Numerical study of maximum norm a posteriori error estimates for singularly perturbed parabolic problems ⋆

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Abstract. A second-order singularly perturbed parabolic equation in one space dimension is considered. For this equation, we give computable a posteriori error estimates in the maximum norm for two semidiscretisations in time and a full discretisation using P_1 FEM in space. Both the Backward-Euler method and the Crank-Nicolson method are considered, and certain critical details of the implementation are addressed. Based on numerical results we discuss various aspects of the error estimators in particular their effectiveness.

Keywords: a posteriori error estimate, maximum norm, singular perturbation, elliptic reconstruction, backward Euler, Crank-Nicolson, parabolic equation, reaction-diffusion

1 Introduction

The authors' recent paper [4] gives certain maximum norm a posteriori error estimates for time-dependent semilinear reaction-diffusion equations in 1-3 space dimensions, applicable in both regular and singularly perturbed regimes. The purpose of the present paper is to numerically investigate the sharpness and robustness of the theoretical results [4] when applied to a relatively simple equation. Our test problem will be a singularly perturbed equation in the form

$$\mathcal{M}u := u_t + \mathcal{L}u = f \quad \text{in} \quad \Omega \times (0,T], \quad \Omega := (0,1), \quad \mathcal{L}u := -\varepsilon^2 u_{xx} + ru, \tag{1a}$$

with a small positive perturbation parameter ε and functions $r : \bar{\Omega} \to \mathbb{R}$, $r \geq \varrho^2$, $\varrho > 0$, $f : \bar{\Omega} \times [0,T] \to \mathbb{R}$, subject to the initial and Dirichlet boundary conditions

$$u(x,0) = \varphi(x) \quad \text{for} \quad x \in \bar{\Omega}, \quad u(0,t) = u(1,t) = 0 \quad \text{for} \quad t \in [0,T]. \tag{1b}$$

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Solutions to (1) typically exhibit sharp layers of width $\mathcal{O}\left(\varepsilon \ln \varepsilon^{-1}\right)$ at the two end points of the spatial domain. Interior layers may also be present depending on the right-hand side and on the initial condition. These layers form challenges for any numerical method; see [9] for an overview.

Recently much attention has been paid to the design of adaptive methods for partial differential equations that automatically adapt the discretisation to the features of the solution, see e.g. [10]. The main ingredients of such methods are reliable a posteriori error estimators.

In [4, 5] error estimators in the maximum norm for singularly perturbed parabolic problems like (1) have been derived. The crucial issue when analysing methods for such problems is to carefully monitor any dependence of constants on the perturbation parameter. In the present paper we shall numerically investigate the sharpness of the a posteriori error bounds derived in [4].

The outline of the paper is as follows. In §2 we review properties of the Green's function of (1) which are the basis for the analysis in [4]. Semidiscretisations in time are studied in §3. Both the implicit Euler method and the Crank-Nicolson method will be considered. §4 is concerned with full discretisations which are obtained by applying a FEM to the semidiscretisations. Finally, in §5 the effects of changing the spatial mesh are studied.

2 The Green's function

The main tool for deriving a posteriori error estimators in [4] is the Green's function $\mathcal{G}$ associated with the differential operator $\mathcal{M}$ of (1). It can be used to express the error of a numerical approximation in terms of its residual.

For definitions and properties of fundamental solutions and Green's functions of parabolic operators, we refer the reader to [2, Chap. 1 and §7 of Chap. 3]. Any given function v of sufficient regularity can be represented as

$$v(x,t) = \int_\Omega \mathcal{G}(x,t;\xi,0)\, v(\xi,0)\,\mathrm{d}\xi + \int_0^t \int_\Omega \mathcal{G}(x,t;\xi,s)\,\big(\mathcal{M}v\big)(\xi,s)\,\mathrm{d}\xi\,\mathrm{d}s. \qquad (2)$$

Theorem 1 ([5, Th. 2.1]). *Let $r \in C^1(\bar{\Omega})$. Assume $\varrho^2 \le r$ on $\bar{\Omega}$ with some constant $\varrho > 0$. Then, for the Green's function $\mathcal{G}$ one has*

$$\int_\Omega \left|\mathcal{G}(x,t;\xi,s)\right|\,\mathrm{d}\xi \le \mathrm{e}^{-\varrho^2(t-s)},$$

$$\int_\Omega \left|\partial_\xi^k \mathcal{G}(x,t;\xi,s)\right|\,\mathrm{d}\xi \le \frac{\gamma_k\,\mathrm{e}^{-\varrho^2(t-s)}}{\varepsilon^k (t-s)^{k/2}} + \mathcal{O}\left(\varepsilon^{k-1}\right), \quad \text{for } k = 1, 2,$$

and

$$\int_\Omega \left|\partial_s \mathcal{G}(x,t;\xi,s)\right|\,\mathrm{d}\xi \le \left(\frac{\gamma_2}{t-s} + \|r\|_\infty\right)\mathrm{e}^{-\varrho^2(t-s)} + \mathcal{O}\left(\varepsilon\right)$$

with constants $\gamma_1 = 1/\sqrt{\pi}$ and $\gamma_2 = \sqrt{2/(\pi\mathrm{e})}$.

