Uniform Convergence with Respect to a Small Parameter of a Scheme with Central Difference on Refining Grids

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Abstract—For a singularly perturbed one-dimensional time-independent divergence equation of diffusion-convection, a scheme is analyzed that approximates the first-order derivative by the central difference. It is proved that this scheme is uniformly convergent with respect to a small parameter in the difference norm $L^1_{\infty}$ on the Bakhvalov and Shishkin grids refined in the boundary layer; the convergence rate is $O(N^{-1})$ and $O(N^{-1} \ln^2 N)$, respectively, where $N$ is the number of grid points. The smoothness condition on the Bakhvalov grid is replaced by a weaker condition.

1. INTRODUCTION

Consider the simplest boundary value problem for the ordinary second-order differential equation:

$$Lu = -\varepsilon(p(x)u')' - (r(x)u)' + q(x)u = f(x), \quad 0 < x < 1, \quad u(0) = g_0, \quad u(1) = g_1,$$

(1.1)

where

$$p(x) \geq p_0 = \text{const} > 0, \quad r(x) \geq r_0 = \text{const} > 0, \quad q(x) \geq 0,$$

(1.2)

and $\varepsilon \in (0, 1]$ is a small parameter.

It is well known [1] that for small $\varepsilon$ a so-called boundary layer is formed in a neighborhood of the point $x = 0$, where the solution $u(x)$ to problem (1.1), (1.2) changes rapidly and its derivatives are not uniformly bounded in $\varepsilon$. Due to this fact, the accuracy of classic numerical methods [2], which do not take into account the presence of the small parameter, becomes dependent not only on the grid step, but also on the value of the small parameter. For this reason, to achieve high accuracy, grids with a great number of points must be used, because the solution found with the use of a coarse grid often has nothing in common with the solution to the initial problem.

For this reason, special methods have been developing for solving singularly perturbed problems [2-4]. These methods are uniformly convergent with respect to the small parameter, and their accuracy depends only on the number of points in the grid and is independent of the magnitude of the small parameter. One possible way of constructing such methods is in using classical difference schemes on a grid that is refining in the boundary layer. This approach was suggested and implemented by N.S. Bakhvalov in [5] for solving second-order differential equations without the terms with the first-order derivative. As applied to problem (1.1), (1.2), the Bakhvalov grid is as follows:

$$x_i = x(i/N), \quad i = 0, 1, ..., N,$$

(1.3)

where $x(t)$ is a continuous function defined as

$$x(t) = \begin{cases} \varepsilon C \ln(b/(b-t)), & 0 \leq t \leq \varepsilon_0, \\ 1 - d/(1-t), & \varepsilon_0 \leq t \leq 1, \\ \tau, & \varepsilon > \varepsilon_0, \end{cases}$$

(1.4)

$$d = d(\varepsilon) = (1 - \varepsilon C \ln(b/(b-\varepsilon_0)))/(1 - \varepsilon_0),$$

where $C$, $0 < \Theta < b < 1$, and $\varepsilon_0 \leq b/C$ are constants. In [6], the grid proposed in [5] was adapted for solving problems of type (1.1), (1.2). For the numerical method developed in [6], the uniform convergence with respect to the small parameter with the rate of $O(N^{-1})$ was proved. In [7, 8], the $O(N^{-2})$ uniform convergence
of difference schemes on the grid from [5] with respect to the small parameter was proved for problems of type (1.1), (1.2). In these works, grid (1.3), (1.4) was assumed to be smooth; i.e., the function \( x(t) \) was assumed to be continuously differentiable, and \( \dot{\theta} = \ddot{\theta} \) was determined by the nonlinear equation

\[
\dot{\theta} = b - \varepsilon C l d(\ddot{\theta})
\]  

(1.5)

and could be calculated by the following iterative process [5]:

\[
\theta^{(0)} = 0, \quad \theta^{(k)} = b - \varepsilon C l d(\theta^{(k-1)}), \quad \lim_{k \to \infty} \theta^{(k)} = \ddot{\theta}, \quad 0 = \theta^{(0)} < \theta^{(1)} < \ldots < \theta^{(n)}
\]  

(1.6)

Note that the necessity of solving the nonlinear equation by an iterative process may be considered as a drawback (see, e.g., [4]). In this paper, we replace the smoothness requirement by a weaker condition

\[
b - \varepsilon \delta < \theta < b - \varepsilon c_0,
\]  

(1.7)

where \( 0 < c_0 < \delta \) are constants. Here, the right-hand inequality for grid (1.3), (1.4) is equivalent to the condition \( \max_i h_i = O(N^{-1}) \), and the left-hand inequality guarantees the approximation properties and convergence. Note that the choice of \( \theta = \tilde{\theta} \) is a particular case of (1.7), just as the choice of

\[
\theta = \theta^{(1)} = b - \varepsilon C
\]  

(1.8)

i.e., the choice of the value obtained after the first iteration step (1.6). The advantage of the last choice as compared to the general case (1.7) is the simplicity of calculation and the reasonable property \( h_i \leq h_{i+1} \) of the grid obtained (this property holds for \( \theta \leq \tilde{\theta} \)).

