{\mathrm{I}_1}(x_1, x_2^{[j]}, x_3^{[l]})$, $q^{\mathrm{I}_1}(x_1) := q_1^{\mathrm{I}_1}(x_1, x_2^{[j]}, x_3^{[l]})$, $q_{1,i} := q_{1,ij\,l}$, and $F_1(x_1) := F_1(x_1, x_2^{[j]}, x_3^{[l]})$. Furthermore, for any function v, let $v' := \partial v/\partial x_1$. Thus we intend to show that $-\varepsilon^2 (U^{\mathrm{I}_1})'' + q_1^{\mathrm{I}_1} = F_1'$.

First, note that

$$-\varepsilon^2 (U^{\mathrm{I}_1})'' + q_1^{\mathrm{I}_1} = -\big[\,\varepsilon^2 (U^{\mathrm{I}_1})' + \int_{x_1}^1 q_1^{\mathrm{I}_1}(t)\,dt\big]'. \tag{5.2}$$

Recalling (2.3) and (2.7), we observe that

$$\varepsilon^2 (U^{\mathrm{I}_1})' = \varepsilon^2 D_1^- U_i^{\mathrm{I}_1} = \varepsilon^2 D_1^- U_N^{\mathrm{I}_1} - \sum_{m=i}^{N-1} \tfrac12 (h_m + h_{m+1})\, q_{1,m}\,, \quad x_1 \in (x_1^{[i-1]}, x_1^{[i]}),$$

where $i = 1, \ldots, N$. Now, substituting the above representation in (5.2) and omitting the derivative of the constant $\varepsilon^2 D_1^- U_N^{\mathrm{I}_1}$, we arrive at $-\varepsilon^2 (U^{\mathrm{I}_1})'' + q_1^{\mathrm{I}_1} = \tilde F_1'$, where

$$\tilde F_1(x) := \sum_{m=i}^{N-1} \tfrac12 (h_m + h_{m+1})\, q_{1,m} - \int_{x_1}^1 q_1^{\mathrm{I}_1}(t)\,dt, \quad x_1 \in (x_1^{[i-1]}, x_1^{[i]}),\ i = 1, \ldots, N.$$

A calculation shows that

$$\sum_{m=i}^{N-1} \tfrac12 (h_m + h_{m+1})\, q_{1,m} = \tfrac12 h_i\, q_{1,i} + \int_{x_1^{[i]}}^1 q_1^{\mathrm{I}_1}(t)\,dt - \tfrac12 h_N\, q_{1,N},$$

and, omitting the derivative of another constant $\frac12 h_N\, q_{1,N}$, we obtain $\tilde F_1' = F_1'$, where

$$F_1(x_1) := \tfrac12 h_i\, q_{1,i} - \int_{x_1}^{x_1^{[i]}} q_1^{\mathrm{I}_1}(t)\,dt\,, \quad x_1 \in (x_1^{[i-1]}, x_1^{[i]}),\ i = 1, \ldots, N. \tag{5.3}$$

Thus we have obtained the desired relation $-\varepsilon^2 (U^{\mathrm{I}_1})'' + q_1^{\mathrm{I}_1} = F_1'$, in which F_1 is defined by (5.3), and it remains to show that the definition of F_1 in (5.1a) is equivalent to (5.3). Indeed, by computing the term $\int_{x_1}^{x_1^{[i]}} q_1^{\mathrm{I}_1}(t)\,dt$ in (5.3) using the relation $q_1^{\mathrm{I}_1}(x_1) = q_{1,i} - (x_1^{[i]} - x_1)\, D_1^- q_{1,i}$, we get (5.1a). □