Let $\tilde{u}$ be an approximation of the exact solution u of (1). Replacing v by $u - \tilde{u}$ in (2), we get the error representation

$$
\begin{aligned}
(u - \tilde{u})(x,t) = \int_\Omega \mathcal{G}(x,t;\xi,0)\ (\varphi - \tilde{u})(\xi,0)\,\mathrm{d}\xi \\
+ \int_0^t \int_\Omega \mathcal{G}(x,t;\xi,s)\big(f - \tilde{u}_s + \mathcal{L}\tilde{u}\big)(\xi,s)\,\mathrm{d}\xi\,\mathrm{d}s.
\end{aligned}
\tag{3}
$$

The main idea when deriving a posteriori error estimates is the use of the Hölder inequality, the L_1-norm bounds for the Green's function in Theorem 1 and maximum-norm bounds for the residuum.

In the case of the backward-Euler discretisation the approximation $\tilde{u}$ is considered to be piecewise constant in time. Therefore, $\tilde{u}$ will be discontinuous in time and $\tilde{u}_s$ has to be read in the context of distributions. Further discontinuities will occur when the spatial discretisation mesh changes between time levels.

To deal with these discontinuities, integration by parts is applied to the second integral in (3):

$$
\begin{aligned}
\int_0^t \mathcal{G}(x,t;\xi,s)\,\tilde{u}_s(\xi,s)\,\mathrm{d}s = \mathcal{G}(x,t;\xi,s)\,\tilde{u}(\xi,s)\Big|_{s=0}^{t-\tau} \\
- \int_0^{t-\tau} \mathcal{G}_s(x,t;\xi,s)\,\tilde{u}(\xi,s)\,\mathrm{d}s + \int_{t-\tau}^t \mathcal{G}(x,t;\xi,s)\,\tilde{u}_s(\xi,s)\,\mathrm{d}s.
\end{aligned}
$$

For the L_1 norm of $\mathcal{G}_s$, Theorem 1 yields the bound

$$
\int_0^t \int_\Omega |\mathcal{G}_s(x,t;\xi,s)|\,\mathrm{d}\xi\,\mathrm{d}s \le \gamma_2 \ell(\tau,t) + \bar{\varrho} + \mathcal{O}(\varepsilon), \quad 0 < \tau < t \le T,
$$

where $\ell(\tau,t) := \int_\tau^t s^{-1}\mathrm{e}^{-\varrho^2 s/2}\,\mathrm{d}s \le \ln(t/\tau)$ and $\bar{\varrho} := \varrho^{-2}\|r\|_\infty$.

3 Semidiscretisation in time

Let $\omega_t : 0 = t_0 < t_1 < t_2 < \cdots < t_M = T$, be an arbitrary nonuniform mesh in time direction with step sizes $\tau_j := t_j - t_{j-1}$ and mesh intervals $J_j := (t_{j-1}, t_j)$, $j = 1, \ldots, M$. Set $f^j := f(\cdot, t_j)$.

Given an arbitrary function $v : \omega_t \to H_0^1(\Omega) : t_j \mapsto v^j$, we introduce its standard piecewise linear interpolant

$$
(I_{1,t}v)(\cdot,t) := \frac{t_j - t}{\tau_j}\,v^{j-1} + \frac{t - t_{j-1}}{\tau_j}\,v^j \quad \text{for } t \in \bar{J}_j, \quad j = 1,\ldots,M,
$$

and the piecewise-constant interpolant

$$
(I_{0,t}v)(\cdot,t) := v^j \quad \text{for } t \in (t_{j-1}, t_j], \quad j = 1\ldots,M; \quad (I_{0,t}v)(\cdot,0) := v^1;
$$

so $I_{0,t}v$ is continuous on $[t_0, t_1]$. Furthermore, introduce the difference quotient

$$
\delta_t v^j := \frac{v^j - v^{j-1}}{\tau_j}
$$

as an approximation of the first-order time derivative.

3.1 Backward Euler

We associate an approximate solution $U^j \in H_0^1(\Omega)$ with the time level t_j and require it to satisfy

$$\delta_t U^j + \mathcal{L}U^j = f^j \quad \text{in } \Omega, \quad j = 1, \ldots, M; \quad U^0 = \varphi, \tag{4}$$

Using (3) with $\tilde{u} = I_{0,t}U$, the following a posteriori error estimate was obtained in [4].

Theorem 2. *For $m = 1, \ldots, M$, the maximum-norm error satisfies*

$$\left\| U^m - u(\cdot, t_m) \right\|_{\infty,\Omega} \leq \eta^{\text{bE}} := \eta^{\text{bE}}_{\text{osc}} + \eta^{\text{bE}}_{\text{t}} + \eta^{\text{bE}}_{\text{t},*} \tag{5}$$

with

$$\eta^{\text{bE}}_{\text{osc}} := \sum_{j=1}^{m} \tau_j \mathrm{e}^{-\varrho^2(t_m - t_j)} \left\| f - I_{0,t} f \right\|_{\infty,\Omega\times J_j}, \quad \eta^{\text{bE}}_{\text{t},*} := 2\tau_m \left\| \delta_t U^m \right\|_{\infty,\Omega},$$

$$\eta^{\text{bE}}_{\text{t}} := \left(\gamma_2 \ln \frac{t_m}{\tau_m} + \bar{\varrho} + \mathcal{O}(\varepsilon) \right) \max_{j=1,\ldots,m-1} \tau_j \left\| \delta_t U^j \right\|_{\infty,\Omega}.$$

Remark 1. In practice, for a singularly perturbed problem the $\mathcal{O}(\varepsilon)$ term is small (compared to $\bar{\varrho}$). Therefore, it will be neglected.