Another method of grid refining was suggested in [9]; it was used to construct uniformly convergent (with respect to the small parameter) methods for solving a wide class of singularly perturbed equations. As applied to problem (1.1), (1.2), the grid described in [9] is as follows:

\[
\overline{\Omega} = \left\{ x_i \mid x_i = \begin{cases} i h, & i = 0, 1, \ldots, n, \\ x_n + (i - n) H, & i = n + 1, n + 2, \ldots, N, \end{cases} \right\},
\]  

(1.9)

where \( h = \delta / n, \quad H = (1 - \delta) / (N - n), \quad n / N = b, \quad \delta = \min(\varepsilon C \ln N, \Lambda) \),

where \( A, b \in (0, 1) \) and \( C \) are constants. The results presented in [9] give the accuracy of \( O(N^{-1} \ln^2 N) \). In recent works [8, 10–12] (see also [13]), grid (1.9) was used to analyze other schemes for solving problems of type (1.1), (1.2). These schemes are uniformly convergent with the rate of \( O(N^{-2} \ln^2 N) \).

In this paper, for problem (1.1), (1.2), we consider a scheme that uses a central difference approximation of the first-order derivative on nonsmooth grids (1.3), (1.4), (1.7), and (1.9):

\[
(L^h u^h) = -\varepsilon(p_{i+1/2}^h + p_{i-1/2}^h) = -\varepsilon(r_{i+1/2}^h - r_{i-1/2}^h) - q_i^h u_i^h = f_i^h, \quad i = 1, 2, \ldots, N - 1,
\]  

(1.10)

where

\[
p_i^h = p(x_i), \quad r_i = r(x_i), \quad q_i^h = q(x_i), \quad f_i^h = f(x_i).
\]  

(1.11)

Here and in what follows, we use the conventional notation from [14]:

\[
h_i = x_i - x_{i-1}, \quad \dot{h}_i = \frac{h_i + \dot{h}_{i+1}}{2}, \quad x_{i-1/2} = x_i - h_i / 2.
\]

\[
\nu_{i+1} = \frac{v_{i+1} - v_{i+1/2}}{\dot{h}_i}, \quad \nu_{i} = \frac{v_i - v_{i-1/2}}{\dot{h}_i}, \quad \nu_{i+1} = \frac{v_{i+1} - v_{i-1}}{2h_i},
\]

(1.12)

and assume that

\[
\varepsilon \leq \varepsilon_0 = \varepsilon_0(p, r, q), \quad \max_i h_i \leq h_0 = h_0(p, r).
\]  

(1.13)
Note that in [10], the central difference scheme on the Shishkin grid was analyzed for the problem of type (1.1), (1.2) in the nondivergent form, and the uniform convergence of this scheme with respect to the small parameter with the rate of $O(N^{-2} \ln^2 N)$ was established.

The main result of this paper is the uniform convergence (with respect to the small parameter) of scheme (1.10), (1.11) on nonuniform grids of type (1.3), (1.4), (1.7) and (1.9) that refine in the boundary layer. The convergence rate of this scheme is $O(N^{-2})$ and $O(N^{-2} \ln^2 N)$ on grids (1.3), (1.4), (1.7) and (1.9), respectively. This result is established in Theorems 4 and 5.

2. GRID GREEN'S FUNCTION AND A PRIORI ESTIMATES FOR $q(x) = 0$

Introduce an arbitrary nonuniform grid on the interval $[0, 1]$: $$
\mathcal{G} = \{ x_0 = x_1 < x_2 < \ldots < x_{N-1} < x_N = 1 \}.
$$

For grid functions defined on $\mathcal{G}$ and vanishing for $i = 0$ and $i = N$, define a scalar multiplication as

$$
(u, v) = \sum_{i=1}^{N-1} u_i v_i 
$$

and norms

$$
\| v \|_{L^1} = \sqrt{\langle v, v \rangle}, \quad \| v \|_{L^2} = \langle |v|, 1 \rangle = \sum_{i=1}^{N-1} |v_i|, \quad \| v \|_{L^\infty} = \max_{i=1} |v_i|,
$$

which satisfy the obvious inequalities

$$
\| v \|_{L^1} \leq \| v \|_{L^2} \leq \| v \|_{L^\infty}.
$$

On the grid $\mathcal{G}$, consider the difference scheme (1.10) with zero boundary value conditions. Using the notation

$$
(A^h v)_i = \sum_{j=1}^{N-1} A_{ij}^h v_j + (r_{i-1}^h v_i + r_{i+1}^h v_{i+1})/2
$$

this scheme can be written as

$$
(L^h v)_i = -(A^h v)_i + q_i^h v_i = f_i, \quad i = 1, 2, \ldots, N-1, \quad v_0 = v_N = 0.
$$

It is clear that the difference expression $A^h v$ approximates the differential expression $A v \equiv xp(x) v' + r(x) v$, and $L v = -(A v) + q(x) v$.