Remark 5.2. One can easily check that F_s, for $s = 1, 2, 3$, of (5.1) allow an alternative representation:

$$\begin{aligned}
F_1 &= [q_1]_{i-1,j,l}\, (x_1 - x_1^{[i-1/2]}) + [D_1^- q_{1,ij\,l}]\, O(h_i^2), && x_1 \in (x_1^{[i-1]}, x_1^{[i]}),\\
F_2 &= [q_2]_{i,j-1,l}\, (x_2 - x_2^{[j-1/2]}) + [D_2^- q_{2,ij\,l}]\, O(\tau_j^2), && x_2 \in (x_2^{[j-1]}, x_2^{[j]}),\\
F_3 &= [q_3]_{i,j,l-1}\, (x_3 - x_3^{[l-1/2]}) + [D_3^- q_{3,ij\,l}]\, O(k_l^2), && x_3 \in (x_3^{[l-1]}, x_3^{[l]}).
\end{aligned}$$

Here, e.g., the new representation of F_1 follows from $q_{1,ij\,l} = [q_1]_{i-1,j,l} + h_i\, [D_1^- q_{1,ij\,l}]$.

Proof of Theorem 2.2 (continued from §2). Extend F_1 and F_2 of Lemma 5.1 onto the whole domain $\bar\Omega$ by the trilinear interpolation

$$F_1(x) := \big[F_1(x_1, x_2^{[j]}, x_3^{[l]})\big]^{\mathrm{I}_2\mathrm{I}_3}, \qquad F_2(x) := \big[F_2(x_1^{[i]}, x_2, x_3^{[l]})\big]^{\mathrm{I}_1\mathrm{I}_3},$$
$$F_3(x) := \big[F_3(x_1^{[i]}, x_2^{[j]}, x_3)\big]^{\mathrm{I}_1\mathrm{I}_2}.$$

Now, noting that any operator $\partial/\partial x_s$ is commutative with I_t for $t \neq s$, we obtain the representation (2.9) for the residual $TU^{\mathrm{I}} - Tu$. Now, invoking Theorem 4.1 yields

$$\|U^{\mathrm{I}} - u\|_\infty \le C\varepsilon^{-2}\big[E_1 + E_2 + E_3\big]\, \ln\!\big(2 + \varepsilon/\kappa\big) + C\varepsilon^{-1}\bar E + \beta^{-1}\|q - q^{\mathrm{I}}\|_\infty, \tag{5.4}$$

where

$$E_1 = \max_{i=1,\dots,N} \bigl\{ h_i^2 \max_{x_2,x_3\in[0,1]} |(q_{1,ij\,l})^{\mathrm{I}_2\mathrm{I}_3}| \bigr\} = \max_{\substack{i=1,\dots,N\\ j,l=0,\dots,N}} \bigl\{ h_i^2 |q_{1,ij\,l}| \bigr\}, \tag{5.5a}$$

and similarly

$$E_2 = \max_{\substack{j=1,\dots,N\\ i,l=0,\dots,N}} \bigl\{ \tau_j^2 |q_{2,ij\,l}| \bigr\}, \qquad E_3 = \max_{\substack{l=1,\dots,N\\ i,j=0,\dots,N}} \bigl\{ k_l^2 |q_{3,ij\,l}| \bigr\}, \tag{5.5b}$$

while

$$\bar{E} = \max_{\substack{i=1,\dots,N\\ j,l=0,\dots,N}} \bigl\{ h_i^2 |D_1^- q_{1,ij\,l}| \bigr\} + \max_{\substack{j=1,\dots,N\\ i,l=0,\dots,N}} \bigl\{ \tau_j^2 |D_2^- q_{2,ij\,l}| \bigr\} + \max_{\substack{l=1,\dots,N\\ i,j=0,\dots,N}} \bigl\{ k_l^2 |D_3^- q_{3,ij\,l}| \bigr\}.$$