The term $\eta^{\text{bE}}_{\text{osc}}$ captures the data oscillations. Therefore, it cannot be evaluated exactly and needs to be approximated. In our experiments this is done as follows:

$$\left\| f - I_{\nu,t} f \right\|_{\infty,\Omega\times J_j} \approx \max_{k=0,\ldots,3} \left\| (f - I_{\nu,t} f)(\cdot, t_{j-1} + k\tau_j/4) \right\|_{\infty,\Omega}, \quad \nu = 0, 1, \tag{6}$$

i.e., the difference between the right-hand side f and it piecewise constant (and later linear) interpolant is sampled at 4 equally spaced points per time interval.

We present numerical results for the following *test problem*:

$$\begin{aligned} u_t - \varepsilon^2 u_{xx} + (1 + x)u &= 1 - \cos 10xt^2, \quad \text{in } \Omega \times (0, T], \\ u(x, 0) = \sin \pi x, \quad x \in [0, 1], \quad & u(0, t) = u(1, t) = 0, \quad t \in (0, T], \end{aligned} \tag{7}$$

with $\varepsilon = 10^{-6}$. This is a sufficiently small value to bring out the singular-perturbation nature of the problem. The exact solution is not available. Instead the true errors are approximated by means of a numerical solution on a very fine layer-adapted mesh. Errors arising from the spatial discretisation can be neglected.

In Table 1 we present results for the semi-discretisation error at final time $T = 1$ and compare it with the a posteriori error estimator of Theorem 2. In time we use a mesh with M mesh intervals and varying step sizes:

$$\tau_j = \begin{cases} \frac{2}{3M} & \text{if } j \text{ is odd}, \\ \frac{4}{3M} & \text{if } j \text{ is even}. \end{cases}$$

M	$\|u-U\|_\infty$	rate	η^{bE}	$\frac{C_{\mathrm{eff}}}{\ln(1/\tau_M)}$	$\eta^{\mathrm{bE}}_{\mathrm{osc}}$	$\eta^{\mathrm{bE}}_{\mathrm{t}}$	$\eta^{\mathrm{bE}}_{\mathrm{t},*}$
2^{11}	1.401e-4	1.00	3.793e-3	2.35	1.091e-4	*2.767e-3*	9.171e-4
2^{12}	7.006e-5	1.00	1.981e-3	2.46	5.455e-5	*1.468e-3*	4.586e-4
2^{13}	3.503e-5	1.00	1.032e-3	2.56	2.727e-5	*7.758e-4*	2.293e-4
2^{14}	1.751e-5	1.00	5.371e-4	2.67	1.363e-5	*4.088e-4*	1.147e-4
2^{15}	8.757e-6	1.00	2.790e-4	2.77	6.817e-6	*2.149e-4*	5.733e-5
2^{16}	4.379e-6	—	1.447e-4	2.88	3.408e-6	*1.127e-4*	2.867e-5

Table 1. Semidiscretisation by the backward Euler method, $\varepsilon = 10^{-6}$.

The table contains the maximum errors at time $T = 1$, the error bounds obtained by the error estimator η^{bE}, the efficiency index

$$C_{\mathrm{eff}} := \eta^{\mathrm{bE}} \Big/ \left\|U^M - u(\cdot, T)\right\|_{\infty,\Omega}$$

and the various parts of the error estimator. The dominant term $\eta^{\mathrm{bE}}_{\mathrm{t}}$ in the estimator is highlighted in the table. It does not converge with first order because of the presence of the $\ln(1/\tau_m)$ term (which also appears in [5]). Also, note that the efficiency slightly deteriorates with increasing M. C_{eff} is approximately proportional to $\ln(1/\tau_M)$. We conjecture that the factor $\ln(1/\tau_m)$ appearing in $\eta^{\mathrm{bE}}_{\mathrm{t},*}$ is merely an artifact of the analysis. Apart from this the estimator is quite efficient with $\frac{C_{\mathrm{eff}}}{\ln(1/\tau_M)} \approx 2.5 \ldots 3.0$.

3.2 Crank-Nicolson

An approximate solution $U^j \in H^1_0(\Omega)$ is associated with the time level t_j. It satisfies

$$\delta_t U^j + \frac{\mathcal{L}U^j + \mathcal{L}U^{j-1}}{2} = \frac{f^{j-1} + f^j}{2} \quad \text{in } \Omega \quad j = 1, \ldots, M; \quad U^0 = \varphi.$$

For the Crank-Nicolson method the following error bound is given in [4]:

$$\left\|U^m - u(\cdot, t_m)\right\|_{\infty,\Omega} \le \eta^{\mathrm{CN}} := \eta^{\mathrm{CN}}_{\mathrm{osc}} + \eta^{\mathrm{CN}}_{\mathrm{t}} + \eta^{\mathrm{CN}}_{\mathrm{t},*} \tag{8}$$

where

$$\eta^{\mathrm{CN}}_{\mathrm{osc}} := \sum_{j=1}^{m} \tau_j \mathrm{e}^{-\varrho^2(t_m - t_j)} \left\|f - I_{1,t} f\right\|_{\infty,\Omega\times J_j}, \quad \eta^{\mathrm{CN}}_{\mathrm{t},*} := \frac{5\tau_m}{8} \left\|\delta_t \psi^m\right\|_{\infty,\Omega},$$

$$\eta^{\mathrm{CN}}_{\mathrm{t}} := \frac{1}{8}\Big(\gamma_2 \ln \frac{t_m}{\tau_m} + \bar{\varrho} + \mathcal{O}(\varepsilon)\Big) \max_{j=1,\ldots,m-1} \tau_j \left\|\delta_t \psi^j\right\|_{\infty,\Omega}$$

and $\psi^j := \mathcal{L}U^j - f(\cdot, t_j)$.