It is well known that the solution to problem (2.5) can be written as

$$
v_i = (G(x_i, \xi_j), f_j) = \sum_{j=1}^{N-1} G(x_i, \xi_j) f_j \delta^h_{ij},
$$

where $G(x_i, \xi_j)$ is Green's function of the difference operator $L^h$. This function considered as a function of $x_i$ for a fixed $\xi_j$ is defined by the following formulas:

$$
L^h G(x_i, \xi_j) = \delta^h(x_i, \xi_j), \quad i, j = 1, 2, \ldots, N-1,
$$

$$
G(0, \xi_j) = G(1, \xi_j) = 0, \quad j = 1, 2, \ldots, N-1,
$$

where

$$
\delta^h(x_i, \xi_j) = \begin{cases}
\hat{h}^{-1}, & x_i = \xi_j, \\
0, & x_i \neq \xi_j,
\end{cases}
$$

is the grid analogue of Dirac's delta function.

**Theorem 1.** Let $q(x) \equiv 0$, the coefficients $p(x)$ and $r(x)$ in equation (1.1) satisfy conditions (1.2) and

$$
p(x) \leq \bar{p}, \quad r(x) \leq \bar{r}.
$$

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and, in addition, \( r(x) \) satisfy the Lipschitz condition with a constant \( \bar{R} \). Then, if the grid \( \bar{\omega} \) is such that the following conditions are satisfied:

(i) \[
\prod_{l=1}^{N} \left( \frac{\varepsilon p_l^h}{h_l r_{l-1}^h} + \frac{1}{2} \right)^{-1} \leq \text{const} = 1/4,
\]

(ii) \[ h_i \leq \text{const} h_j \quad \text{for} \quad i \leq j, \]

then (2.5), (2.4)—the difference Green's function (2.7) of the operator \( L^h \) is uniformly bounded in \( \varepsilon \):

\[
|G(x_n, \xi_j)| \leq c = 81/(4r_0).
\]

Here and in what follows, the same notation \( c \) and const is used to denote different constants independent of \( \varepsilon \) and \( N \).

**Proof.** Define \( v_j = G(x_n, \xi_j) \) for a fixed \( \xi_j \) and prove that, for any \( \xi_j \) (\( j = 1, 2, ..., N-1 \)), the estimate

\[
|v_i| = |G(x_n, \xi_j)| \leq c
\]

holds, where \( c \) is a constant independent of \( j, \varepsilon \) and \( N \). Thus,

\[
(L^h v)_i = -(A^h v)_{i+1} + s^h(x_n, \xi_j), \quad i = 1, 2, ..., N-1, \quad v_0 = v_N = 0.
\]

Multiply each equation in (2.9) by \( h_i \) to obtain

\[
(A^h v)_{i+1} - (A^h v)_i = \begin{cases} 
-1, & i = j, \\
0, & i \neq j, \quad i = 1, 2, ..., N-1.
\end{cases}
\]

i.e.,

\[
(A^h v)_i = a - \begin{cases} 
0, & i \leq j, \\
1, & i > j, \quad i = 1, 2, ..., N,
\end{cases}
\]

where \( a \) is a constant. The solution to this difference equation can be written as \( v_i = aV_i - W_i \), where \( V_i \) and \( W_i \) are solutions to the following difference problems:

\[
(A^h V)_i = 1, \quad i = 1, 2, ..., N, \quad V_0 = 0.
\]

\[
(A^h W)_i = \begin{cases} 
0, & i \leq j, \\
1, & i > j, \quad i = 1, 2, ..., N, \quad W_0 = 0.
\end{cases}
\]

Taking into account the boundary condition \( v_N = 0 \), we have \( a = W_N/V_N \) and

\[
v_i = W_0 V_i/V_N - W_i.
\]

Let us obtain estimates for \( V_i \) and \( W_i \). From (2.4) we have

\[
(A^h v)_i = \left( \frac{\varepsilon p_i^h}{h_i} + \frac{r_i^h}{2} \right) v_i - \left( \frac{\varepsilon p_i^h}{h_i} - \frac{r_i^h}{2} \right) v_{i-1}.
\]

Rewrite the difference problems (2.10) and (2.11) as

\[
V_0 = 0, \quad \left( \frac{\varepsilon p_i^h}{h_i} + \frac{1}{2} \right) [r_i^h V_i - 1] - \left( \frac{\varepsilon p_i^h}{h_i} - \frac{1}{2} \right) [r_{i-1}^h V_{i-1} - 1] = \alpha_i, \quad i = 1, 2, ..., N,
\]

\[
W_0 = ... = W_j = 0, \quad \left( \frac{\varepsilon p_i^h}{h_i} + \frac{1}{2} \right) [r_i^h W_i - 1] - \left( \frac{\varepsilon p_i^h}{h_i} - \frac{1}{2} \right) [r_{i-1}^h W_{i-1} - 1] = \alpha_i, \quad i = j + 1, j + 2, ..., N,
\]
where
\[ \alpha_i = \frac{\varepsilon p_i}{h_i} \left( 1 - \frac{1}{r_i} \right). \] (2.14)