Combining (5.4) with (2.7) and the trilinear interpolation estimate

$$\begin{aligned} \|q - q^{\mathrm{I}}\|_\infty \le C\Bigl[&\max_{\substack{i=1,\dots,N\\ j,l=0,\dots,N}} \bigl\{ h_i^2 (1 + |D_1^- U_{ij\,l}|^2) \bigr\} \\ &+ \max_{\substack{j=1,\dots,N\\ i,l=0,\dots,N}} \bigl\{ \tau_j^2 (1 + |D_2^- U_{ij\,l}|^2) \bigr\} + \max_{\substack{j=1,\dots,N\\ i,l=0,\dots,N}} \bigl\{ k_l^2 (1 + |D_3^- U_{ij\,l}|^2) \bigr\} \Bigr], \end{aligned} \tag{5.6}$$

we obtain a version of the desired a posteriori error estimate of Theorem 2.2 in which the quantity $\min\bigl\{|D_1^2 U_{i-1,j,l}|, |D_1^2 U_{ij\,l}|\bigr\}$ is replaced by $|D_1^2 U_{ij\,l}|$, and similarly the quantities $\min\bigl\{|D_2^2 U_{i,j-1,l}|, |D_2^2 U_{ij\,l}|\bigr\}$ and $\min\bigl\{|D_3^2 U_{i,j,l-1}|, |D_3^2 U_{ij\,l}|\bigr\}$ are replaced by $|D_2^2 U_{ij\,l}|$ and $|D_3^2 U_{ij\,l}|$, respectively.

Comparing (5.1) with Remark 5.2, we observe that the quantities $|q_{s,ij\,l}|$, for $s = 1,2,3$, in (5.5) can be replaced by $\min\{|q_{1,i-1,j,l}|, |q_{1,ij\,l}|\}$, $\min\{|q_{2,i,j-1,l}|, |q_{2,ij\,l}|\}$ and $\min\{|q_{3,i,j,l-1}|, |q_{3,ij\,l}|\}$, respectively. This yields a sharper version of (5.4), which we combine with (2.7) and (5.6) to get the precise assertion of Theorem 2.2.

Finally, note that the interpolation error estimate (5.6), which we used, follows from $q - q^{\mathrm{I}} = [q - q^{\mathrm{I}_1}] + [q^{\mathrm{I}_1} - (q^{\mathrm{I}_1})^{\mathrm{I}_2}] + [q^{\mathrm{I}_1\mathrm{I}_2} - (q^{\mathrm{I}_1\mathrm{I}_2})^{\mathrm{I}_3}]$ combined with the observation that $|\partial^2 q/\partial x_s^2| \le C(1 + |D_s^- U_{ij\,l}|^2)$ in each mesh cell $(x_1^{[i-1]}, x_1^{[i]}) \times (x_2^{[j-1]}, x_2^{[j]}) \times (x_3^{[l-1]}, x_3^{[l]})$; see [2, Comment 2.15] for a similar argument. □

6. Numerical results

The maximum norm *a posteriori* error estimate of Theorem 2.2 implies that

$$e := \|U^{\mathrm{I}} - u\|_\infty \le \tilde{C}\eta, \qquad \eta := \max\{\eta_0, \eta_1, \eta_2, \eta_3\}, \tag{6.1}$$

$$\eta_n := \max\Bigl\{ \max_{\substack{i=1,\dots,N_1\\ j=0,\dots,N_2\\ l=0,\dots,N_3}} \bigl\{ h_i^2 M_{1,ij\,l}^{(n)} \bigr\} \;;\; \max_{\substack{i=0,\dots,N_1\\ j=1,\dots,N_2\\ l=0,\dots,N_3}} \bigl\{ \tau_j^2 M_{2,ij\,l}^{(n)} \bigr\} \;;\; \max_{\substack{i=0,\dots,N_1\\ j=0,\dots,N_2\\ l=1,\dots,N_3}} \bigl\{ k_l^2 M_{3,ij\,l}^{(n)} \bigr\} \Bigr\},$$

for $n = 0,1,2,3$, where $\tilde{C} = C \ln\bigl(2 + \varepsilon/\kappa\bigr)$,

$$M_{1,ij\,l}^{(2)} := \min\bigl\{|D_1^2 U_{i-1,j,l}|, |D_1^2 U_{ij\,l}|\bigr\}, \quad M_{2,ij\,l}^{(2)} := \min\bigl\{|D_2^2 U_{i,j-1,l}|, |D_2^2 U_{ij\,l}|\bigr\},$$