The term $\eta^{\mathrm{CN}}_{\mathrm{osc}}$ captures the data oscillations and needs to be approximated. This is done by means of (6). The results of our test computations can be found in Table 2. They are in aggrement with the theoretical results. Again, we have highlighted the dominant term of the estimator in the table. Its second order convergence is affected of the presence of the $\ln(1/\tau_m)$ term.

M	$\|u-U\|_\infty$	rate	η^{CN}	$\frac{C_{\mathrm{eff}}}{\ln(1/\tau_M)}$	$\eta_{\mathrm{osc}}^{\mathrm{CN}}$	$\eta_{\mathrm{t}}^{\mathrm{CN}}$	$\eta_{\mathrm{t},*}^{\mathrm{CN}}$
2^7	4.108e-6	2.00	2.231e-4	6.23	2.486e-5	*9.239e-5*	1.058e-4
2^8	1.028e-6	2.00	5.755e-5	6.42	6.189e-6	*2.494e-5*	2.642e-5
2^9	2.570e-7	2.00	1.484e-5	6.62	1.544e-6	*6.695e-6*	6.600e-6
2^{10}	6.427e-8	2.00	3.824e-6	6.82	3.856e-7	*1.789e-6*	1.649e-6
2^{11}	1.607e-8	2.00	9.846e-7	7.02	9.635e-8	*4.760e-7*	4.123e-7
2^{12}	4.018e-9	—	2.533e-7	7.23	2.408e-8	*1.262e-7*	1.031e-7

Table 2. Semidiscretisation by the Crank-Nicolson method, $\varepsilon = 10^{-6}$.

4 Full discretisations

In this section we describe our results for full discretisations of the parabolic problem (1). To this end we apply piecewise linear P_1 finite elements to the semidiscrete backward Euler and Crank-Nicolson methods.

4.1 The spatial discretisation

Consider a steady-state version of the abstract parabolic problem (1):

$$\mathcal{L}v = -\varepsilon^2 v_{xx} + rv = g \quad \text{in } \Omega, \quad v(0) = v(1) = 0, \tag{9}$$

with $0 < \varepsilon \ll 1$ and $r \geq \varrho^2$ on Ω, $\varrho > 0$. The corresponding variational formulation is: Find $u \in H_0^1(0,1)$ such that

$$a(u,v) := \varepsilon^2 \langle u_x, v_x \rangle + \langle ru, v \rangle = \langle f, v \rangle \quad \forall\, w \in H_0^1(\Omega),$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in $L_2(\Omega)$.

An approximate solution of (9) is obtained by means of the P_1-Galerkin FEM. Let V_h be the space of continuous piecewise-linear finite element functions on an arbitrary nonuniform mesh $\bar{\omega}_x = \{x_i\}_{i=0}^N$ with $0 = x_0 < x_1 < \cdots < x_N = 1$, $h_i := x_i - x_{i-1}$ and $I_i := (x_{i-1}, x_i)$. Note that here we make absolutely no mesh regularity assumptions. As solutions of our problem typically exhibit sharp layers so a suitable mesh is expected to be highly-nonuniform; see, e.g., [7].

Our discretisation of (9) is: Find $v_h \in \mathring{V}_h := V_h \cap H_0^1(\Omega)$ such that

$$a_{V_h}(v_h, w_h) := \varepsilon^2 \left\langle v_h', w_h' \right\rangle + \left\langle r v_h, w_h \right\rangle_{V_h} = \left\langle g, w_h \right\rangle_{V_h} \quad \forall\, w_h \in \mathring{V}_h, \tag{10}$$

where $\langle \psi, w \rangle_{V_h} := \langle I_{1,x}\psi, w \rangle$ with the standard piecewise-linear nodal interpolation $I_{1,x} : C(\bar{\Omega}) \to V_h$. For the resulting FEM we cite the following a posteriori error bound from [7].

Theorem 3. *Let v be the solution of* (9) *and v_h its finite element approximation defined by* (10)*. Then the maximum-norm error satisfies.*

$$\|v - v_h\|_{\infty,\Omega} \leq \eta_\varepsilon\big(V_h, rv_h - f\big)$$

with the a posteriori error estimator

$$\eta_\varepsilon\big(V_h, q\big) := \max_{i=1,\dots,N} \left\{ \frac{h_i^2}{4\varepsilon^2} \left\| I_{1,x} q \right\|_{\infty, I_i} \right\} + \varrho^{-2} \left\| q - I_{1,x} q \right\|_{\infty,\Omega}. \tag{11}$$

4.2 Fully discrete backward Euler method

A spatial mesh $\bar{\omega}_x^j : 0 = x_0^j < x_1^j < \cdots < x_{N_j}^j = 1$, a finite-element space V_h^j of piecewise-linear functions and a computed solution $u_h^j \in \mathring{V}_h^j := V_h^j \cap H_0^1(\Omega)$ are associated with the time level t_j. By $I_{1,x}^j : C(\bar{\Omega}) \to V_h^j$ we denote the nodal interpolation in V_h^j and by $P_h^j : L^2(\Omega) \to V_h^j$ the L_2 projection onto V_h^j. Given the computed solution u_h^j, we set $\hat{u}_h^{j-1} := P^j u_h^{j-1}$ and

$$\delta_t^* u_h^j := \frac{u_h^j - \hat{u}_h^{j-1}}{\tau_j}.$$

Note that $\hat{u}_h^{j-1} = u_h^{j-1}$ if $V_h^{j-1} \subset V_h^j$, i.e. when the mesh is purely refined. Otherwise, when parts of the mesh are coarsend, one typically has $\hat{u}_h^{j-1} \neq u_h^{j-1}$.