Consider the grid functions
\[ \tilde{V}_i = r_i V_i - 1, \quad \tilde{W}_i = r_i W_i - 1 \] (2.15)
and introduce the notation
\[ Q_i = \left( \frac{\varepsilon p_i}{h_i r_i^2} + \frac{1}{2} \right)^{-1}. \] (2.16)

Now, the difference problems under consideration can be written as
\[ \tilde{V}_0 = -1, \quad \tilde{V}_i = Q_i \tilde{V}_{i-1} + \alpha_i \left( \frac{\varepsilon p_i}{h_i r_i^2} + \frac{1}{2} \right)^{-1}, \quad i = 1, 2, \ldots, N, \]
\[ \tilde{W}_0 = \ldots = \tilde{W}_j = -1, \quad \tilde{W}_i = Q_i \tilde{W}_{i-1} + \alpha_i \left( \frac{\varepsilon p_i}{h_i r_i^2} + \frac{1}{2} \right)^{-1}, \quad i = j + 1, j + 2, \ldots, N. \]

From this, we readily obtain that
\[ \tilde{V}_0 = -1, \quad \tilde{V}_i = -\prod_{k=1}^{i} Q_k + S_{i+1}, \quad i = 1, 2, \ldots, N, \]
\[ \tilde{W}_0 = \ldots = \tilde{W}_j = -1, \quad \tilde{W}_i = -\prod_{k=1}^{i} Q_k + S_{i+1}, \quad i = j + 1, j + 2, \ldots, N, \]
where
\[ S_{i+1} = \alpha_i \left( \frac{\varepsilon p_i}{h_i r_i^2} + \frac{1}{2} \right)^{-1} + \alpha_{i-1} \left( \frac{\varepsilon p_{i-1}}{h_{i-1} r_{i-1}^2} + \frac{1}{2} \right)^{-1} Q_i + \alpha_i \left( \frac{\varepsilon p_{i-1}}{h_{i-1} r_{i-1}^2} + \frac{1}{2} \right)^{-1} Q_{i+1} \cdots Q_{i+1} \cdots Q_{i-1} Q_i. \]

We see that by virtue of (2.12) and (2.15), the function \( v_i \) can be written in terms of \( \tilde{V}_i \) and \( \tilde{W}_i \) as
\[ v_i = \frac{1 + \tilde{W}_i}{1 + \tilde{V}_i} \frac{1}{1 + \tilde{V}_i} - \frac{1}{1 + \tilde{W}_i} \] for variable coefficients, estimates for \( Q_i \) and \( S_{i+1} \) are given by the following lemma.

**Lemma 1.** Under the assumptions of Theorem 1, for sufficiently small \( \varepsilon_0 \) and \( h_0 \), (1.12) entails the estimates
\[ |Q_i| \leq 1, \] (2.17)
\[ |S_{i+1}| \leq 1/4. \] (2.18)

By this lemma, \( \prod_{k=1}^{i} Q_k \leq 1 \), and by condition (i) of the theorem \( |\prod_{k=1}^{i} Q_k| \leq 1/4 \). Therefore,
\[ 1 + \tilde{V}_i \geq 1/2, \quad |1 + \tilde{V}_i| \leq 9/4, \quad |1 + \tilde{W}_i| \leq 9/4 \] (2.19)
and, consequently,
\[
|v_i| \leq 2(\max_j |1 + V_i| \max_j |1 + W_i|) [r_0(1 + V_i)]^{-1} \leq 8\Upsilon(4r_0),
\]
which proves the theorem.