$$M_{3,ij\,l}^{(2)} := \min\bigl\{|D_3^2 U_{i,j,l-1}|, |D_3^2 U_{ij\,l}|\bigr\},$$

and for $s = 1,2,3$, we define

$$M_{s,ij\,l}^{(0)} = 1, \qquad M_{s,ij\,l}^{(1)} := |D_s^- U_{ij\,l}|^2, \qquad M_{s,ij\,l}^{(3)} := \varepsilon |D_s^- D_s^2 U_{ij\,l}|.$$

Here η_n and $M^{(n)}$, $n = 1,2,3$, involve discrete analogues of nth-order derivatives.

In this section we present numerical results on *a priori* chosen meshes to investigate the efficiency of the upper maximum norm error estimator η in (6.1) and its components η_n. It is also of interest which of η_n is the principal component in η if any. We shall examine the errors e and, more importantly, the quantities η, e/η,

TABLE 6.1. Bakhvalov mesh, $\lambda = 3$: maximum norm error e and the efficiency constant e/η for the upper error estimator η.

	$\varepsilon = 10^{-1}$		$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-k}$, $k = 4, \dots, 10$	
N	e	e/η	e	e/η	e	e/η	e	e/η
16	1.74e-2	1.80e-1	1.71e-2	1.31e-1	1.73e-2	1.31e-1	1.73e-2	1.31e-1
32	4.51e-3	1.82e-1	4.31e-3	1.28e-1	4.36e-3	1.28e-1	4.38e-3	1.28e-1
64	1.17e-3	1.86e-1	1.08e-3	1.26e-1	1.09e-3	1.26e-1	1.10e-3	1.27e-1
128	2.96e-4	1.87e-1	2.70e-4	1.26e-1	2.74e-4	1.26e-1	2.75e-4	1.26e-1

TABLE 6.2. Bakhvalov mesh, $\lambda = 3$: upper maximum norm error estimator η, its components η_1, η_2, η_3, and its efficiency constant e/η.

	$\varepsilon = 10^{-1}$				$\varepsilon = 10^{-k}$, $k = 4, \dots, 10$			
N	η_1	η_2	$\eta_3 = \eta$	e/η	η_1	η_2	$\eta_3 = \eta$	e/η
16	4.43e-2	7.79e-2	9.64e-2	1.80e-1	1.10e-1	1.07e-1	1.33e-1	1.31e-1
32	1.29e-2	2.24e-2	2.48e-2	1.82e-1	3.10e-2	3.09e-2	3.41e-2	1.28e-1
64	3.50e-3	6.00e-3	6.29e-3	1.86e-1	8.26e-3	8.25e-3	8.65e-3	1.27e-1
128	9.07e-4	1.55e-3	1.58e-3	1.87e-1	2.13e-3	2.13e-3	2.18e-3	1.26e-1

η_n, e/η_n and their dependence on ε, numbers of mesh nodes and particular mesh choices.

We consider $\varepsilon = 10^{-k}$, $k = 1, \dots, 10$, and two tensor-product meshes with $N_1 = N_2 = N_3 = N$: a variant of the layer-adapted mesh by Bakhvalov [4] and a simple uniform mesh; see Tables 6.1–6.5. Note that a Bakhvalov-type layer-adapted mesh was chosen for the numerical experiments, since it yields ε-uniform second-order accuracy [4, 16]. Furthermore, we expect a robust adaptive algorithm to generate a mesh that is very close to a Bakhvalov mesh, as in [18, §6 and Figure 2].