We apply the FEM (10) to (4) and obtain the full discretisation: Find $u_h^j \in \mathring{V}_h^j$, $j = 0, \ldots, M$, such that

$$\left\langle \delta_t^* u_h^j, w_h \right\rangle + a_{V_h^j}\left(u_h^j, w_h\right) = \left\langle f^j, w_h \right\rangle_{V_h} \quad \forall\, w_h \in \mathring{V}_h^j,\ j = 1, \ldots, M, \tag{12}$$

with some initial value u_h^0, for example $u_h^0 = I_{1,x}^0 \varphi$.

Elliptic reconstruction. For each time level t_j, $j = 1, \ldots, M$, we follow an idea from [8] and introduce the *elliptic reconstruction* $R^j \in H_0^1(\Omega)$ of u_h^j, which is uniquely defined by

$$a(R^j, w) = \langle f^j - \delta_t^* u_h^j, w \rangle \quad \forall\, w \in H_0^1(\Omega). \tag{13}$$

In view of (12), u_h^j can be interpreted as the finite-element approximation of R^j obtained by (10). Therefore, Theorem 3 applies and yields

$$\left\| u_h^j - R^j \right\|_{\infty,\Omega} \le \eta^j := \eta_\varepsilon\left(V_h^j, q^j\right), \qquad j = 1, \ldots, M, \tag{14a}$$

with

$$q^j := r u_h^j - f^j + \delta_t^* u_h^j, \tag{14b}$$

Remark 2. (i) The second term in the error estimator η_ε, see (11), simplifies to

$$\varrho^{-2} \left\| q^j - I_{1,x} q^j \right\|_{\infty,\Omega} = \varrho^{-2} \left\| r u_h^j - f^j - I_{1,x}\left(r u_h^j - f^j\right) \right\|_{\infty,\Omega},$$

because $I_{1,x}^j \delta_t^* u_h^j = \delta_t^* u_h^j$.

(ii) The first term of η_ε requires to evaluate q^j in the mesh nodes x_i^j. For small ε, its evaluation using (14b) is numerically unstable because rounding errors are amplified. A stable alternative is to determine $q^j \in V_h^j$ such that $q^j(0) = f^j(0)$, $q^j(1) = f^j(1)$ and

$$\left\langle q^j, v_h \right\rangle_{V_h^j} = -\varepsilon^2 \left\langle u_{h,x}^j, v_{h,x} \right\rangle \quad \forall\, v_h \in \mathring{V}_h^j.$$

This requires to invert the standard mass matrix.

A posteriori estimator for the parabolic problem. Consider the error at time t_m. By the triangle inequality, we have

$$\begin{aligned}\|u_h^m - u(\cdot,t_m)\|_{\infty,\Omega} &\le \|u_h^m - R^m\|_{\infty,\Omega} + \|R^m - u(\cdot,t_m)\|_{\infty,\Omega} \\ &\le \eta^m + \|R^m - u(\cdot,t_m)\|_{\infty,\Omega} .\end{aligned}$$

The difference $R^m - u(\cdot,t_m)$ is represented using (3) with $\tilde{u} = I_{0,t}R$. The reconstruction R can be completely eliminated using (13) and (14a). We arrive at the following a posteriori error bound [4].

Theorem 4. *For $m = 1,\dots,M$, the maximum-norm error satisfies*

$$\left\|u_h^m - u(\cdot,t_m)\right\|_{\infty,\Omega} \le \eta^{\mathrm{bE}} := \eta_{\mathrm{init}} + \eta_{\mathrm{osc}}^{\mathrm{bE}} + \eta_{\mathrm{proj}} + \eta_{\mathrm{t}}^{\mathrm{bE}} + \eta_{\mathrm{t},*}^{\mathrm{bE}} + \eta_{\mathrm{ell}}^{\mathrm{bE}} + \eta_{\mathrm{ell},*}^{\mathrm{bE}}$$

with

$$\eta_{\mathrm{init}} := \mathrm{e}^{-\gamma^2 t_m}\left\|u_h^0 - \varphi\right\|_{\infty,\Omega}, \quad \eta_{\mathrm{osc}}^{\mathrm{bE}} := \sum_{j=1}^{m} \tau_j \mathrm{e}^{-\varrho^2(t_m-t_j)}\left\|f - I_{0,t}f\right\|_{\infty,\Omega\times J_j},$$

$$\eta_{\mathrm{proj}} := \sum_{j=1}^{m} \mathrm{e}^{-\gamma^2(t_m-t_j)}\left\|\hat{u}_h^{j-1} - u_h^{j-1}\right\|_{\infty,\Omega}, \quad \eta_{\mathrm{t},*}^{\mathrm{bE}} := 2\tau_m\left\|\delta_t^* U^m\right\|_{\infty,\Omega},$$

$$\eta_{\mathrm{t}}^{\mathrm{bE}} := \left(\gamma_2 \ln\frac{t_m}{\tau_m} + \bar{\varrho} + \mathcal{O}(\varepsilon)\right) \max_{j=1,\dots,m-1} \tau_j\left\|\delta_t^* U^j\right\|_{\infty,\Omega},$$

$$\eta_{\mathrm{ell}}^{\mathrm{bE}} := \left(\gamma_2 \ln\frac{t_m}{\tau_m} + \bar{\varrho} + \mathcal{O}(\varepsilon)\right) \max_{j=1,\dots,m-1} \eta^j \quad \text{and} \quad \eta_{\mathrm{ell},*}^{\mathrm{bE}} := 2\eta^m.$$

Remark 3. Comparing with Theorem 2, we notice four new terms.