**Proof of Lemma 1.** For \( Q \leq 0 \), estimate (2.17) is obvious. For \( Q > 0 \), inequality (2.17) is equivalent to the inequality \( \alpha_i < 1 \) (by virtue of (2.14) and (2.16)). By virtue of (2.8), we have
\[
|\alpha_i| \leq \varepsilon P R/r_i^2 = c\varepsilon.
\]
Therefore, we can find \( \varepsilon_0 \) in (1.12) such that
\[
|\alpha_i| \leq \frac{1}{2}.
\]
For this \( \varepsilon_0 \), (2.17) and the following inequality are true:
\[
\left( \frac{\varepsilon P_i^h}{h_i r_i^2} + \frac{1}{2} \right)^{-1} = \frac{1 - Q_i}{1 - \alpha_i} \leq 2(1 - Q_i).
\]
We proceed with proving (2.18). Let \( (m + 1) \) be the number of the first negative element in the sequence \( \{Q_i\} \) (it is quite possible that \( m + 1 \) = i); i.e.,
\[
Q_1, \ldots, Q_m \geq 0, \quad Q_{m+1} < 0.
\]
Notice that for \( k < m < i \)
\[
S_{kl} = S_{m+1, i} + Q_{m+1} Q_{m+2} \ldots Q_{kl}.
\]
Therefore, by virtue of (2.17), we have
\[
\max |S_{kl}| \leq \max |S_{k|} + \max |S_{k|}|
\]
For \( k \leq i \leq m \), we have, by virtue of (2.20), (2.17), (2.23), and (2.22), that
\[
|S_{kl}| \leq c\varepsilon \left[ (1 - Q_i) + (1 - Q_{i-1}) Q_i + \ldots + (1 - Q_k) Q_{k+1} Q_{k+2} \ldots Q_i \right] \leq c\varepsilon \left( 1 - Q_k Q_{k+1} Q_{k+2} \ldots Q_i \right) \leq c\varepsilon.
\]
For \( m \leq k \leq i \), by virtue of (2.20) and the obvious inequality
\[
\left( \frac{\varepsilon P_i^h}{h_i r_i^2} + \frac{1}{2} \right)^{-1} \leq 2
\]
we have
\[
|S_{kl}| \leq c\varepsilon \left[ 1 + |Q_i| + \ldots + |Q_{k-1} Q_{k+2} \ldots Q_{i-1} Q_i| \right] \leq c\varepsilon \sum_{l=0}^{\infty} (\max_{l \geq 0} |Q|)^l = c\varepsilon \max_{l \geq m} \frac{1}{1 - |Q|}.
\]
Therefore,
\[
\max_{k \leq i} |S_{kl}| \leq c\varepsilon \max_{l \geq m} \left( 1 + \frac{1}{1 - |Q|} \right).
\]
For \( Q_i \leq 0 \), we have \( \varepsilon P_i^h/(h_i r_i^2) \leq 1/2 \) and, thus,
\[
\frac{1}{1 - |Q|} \leq 1 + \left( 1 - \frac{\varepsilon P_i^h}{h_i r_i^2} \right) \left( \frac{\varepsilon P_i^h}{h_i r_i^2} + \frac{\varepsilon P_i^h}{h_i r_i^2} \right)^{-1} \leq 1 + \frac{1}{2} \left( \frac{\varepsilon P_i^h}{h_i r_i^2} + \frac{\varepsilon P_i^h}{h_i r_i^2} \right)^{-1} \leq c\left( 1 + \frac{1}{\varepsilon} \right).
\]
For \( Q_i < 0 \) and \( i > m \), by virtue of (2.22), we have
\[
\frac{1}{1 - |Q|} = \left( \frac{\varepsilon P_i^h}{h_i r_i^2} + \frac{1}{2} \right)(1 - \alpha_i)^{-1} \leq \frac{1}{2} \left( \frac{\varepsilon P_i^h}{h_i r_i^2} + \frac{1}{2} \right).
\]

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Taking into account condition (ii) of Theorem 1 and inequality $\varepsilon p_{m+1}^{h_i}/(h_{m+1} + r^{h_i}) < 1/2$, which follows from (2.23), we obtain
\[
\frac{\varepsilon p_i^h}{h_i r_i^h} \leq \frac{\varepsilon}{r_0 h_i} \leq \text{const} \frac{\varepsilon}{h_{m+1}} \leq \text{const}.
\]
This implies
\[
\max_{x \in I} |S(x)| \leq c(\varepsilon + \max_x h_i) \leq c(\varepsilon_0 + h_0),
\]
and, for sufficiently small constant $\varepsilon_0 = \varepsilon_{0, p, r}$ and $h_0 = h_{0, p, r}$ from (1.12), we obtain (2.18), which completes the proof of the lemma.

Since the solution of (2.4), (2.5) can be written as (2.6), Theorem 1 (with regard to (2.2) entails the following corollary.

**Corollary 1.** Let the assumptions of Theorem 1 be satisfied. If $v_i$ is the solution to problem (2.4), (2.5), then
\[
\|v_i\|_{L^\infty} \leq c\|f_i\|_{L^1}.
\]

Note that under certain conditions, a stronger a priori estimate than that given in Theorem 1 is true (compare with the estimate presented in [8]). This estimate is established in the following theorem.

**Theorem 2.** Let the assumptions of Theorem 1 be satisfied and $(m+1)$ be the number of the first negative element in the sequence $\{q_i\}$ (see (2.16)); i.e., (2.23) is satisfied. If $v_i$ is the solution to the difference equation
\[
(L^h v)_i = \begin{cases} (\eta)_i, & i = 1, 2, \ldots, m, \\ 0, & i = m+1, \ldots, N-1, \\ \eta_{0} = v_{N} = 0, \end{cases}
\]
and, in addition, $\eta_{m+1} = 0$, then the following estimate holds:
\[
\|v_i\|_{L^\infty} \leq \frac{11}{2} \|\eta_i\|_{L^\infty}.
\]