To be precise, if $\varepsilon \le \bar{\varepsilon}$, our Bakhvalov-type mesh is given by $x_1^{[i]} = x_2^{[i]} = x_3^{[i]} := \varphi(i/N)$ for $i = 0, 1, \dots, N$, where $\varphi(t) := \varepsilon\lambda \ln[b/(b-t)]$ for $t \in [0, \theta]$, $\varphi(1) := 1$, and $\varphi(t)$ is continuous on $[0, 1]$ and linear on $[\theta, 1]$. We use the constants $b = 1/2$, $\bar{\varepsilon} = b/\lambda$, and $\theta = b - \varepsilon\lambda$. The constant λ will be specified later. For $\varepsilon > \bar{\varepsilon}$, the Bakhvalov mesh is defined to be a simple uniform mesh.

Our test problem is the linear problem (3.1), in which $p(x) := 1$, and whose exact solution

$$u(x) = \big(\cos(\tfrac{1}{2}\pi x_1) - \mu(x_1)\big)\big(1 - x_2 - \mu(x_2)\big)\big(1 - x_3^2 - \mu(x_3)\big), \; \mu(t) = \frac{e^{-t/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}},$$

exhibits boundary and corner layers.

Tables 6.1 and 6.2 present numerical results for the Bakhvalov mesh with $\lambda = 3$. This mesh yields ε-uniform second-order accuracy in the maximum norm, i.e. ultimately, we would like to be able to construct a similar adaptive mesh. We observe agreement with our theoretical estimate (6.1). Not only does e/η stabilize—see Table 6.1—but it becomes close to the linear interpolation error constant $1/8 = 1.25e - 1$. Table 6.2 is given to compare the components η_n of η. We observe that $\eta_2 \approx \eta_3 = \eta$. Furthermore, for $\varepsilon \le 10^{-2}$ we have $\eta_1 \approx \eta_2 \approx \eta_3$, while for $\varepsilon = 10^{-1}$ the quantity η_1 is dominated by η_2 and η_3. The quantity η_0 is not presented, since it is negligible and, furthermore, known a priori.

TABLE 6.3. Uniform mesh: maximum norm error e and the efficiency constant e/η_2 for the component η_2 of the upper maximum norm error estimator η.

	$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-k}$, $k = 5, \dots, 10$	
N	e	e/η_2	e	e/η_2	e	e/η_2	e	e/η_2
16	7.36e-1	8.02e-1	9.90e-1	9.99e-1	9.99e-1	1.01e+0	1.00e+0	1.01e+0
32	4.35e-1	5.31e-1	9.80e-1	9.87e-1	9.99e-1	1.00e+0	1.00e+0	1.00e+0
64	1.64e-1	2.86e-1	9.43e-1	9.53e-1	9.99e-1	9.99e-1	1.00e+0	1.00e+0
128	5.61e-2	2.00e-1	8.24e-1	8.52e-1	9.97e-1	9.98e-1	1.00e+0	1.00e+0

TABLE 6.4. Uniform mesh: the components η_2 and η_3 of the upper maximum norm error estimator η and the efficiency constant e/η_2 for η_2.

	$\varepsilon = 10^{-4}$			$\varepsilon = 10^{-7}$			$\varepsilon = 10^{-10}$		
N	η_2	$\eta_3 = \eta$	e/η_2	η_2	$\eta_3 = \eta$	e/η_2	η_2	$\eta_3 = \eta$	e/η_2
16	9.91e-1	6.20e+2	1.01e+0	9.91e-1	6.20e+5	1.01e+0	9.91e-1	6.20e+8	1.01e+0
32	9.98e-1	3.12e+2	1.00e+0	9.98e-1	3.12e+5	1.00e+0	9.98e-1	3.12e+8	1.00e+0
64	9.99e-1	1.56e+2	9.99e-1	9.99e-1	1.56e+5	1.00e+0	9.99e-1	1.56e+8	1.00e+0
128	9.99e-1	7.81e+1	9.98e-1	1.00e+0	7.81e+4	1.00e+0	1.00e+0	7.81e+7	1.00e+0

TABLE 6.5. Bakhvalov mesh, $\lambda = 1$: maximum norm error e, upper maximum norm error estimator η, its components η_1, η_2, η_3, and its efficiency constant e/η.