- η_{init}: the error in approximating the initial condition,
- η_{proj}: the accumulated erros due to projections when the mesh is coarsend,
- $\eta_{\mathrm{ell}}^{\mathrm{bE}}$ and $\eta_{\mathrm{ell},*}^{\mathrm{bE}}$: elliptic error estimates for the spatial discretisation.

Numerical results. In order of balancing the accuracy in space and time, we use a Bakhvalov mesh with $N = [\sqrt{8M}]$ mesh points in space. We do so, because the method is formally 1st order in time and 2nd order in space. The Bakhvalov mesh [1] is given by

$$x_i^j = x_i = \mu(\xi_i), \quad \xi_i = i/N$$

with the mesh generating function

$$\mu(\zeta) = \begin{cases} \vartheta(\zeta) := \dfrac{\sigma\varepsilon}{\varrho} \ln\dfrac{\alpha}{\alpha-\zeta} & \zeta \in [0,\zeta^*], \\ \vartheta(\zeta^*) + \vartheta'(\zeta^*)(\zeta-\zeta^*) & \zeta \in [\zeta^*,1/2], \\ 1 - \mu(1-\zeta) & \zeta \in [1/2,1]. \end{cases}$$

M	N	error η^{bE}	rate $\frac{C_{\mathrm{eff}}}{\ln(1/\tau_M)}$	η_{init} $\eta_{\mathrm{osc}}^{\mathrm{bE}}$	rate rate	$\eta_{\mathrm{t}}^{\mathrm{bE}}$ $\eta_{\mathrm{ell}}^{\mathrm{bE}}$	rate rate	$\eta_{\mathrm{t},*}^{\mathrm{bE}}$ $\eta_{\mathrm{ell},*}^{\mathrm{bE}}$	rate rate
2^{11}	362	1.401e-4	1.00	2.283e-5	1.00	*2.767e-3*	*0.91*	9.171e-4	1.00
		1.062e-2	6.59	1.091e-4	1.00	*4.998e-3*	*0.92*	1.803e-3	1.00
2^{12}	512	7.006e-5	1.00	1.142e-5	1.00	*1.468e-3*	*0.92*	4.586e-4	1.00
		5.534e-3	6.87	5.455e-5	1.00	*2.643e-3*	*0.92*	8.985e-4	1.00
2^{13}	724	3.503e-5	1.00	5.709e-6	1.00	*7.758e-4*	*0.92*	2.293e-4	1.00
		2.881e-3	7.15	2.727e-5	1.00	*1.394e-3*	*0.93*	4.483e-4	1.00
2^{14}	1024	1.751e-5	1.00	2.854e-6	1.00	*4.088e-4*	*0.93*	1.147e-4	1.00
		1.497e-3	7.44	1.363e-5	1.00	*7.335e-4*	*0.93*	2.238e-4	1.00
2^{15}	1448	8.757e-6	1.00	1.427e-6	1.00	*2.149e-4*	*0.93*	5.733e-5	1.00
		7.775e-4	7.72	6.817e-6	1.00	*3.852e-4*	*0.93*	1.118e-4	1.00
2^{16}	2048	4.379e-6	—	7.135e-7	—	*1.127e-4*	—	2.867e-5	—
		4.031e-4	8.01	3.408e-6	—	*2.018e-4*	—	5.584e-5	—

Table 3. Backward Euler and linear FEM, $\varepsilon = 10^{-6}$.

The transition point ζ^* satisfies $(1-2\zeta^*)\vartheta'(\zeta^*) = 1 - 2\vartheta(\zeta^*)$ which implies $\mu \in C^1[0,1]$. For the mesh parameters are chosen we take $\sigma = 4$ and $\alpha = 1/4$.

Table 3 displays the results of our test computations for (7). It contains the error at final time $T = 1$, the a posteriori error estimator, the efficiency index and the various components of the error estimator together with their respective rate of convergence.

While the results are in agreement with Theorem 4, we observe that the terms $\eta_{\mathrm{t}}^{\mathrm{bE}}$ and $\eta_{\mathrm{ell}}^{\mathrm{bE}}$ (highlighted in the table) dominate and converge slower than all other terms. This is because of the factor $\ln \frac{t_m}{\tau_m}$ in their definition. As for the semidiscretisation we conjecture that this factor is an artifact of the error analysis in [4].

Note that $\eta_{\mathrm{proj}} \equiv 0$ because the mesh does not change with time. The effect of mesh adaptivity will be discussed in more detail in §5.

In Table 4 we present computational results for a uniform mesh in space. The method does not converge. This has to be expected because the mesh is not adapted to the layer structure. Examining the various terms of the error estimator, we see that the terms $\eta_{\mathrm{ell}}^{\mathrm{bE}}$ and $\eta_{\mathrm{ell},*}^{\mathrm{bE}}$ dominate. Thus, the source of the bad behaviour is correctly attributed to a wrong spatial resolution.

4.3 Fully discrete Crank-Nicolson method

With each time level t_j, $j = 0, \dots, M$, we associate an approximation $u_h^j \in \mathring{V}_h^j$ of $u(\cdot, t_j)$ that satisfies

$$\left\langle \delta_t^* u_h^j, w_h \right\rangle + \tfrac{1}{2} a_{V_h^j}\left(u_h^j + \hat{u}_h^{j-1}, w_h\right) = \tfrac{1}{2}\left\langle f^j + f^{j-1}, w_h \right\rangle_{V_h} \quad \forall\, w_h \in \mathring{V}_h^j, \qquad j = 1, \dots, M,$$

with some initial value u_h^0, e.g., $u_h^0 = I_{1,x}^0 \varphi$.