**Proof.** Multiply each equation in (2.24) by $h_i$ to obtain, by virtue of (2.4) and (2.5), that
\[
(A^h v)_i = a - \begin{cases} \eta_i, & i = 1, 2, \ldots, m, \\ 0, & i = m+1, \ldots, N, \end{cases}
\]
where $a$ is a constant. The proof proceeds similar to the reasoning used in the proof of Theorem 1. We write the solution to this difference equations as $v_i = w_i - V_i$, where $V_i$ is the solution to the difference problem (2.10) and $w_i$ is the solution to the following difference problem:
\[
(A^h w)_i = \begin{cases} \eta_i, & i = 1, 2, \ldots, m, \\ 0, & i = m+1, \ldots, N, \end{cases} \quad w_0 = 0.
\]

Taking into account the boundary condition $\nu_N = 0$, we obtain that
\[
a = w_N / V_N, \quad v_i = w_N V_i / V_N - w_i = w_N (1 + V_i / V_N - w_i),
\]
where $V_i$ is the grid function (2.15). From this, by virtue of (2.19), we obtain
\[
\|v_i\|_{L^\infty} \leq \left(\frac{9}{2} + 1\right) \|w\|_{L^\infty} = \frac{11}{2} \|w\|_{L^\infty}.
\]
It remains to estimate $\|w\|_{L^\infty}$. By virtue of (2.13), equation (2.25) $(i = 1, 2, \ldots, m)$ supplemented with the boundary condition $w_0 = 0$ makes up a system of linear algebraic equations with a two-diagonal matrix with positive diagonal elements and nonpositive off-diagonal elements (by virtue of assumption (2.23) of the the-
For any row \( i \leq m \), the diagonal predominance is \( (r^h_i + r^h_{i-1})/2 \geq r_0 \). Hence (see [14, 15]), the matrix of this system of \( m \) equations is monotone, and the maximum principle holds for this difference problem. By this principle, for \( i \leq m \), we obtain
\[
|w_i| \leq ||\eta||_{L^r}^{1/r_0} (2.27)
\]
For \( i > m \), we obtain from (2.25), (2.13), and (2.16) that \( w_i = Q_{w_{i-1}} \). By virtue of (2.17), estimate (2.27) holds for any \( i \). Substitution of this estimate in (2.26) proves the theorem.

3. A PRIORI ESTIMATES FOR \( q(x) = 0 \)

Consider a more general case \( (q(x) = 0) \). Theorem 3 is the main result of this section.

**Theorem 3.** Let the assumptions of Theorems 1 and 2 be satisfied and
\[
q(x) \leq \bar{q} = \text{const.}
\]
If \( v_i \) is the solution to the difference problem
\[
(L^h v_i) = (\eta)_{i+1} f_i, \quad i = 1, 2, \ldots, N - 1, \quad v_0 = v_N = 0, \tag{3.1}
\]
where
\[
\eta_i = 0, \quad i = M + 1, M + 2, \ldots, N,
\]
and \( M \) differs from \( m \) in (2.23) by a finite positive \( m_0 \) independent of \( \varepsilon \) and \( N \) (i.e., \( M = m + m_0 \)), then the following estimate holds:
\[
||v||_{L^2} \leq c(||\eta||_{L^r} + ||f||_{L^r}).
\]

**Proof.** For \( q(x) = 0 \), the assertion of the theorem readily follows from Corollary 1 and Theorem 2 since
\[
\eta_{ki} = \bar{\eta}_{ki} + \bar{\zeta}_{ki}, \quad \text{where}
\]
\[
\bar{\eta}_i = \begin{cases} \eta_i, & i \leq m, \\ 0, & i > m, \end{cases}
\]
\[
\bar{\zeta}_i = \begin{cases} 0, & i < m, i > M, \\ \eta_{m+1}/H_1, & i = m, \\ \eta_{M+1}, & m < i \leq M, \end{cases}
\]
and
\[
||\bar{\zeta}||_{L^r} = ||\eta_{m+1}|| + \sum_{i = m + 1}^M ||\eta_{i+1} - \eta_{i}|| \leq \text{const} ||\eta||_{L^r}
\]
(the sum involves a finite number of terms).

Let \( q(x) \geq 0 \). It is easily seen that the solution to problem (3.1) can be written as
\[
v_i = v_i + W_i,
\]
where the grid functions \( V_i \) and \( W_i \) are the solutions to the following difference problems:
\[
- \varepsilon (p^h V_i)_{i+1} - (r^h V_i)_{i+1} = f_i + (\eta)_{i+1}, \quad i = 1, 2, \ldots, N - 1, \quad V_0 = V_N = 0,
\]
\[
- \varepsilon (p^h W_i)_{i+1} - (r^h W_i)_{i+1} + q^h W_i = - q^h V_i, \quad i = 1, 2, \ldots, N - 1, \quad W_0 = W_N = 0.
\]
Since the theorem is true for \( q(x) = 0 \), we have
\[
||v||_{L^r} \leq ||V||_{L^r} + ||W||_{L^r}, \quad ||V||_{L^r} \leq c(||\eta||_{L^r} + ||f||_{L^r}) \tag{3.3}
\]
\[
||W||_{L^r} \leq c(||q^h V||_{L^r} + ||q^h W||_{L^r}).
\]
By virtue of (2.3) and taking into account $\|q^h W\|_{L^2}^2 \leq \tilde{q}(q^h W, W)$, we have

$$\|W\|_{L^2}^2 \leq c\|V\|_{L^2}^2 + (q^h W, W).$$  \hspace{1cm} (3.4)