	$\varepsilon = 10^{-5}$					$\varepsilon = 10^{-10}$				
N	e	η_1	η_2	$\eta_3 = \eta$	e/η	e	η_1	η_2	$\eta_3 = \eta$	e/η
16	1.75e-1	5.49e-2	3.77e-2	4.36e-1	4.02e-1	2.24e-1	5.49e-2	3.77e-2	4.83e-1	4.64e-1
32	1.02e-1	1.47e-2	1.67e-2	2.13e-1	4.81e-1	1.31e-1	1.47e-2	1.58e-2	2.43e-1	5.40e-1
64	5.44e-2	3.78e-3	8.37e-3	1.05e-1	5.19e-1	7.06e-2	3.79e-3	7.94e-3	1.21e-1	5.82e-1
128	2.75e-2	9.61e-4	4.22e-3	5.16e-2	5.34e-1	3.65e-2	9.61e-4	4.00e-3	6.00e-2	6.08e-1

When uniform meshes are used—see Tables 6.3 and 6.4—the boundary layers are not resolved and $e = O(1)$. This is indicated by $\eta = \eta_3$ blowing up even more significantly than e. Unlike η_3 the component η_2 remains bounded. Thus both η_2 and η_3 not being small correctly indicate that the method is inaccurate. But η_2 better reflects the actual errors since $e/\eta_2 \approx \text{const} = 1.0$ in Table 6.4.

Finally we consider the Bakhvalov mesh with $\lambda = 1$; see Table 6.5. Since the condition $\lambda > 2$, which implies ε-uniform second-order accuracy for our test problem [4, 16], is violated, the errors slightly increase as $\varepsilon \to 0$. We observe that η_1 is too small compared to η and e.

In summary, for our test problem on the meshes considered, the error estimator η indicates correctly whether or not the method is ε-uniformly accurate. Furthermore, we observe that the quantity $\eta = \eta_3$ might blow up; see Table 6.4, while the component η_1 is sometimes too optimistic; see Table 6.5. The component η_2 seems the most relevant estimator for the actual error e. Besides, η_2 does not blow up, like η_3, and hence seems a suitable error indicator for *a posteriori* mesh construction.

References

[1] M. Ainsworth and I. Babuška, *Reliable and robust a posteriori error estimating for singularly perturbed reaction-diffusion problems*, SIAM J. Numer. Anal., 36 (1999), 331–353.

[2] T. Apel, *Anisotropic finite elements: Local estimates and applications*, Teubner, Stuttgart, 1999.

[3] I. Babuška and T. Strouboulis, *The finite element method and its reliability*, Clarendon Press, Oxford University Press, New York, 2001.

[4] N.S. Bakhvalov, *Towards optimization of methods for solving boundary value problems in the presence of a boundary layer*, Zh. Vychisl. Mat. i Mat. Fiz., 9 (1969), 841–859 (in Russian).

[5] I. Boglaev, *On monotone iterative methods for a nonlinear singularly perturbed reaction-diffusion problem*, J. Comput. Appl. Math., 162 (2004), 445–466.

[6] M.J. Castro-Diaz, F.F. Hetch, B. Mohammadi and O. Pironneau, *Anisotropic unstructed mesh adaptation for flow simulations*, Internat. J. Numer. Methods Fluids, 25 (1997), 475–491.

[7] L. Chen, P. Sun, and J. Xu, *Optimal anisotropic simplicial meshes for minimizing interpolation errors in L_p-norm*, Math. Comp., 76 (2007), 179–204.

[8] C. Clavero, J.L. Gracia and E. 'Riordan, *A parameter robust numerical method for a two dimensional reaction-diffusion problem*, Math. Comp., 74 (2005), 1743–1758.

[9] E.F. D'Azevedo, *Optimal triangular mesh generation by coordinate transformation*, SIAM J. Sci. Statist. Comput., 12 (1991), 755–786.