M	N	error	rate	η_{init}	rate	$\eta_{\mathrm{t}}^{\mathrm{bE}}$	rate	$\eta_{\mathrm{t},*}^{\mathrm{bE}}$	rate
		η^{bE}	$\frac{C_{\mathrm{eff}}}{\ln(1/\tau_M)}$	$\eta_{\mathrm{osc}}^{\mathrm{bE}}$	rate	$\eta_{\mathrm{ell}}^{\mathrm{bE}}$	rate	$\eta_{\mathrm{ell},*}^{\mathrm{bE}}$	rate
2^{11}	362	5.371e-2	0.00	5.710e-6	1.00	3.208e-3	0.91	1.157e-3	1.00
		33.98	55.03	1.091e-4	1.00	24.97	-0.09	9.006	0.00
2^{12}	512	5.363e-2	0.00	2.854e-6	1.00	1.705e-3	0.92	5.796e-4	1.00
		35.50	57.58	5.455e-5	1.00	26.49	-0.08	9.006	0.00
2^{13}	724	5.359e-2	0.00	1.428e-6	1.00	9.021e-4	0.92	2.901e-4	1.00
		37.01	60.08	2.727e-5	1.00	28.01	-0.08	9.006	0.00
2^{14}	1024	5.357e-2	0.00	7.136e-7	1.00	4.758e-4	0.93	1.451e-4	1.00
		38.53	62.57	1.363e-5	1.00	29.52	-0.07	9.006	0.00
2^{15}	1448	5.356e-2	0.00	3.569e-7	1.00	2.502e-4	0.93	7.261e-5	1.00
		40.04	65.03	6.817e-6	1.00	31.03	-0.07	9.006	0.00
2^{16}	2048	5.355e-2	—	1.784e-7	—	1.312e-4	—	3.632e-5	—
		41.55	67.50	3.408e-6	—	32.54	—	9.006	—

Table 4. Backward Euler and linear FEM, $\varepsilon = 10^{-6}$, uniform mesh.

Using elliptic reconstructions and piecewise *linear* interpolation in time, the following a posteriori error bound was derived in [4].

Theorem 5. *For $m = 1, \dots, M$, the maximum-norm error satisfies*

$$\left\|u_h^m - u(\cdot, t_m)\right\|_{\infty,\Omega} \le \eta^{\mathrm{CN}} := \eta_{\mathrm{init}} + \eta_{\mathrm{osc}}^{\mathrm{CN}} + \eta_{\mathrm{proj}} + \eta_{\mathrm{t}}^{\mathrm{CN}} + \eta_{\mathrm{t},*}^{\mathrm{CN}} + \eta_{\mathrm{ell}}^{\mathrm{CN}} + \eta_{\mathrm{ell},*}^{\mathrm{CN}}$$

with η_{init} and η_{proj} as in Theorem 4, and

$$\eta_{\mathrm{osc}}^{\mathrm{CN}} := \sum_{j=1}^{m} \tau_j \mathrm{e}^{-\varrho^2(t_m - t_j)} \left\|f - I_{1,t} f\right\|_{\infty,\Omega\times J_j}, \quad \eta_{\mathrm{t},*}^{\mathrm{CN}} := \frac{5\tau_m^2}{8} \left\|\delta_t^* \psi^m\right\|_{\infty,\Omega},$$

$$\eta_{\mathrm{t}}^{\mathrm{CN}} := \frac{1}{8}\Big(\gamma_2 \ln \frac{t_m}{\tau_m} + \bar{\varrho} + \mathcal{O}(\varepsilon)\Big) \max_{j=1,\dots,m-1} \tau_j^2 \left\|\delta_t^* \psi^j\right\|_{\infty,\Omega},$$

$$\eta_{\mathrm{ell}}^{\mathrm{CN}} := 2\Big(\gamma_2 \ln \frac{t_m}{\tau_m} + \bar{\varrho} + \mathcal{O}(\varepsilon)\Big) \max_{j=1,\dots,m-1} \eta^j \quad \text{and} \quad \eta_{\mathrm{ell},*}^{\mathrm{CN}} := 5\eta^m,$$

where $\psi^j, \hat{\psi}^{j-1} \in \mathring{V}_h^j$ solve

$$\left\langle \psi^j, w_h \right\rangle_{V_h^j} = a_{V_h^j}\left(u_h^j, w_h\right) - \left\langle f^j, w_h \right\rangle_{V_h^j} \quad \forall\, w_h \in V_h^j, \tag{15a}$$

$$\left\langle \hat{\psi}^{j-1}, w_h \right\rangle_{V_h^j} = a_{V_h^j}\left(\hat{u}_h^{j-1}, w_h\right) - \left\langle f^{j-1}, w_h \right\rangle_{V_h^j} \quad \forall\, w_h \in V_h^j. \tag{15b}$$

Furthermore,

$$\eta^j := \max\left\{\eta_\varepsilon\big(V_h^j, r u_h^j - f^j + \psi^j\big), \eta_\varepsilon\big(V_h^j, r\hat{u}_h^{j-1} - f^{j-1} + \hat{\psi}^{j-1}\big)\right\}.$$

Remark 4. The evaluation of the error estimator requires the solutions of the two auxiliary problems (15a) and (15b). With regard to the numerical stability of computing η, Remark 2(ii) applies.