Let us find an estimate for the second term on the right-hand side of this inequality. Take the scalar product (in the sense of (2.1)) of (3.2) with $(r^h v_i)$. For any grid function $v_i$ that is zero for $i = 0$ and $i = N$, the equations $(v_i, v_i) = 0$ and $(w_i, v_i) = -(w, v_i)$ hold, where $(v, w) = \sum_{i=0}^{N} v_i w_i h_i$ is another scalar product and $(v, v) \leq \|v\|_{L^2}^2$. Thus, we have

$$\varepsilon((p_i^h W_i, (r^h W)_{x_i}) + (q^h r^h W, W) = -(q^h V, r^h W).$$

From this, using the identity

$$(r^h W)_{x_i} = r^h_{i-1} W_{x_{i-1}} + r^h_i W_i$$

and inequality

$$\varepsilon((p_i^h r^h_{i-1} W_{x_{i-1}}, W_{x_{i-1}}) \leq \varepsilon((p_i^h r^h_{i-1} W_{x_{i}}, W_{x_{i}}) + \frac{\varepsilon}{4} \left(\frac{p_i^h r^h_{i-1} W_{x_i}}{r^h_{i-1}}\right),$$

we obtain

$$(q^h W, W) \leq \frac{1}{r^h_{i-1}} \left(\frac{p_i^h r^h_{i-1} W_{x_i} W_i}{r^h_{i-1}}\right) + \frac{1}{r^h_{i-1}} \left(\frac{c^*}{r^h_{i-1}}\right) \|W\|_{L^2}^2 + \frac{1}{c^*} \|V\|_{L^2}^2 + 4 \epsilon^2 q^2 r^2 \|V\|_{L^2}^2 + 4 c^* \frac{r^2}{r^h_{i-1}} \|V\|_{L^2}^2.$$

where $c^*$ is an arbitrarily large constant. Substitute this inequality in (3.4), to obtain

$$\|W\|_{L^2}^2 \leq c \left[\left(\frac{\varepsilon}{c^*}\right) \|W\|_{L^2}^2 + \|V\|_{L^2}^2\right].$$

Choose $c^*$ large enough to obtain, for sufficiently small constant $\varepsilon_0 = \varepsilon_0(p, r, q)$ from (1.12), that

$$\|W\|_{L^2}^2 \leq c \|V\|_{L^2}^2.$$

This inequality, with regard to (3.3), proves the theorem.

4. APPROXIMATION

Theorem 4. Let $u(x)$ be the solution to problem (1.1), (1.2) with sufficiently smooth coefficients and the right-hand side, $u^h$ be the solution to problem (1.10), (1.11) on a nonuniform grid $\Omega$ satisfying conditions (i) and (ii) of Theorem 1, and the following condition be satisfied:

(iii) $h_i = H_i$, $i = M, M + 1, \ldots, N$, $\frac{\varepsilon \rho_{h_i}}{h_{i-1}} - \frac{1}{2} \geq 0$, $i = 1, 2, \ldots, m$, $m \leq M \leq m + m_0$,

where $m_0$ is a positive integer independent of $\varepsilon$ and $N$. Then, the following estimate holds uniformly with respect to $\varepsilon$:

$$\|u^h - u(x_i)\|_{L^2} = O\left(\max_i \left[\min\left(\frac{h_i^2}{\varepsilon}, 1\right) \exp\left(-\frac{\gamma x_{i-1}}{\varepsilon}\right) + \max_i \varepsilon^2\right]\right),$$

where $\gamma$ is an arbitrary constant satisfying the condition

$$\gamma = \text{const} < r(0)/p(0).$$  \hspace{1cm} (4.1)