[10] E.F. D'Azevedo and R.B. Simpson, *On optimal interpolation triangle incidences*, SIAM J. Sci. Statist. Comput., 10 (1989), 1063–1075.

[11] V. Dolejší and J. Felcman, *Anisotropic mesh adaptation for numerical solution of boundary value problems*, Numer. Methods Partial Differential Equations, 20 (2004), 576–608.

[12] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, 1998.

[13] W.G. Habashi, J. Dompierre, Y. Bourgault, D. Ait-Ali-Yahia, M. Fortin and M.-G. Vallet, *Anistopic mesh adaptation: towards user independent, mesh independent and solver independent CFD. Part I: General principles*, Internat. J. Numer. Methods Fluids, 32 (2000), 725–744.

[14] N. Kopteva, *Maximum norm a posteriori error estimates for a one-dimensional convection-diffusion problem*, SIAM J. Numer. Anal., 39 (2001), 423-441.

[15] N. Kopteva, *Maximum norm a posteriori error estimates for a one-dimensional singularly perturbed semilinear reaction-diffusion problem*, IMA J. Numer. Anal., 27 (2007), 576-592.

[16] N. Kopteva, *Maximum norm error analysis of a 2d singularly perturbed semilinear reaction-diffusion problem*, Math. Comp., 76 (2007), 631–646.

[17] N. Kopteva, *Maximum norm a posteriori error estimate for a 2d singularly perturbed reaction-diffusion problem*, SIAM J. Numer. Anal., 46 (2008), 1602–1618.

[18] N. Kopteva and M. Stynes, *A robust adaptive method for a quasilinear one-dimensional convection-diffusion problem*, SIAM J. Numer. Anal., 39 (2001), 1446–1467.

[19] G. Kunert and R. Verfürth, *Edge residuals dominate a posteriori error estimates for linear finite element methods on anisotropic triangular and tetrahedral meshes*, Numer. Math., 86 (2000), 283–303.

[20] G. Kunert, *Robust a posteriori error estimation for a singularly perturbed reaction-diffusion equation on anisotropic tetrahedral meshes. A posteriori error estimation and adaptive computational methods*, Adv. Comput. Math., 15 (2001), 237–259.

[21] O.A. Ladyzhenskaya and N.N. Ural'tseva, *Linear and quasilinear elliptic equations*, Academic Press, New York, 1968.

[22] T. Linß, *Maximum-norm error analysis of a non-monotone FEM for a singularly perturbed reaction-diffusion problem*, BIT, 47 (2007), 379–391.

[23] T. Linß and M. Stynes, *The SDFEM on Shishkin meshes for linear convection-diffusion problems*, Numer. Math., 87 (2001), 457–484.

[24] J.J. H. Miller, E. O'Riordan and G.I. Shishkin, *Solution of Singularly Perturbed Problems with ε-uniform Numerical Methods — Introduction to the Theory of Linear Problems in One and Two Dimensions*, World Scientific, Singapore, 1996.

[25] R.H. Nochetto, *Pointwise a posteriori error estimates for elliptic problems on highly graded meshes*, Math. Comp., 64 (1995), 1–22.

[26] A.D. Polyanin, *Handbook of linear partial differential equations for engineers and scientists*, CRC Press, London, 2001.

[27] J. Remacle, X. Li, M.S. Shephard and J.E. Flaherty, *Anisotropic adaptive simulation of transient flows using discountinuous Galerkin methods*, Internat. J. Numer. Methods Engrg., 62 (2005), 899–923.

[28] A.H. Schatz and L.B. Wahlbin, *On the finite element method for singularly perturbed reaction-diffusion problems in two and one dimensions*, Math. Comp., 40 (1983), 47–88.

N.M. Chadha, Department of Mathematics and Statistics, University of Limerick, Limerick, Ireland

E-mail address: naresh.chadha@ul.ie

N. Kopteva, Department of Mathematics and Statistics, University of Limerick, Limerick, Ireland

E-mail address: natalia.kopteva@ul.ie