M	N	error η^{CN}	rate $\frac{C_{\mathrm{eff}}}{\ln(1/\tau_M)}$	η_{init} $\eta_{\mathrm{osc}}^{\mathrm{CN}}$	rate rate	$\eta_{\mathrm{t}}^{\mathrm{CN}}$ $\eta_{\mathrm{ell}}^{\mathrm{CN}}$	rate rate	$\eta_{\mathrm{t},*}^{\mathrm{CN}}$ $\eta_{\mathrm{ell},*}^{\mathrm{CN}}$	rate rate
2^7	2^{11}	1.362e-5	1.99	7.135e-7	2.00	*9.239e-5*	*1.89*	1.058e-4	2.00
		5.938e-4	5.00	2.486e-5	2.01	*2.304e-4*	*1.88*	1.396e-4	2.00
2^8	2^{12}	3.421e-6	2.00	1.784e-7	2.00	*2.494e-5*	*1.90*	2.642e-5	2.00
		1.554e-4	5.21	6.189e-6	2.00	*6.277e-5*	*1.89*	3.486e-5	2.00
2^9	2^{13}	8.570e-7	2.00	4.459e-8	2.00	*6.695e-6*	*1.90*	6.600e-6	2.00
		4.052e-5	5.42	1.544e-6	2.00	*1.692e-5*	*1.90*	8.712e-6	2.00
2^{10}	2^{14}	2.145e-7	2.00	1.115e-8	2.00	*1.789e-6*	*1.91*	1.649e-6	2.00
		1.054e-5	5.64	3.856e-7	2.00	*4.532e-6*	*1.91*	2.177e-6	2.00
2^{11}	2^{15}	5.365e-8	2.00	2.787e-9	2.00	*4.760e-7*	*1.92*	4.123e-7	2.00
		2.739e-6	5.85	9.635e-8	2.00	*1.207e-6*	*1.91*	5.443e-7	2.00
2^{12}	2^{16}	1.342e-8	—	6.967e-10	—	*1.262e-7*	—	1.031e-7	—
		7.103e-7	6.07	2.408e-8	—	*3.202e-7*	—	1.361e-7	—

Table 5. Crank-Nicolson and linear FEM, $\varepsilon = 10^{-6}$.

Numerical results for our test problem (7) are contained in Table 5. The Crank-Nicolson method with linear FEM in space is formally 2nd order both in space and in time. Therefore, N should be chosen proportional to M. We have chosen $N = 16M$ to balance the accuracy. The results are in agreement with our theoretical findings. Again they suggest that the logarithmic term is a mere artifact of the error analysis.

5 Mesh adaptivity, projection errors

So far we have considered discretisations the spatial mesh remains unchanged while we integrate in time. In this final section of the paper we will investigate some effects of changing the spatial discretisation.

A typical approach in mesh adaptivity is enrichment of the spatial discretisation by adding mesh points whenever required. At stages when the discretisation becomes too big it is reduced by removing mesh points where they are not needed anymore. Thus, typically the refinement steps outnumbers the coarsening steps. There are two adavantage of this approach

- no projection errors are introduced during refinement and
- the L_2 projection has to be computed only when the mesh is coarsened.

We model this strategy in our next experiment.

Starting from a Bakhvalov mesh with $2N$ mesh points we coarsen the mesh at time $T/8$ by removing every other mesh point. At time $T/4$ we switch back the mesh with $2N$ points then coarsen again at time $3T/8$ etc. Table 6 gives the results for the backward Euler method, $N = [\sqrt{8M}]$. Again, the estimator predicts the actual errors very well.

M	error η^{bE}	rate $\frac{C_{\mathrm{eff}}}{\ln(1/\tau_M)}$	η_{init} $\eta_{\mathrm{osc}}^{\mathrm{bE}}$	rate rate	$\eta_{\mathrm{t}}^{\mathrm{bE}}$ $\eta_{\mathrm{ell}}^{\mathrm{bE}}$	rate rate	$\eta_{\mathrm{t},*}^{\mathrm{bE}}$ $\eta_{\mathrm{ell},*}^{\mathrm{bE}}$	rate rate	$\eta_{\mathrm{proj}}^{\mathrm{bE}}$	rate
2^{11}	2.577e-4	0.91	5.709e-6	1.00	2.767e-3	0.91	9.171e-4	1.00	2.765e-4	0.98
	9.609e-3	3.24	1.091e-4	1.00	4.998e-3	0.51	5.357e-4	0.97		
2^{12}	1.369e-4	0.93	2.854e-6	1.00	1.468e-3	0.92	4.586e-4	1.00	1.401e-4	0.99
	5.914e-3	3.76	5.455e-5	1.00	3.516e-3	1.33	2.744e-4	0.97		
2^{13}	7.183e-5	0.95	1.427e-6	1.00	7.758e-4	0.92	2.293e-4	1.00	7.069e-5	0.99
	2.638e-3	3.20	2.727e-5	1.00	1.394e-3	0.93	1.396e-4	0.98		
2^{14}	3.711e-5	0.97	7.135e-7	1.00	4.088e-4	0.93	1.147e-4	1.00	3.556e-5	0.99
	1.378e-3	3.23	1.363e-5	1.00	7.335e-4	0.93	7.065e-5	0.99		
2^{15}	1.900e-5	0.97	3.568e-7	1.00	2.149e-4	0.93	5.733e-5	1.00	1.786e-5	1.00
	7.181e-4	3.29	6.817e-6	1.00	3.852e-4	0.93	3.564e-5	0.99		
2^{16}	9.708e-6	—	1.784e-7	—	1.127e-4	—	2.867e-5	—	8.958e-6	—
	3.736e-4	3.35	3.408e-6	—	2.018e-4	—	1.795e-5	—		

Table 6. Euler method on two nested Bakhvalov meshes

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