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Proof. Let \( z_i = u_i^h - u(x_i) \) be the error of the solution. Then
\[
(L^h z_i) = -(A^h z_i) + q_i^h = \psi_i, \quad i = 1, 2, \ldots, N-1, \quad \varepsilon_0 = \varepsilon_N = 0,
\]
where
\[
\psi_i = f_i^h - (L^h u)_i + f_i^h = (A^h u)_i - (L^h u)_i + f_i^h.
\]
Integrate (1.1) on the interval \([x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]\) to obtain
\[
[(A u)(x_{i-\frac{1}{2}})]_{1-i} + \frac{1}{h_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} [-q(x)u(x) + f(x)] dx = 0.
\]
Therefore, the approximation error \( \psi_i \) can be written as
\[
\psi_i = [(A^h u)_i - (A u)(x_{i-\frac{1}{2}})] + \left[ \frac{1}{h_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} q(x)u(x)dx - q(x_i)u(x_i) \right] - \left[ \frac{1}{h_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f(x)dx - f_i^h \right].
\]
To analyze \( \psi_i \), we use the representation of the solution to (1.1), (1.2) as (see, e.g., [9, p. 197])
\[
u(x) = U(x) + V(x),
\]
where \( LU(x) = f(x), LV(x) = 0, 0 < x < 1, \) and
\[
|U^{(\ell)}(x)| \leq \text{const}, \quad |V^{(\ell)}(x)| \leq \text{const} e^{-|x|}, \quad k = 0, 1, 2, 3, 4,
\]
for any constant \( \gamma \) satisfying (4.1). Then
\[
\psi_i = \Psi_i(U) + \Psi_i(V),
\]
where
\[
\Psi_i(U) = [(A^h U)_i - (A U)(x_{i-\frac{1}{2}})] + \left[ \frac{1}{h_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} q(x)U(x)dx - q(x_i)U(x_i) \right] - \left[ \frac{1}{h_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f(x)dx - f_i^h \right],
\]
\[
\Psi_i(V) = [(A^h V)_i - (A V)(x_{i-\frac{1}{2}})] + \left[ \frac{1}{h_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} q(x)V(x)dx - q(x_i)V(x_i) \right].
\]
Using the Taylor expansion, we show that
\[
\Psi_i(U) = \begin{cases} \eta_{i1} + O(h_i^2), & i < M, \\ O(h_i^2), & i \geq M, \end{cases}
\]
where
\[
\eta_i = O(h_i^2), \quad i \leq M.
\]
Indeed, for \( i \geq M \), the grid is uniform, and by virtue of (4.3) the estimate is obvious. If \( i < M \), i.e., the grid is nonuniform in the general case, then
\[
(A^h U)_i - (A U)(x_{i-\frac{1}{2}}) = \frac{e h_i^2}{24} p(x_{i-\frac{1}{2}}) U''''(\xi) + \frac{h_i^2}{8} (r U)''(\xi).
\]
for certain \( \xi_0, \xi_1 \in (x_{i-1}, x_i) \) and using the standard technique described in [14] we have

\[
\frac{1}{h_i} \int_{x_{i-1}}^{x_{i+1}} f(x) dx - f(x_i) = \frac{1}{h_i} \int_{x_{i-1}}^{x_{i+1}} \left[ f(x) - f(x_i) \right] dx - \int_{x_{i-1}}^{x_{i+1}} f(x_i) - f(x) \right] dx
\]

\[= \left[ h_i f'(x_{i-1/2}) \right]_{x_{i-1/2}} + \max_{\xi \in (x_{i-1/2}, x_{i+1/2})} |f''(\xi)| O(h_i^2). \]

Similarly,

\[
\frac{1}{h_i} \int_{x_{i-1}}^{x_{i+1}} q(x) U(x) dx - q(x_i) U(x_i) = \left[ h_i^2 (q U)'(x_{i-1/2}) \right]_{x_{i-1/2}} + \max_{\xi \in (x_{i-1/2}, x_{i+1/2})} |(q U)'(\xi)| O(h_i^2).
\]

From this, taking into account (4.3), we obtain estimate (4.4), (4.5).

Now, we estimate \( \Psi_i(V) \). For \( i < M \), the same reasoning as that used for estimating \( \Psi_i(U) \) yields

\[
(A^i V)_i - (AV)(x_{i-1/2}) = \frac{\epsilon h_i^2}{24} p(x_{i-1/2}) V''(\xi) + \frac{h_i^2}{8} (r V)'(\xi)
\]

for certain \( \xi, \tilde{\xi} \in (x_{i-1}, x_i) \), and

\[
\frac{1}{h_i} \int_{x_{i-1}}^{x_{i+1}} q(x) V(x) dx - q(x_i) V(x_i) = \left[ h_i^2 (q V)'(x_{i-1/2}) \right]_{x_{i-1/2}} + \max_{\xi \in (x_{i-1/2}, x_{i+1/2})} |(q V)'(\xi)| O(h_i^2),
\]

i.e., with regard to (4.3), we have

\[
\Psi_i(V) = \tilde{\Psi}_i + O\left(\frac{h_i^2}{\epsilon} \exp\left(-\frac{\gamma x_{i-1}}{\epsilon}\right)\right), \quad i < M,
\]

where

\[
\tilde{\Psi}_i = O\left(\frac{h_i^2}{\epsilon} \exp\left(-\frac{\gamma x_{i-1}}{\epsilon}\right)\right), \quad i \leq M.
\]

On the other hand,

\[
(A^i V)_i - (AV)(x_{i-1/2}) = [\epsilon p(x_{i-1/2}) V(\xi) + (r V)(\tilde{\xi})] - [\epsilon (p V)(\tilde{\xi}) + (r V)(\tilde{\xi})],
\]

where \( \xi, \tilde{\xi} \in (x_{i-1}, x_i) \); therefore, by virtue of (4.3), we have

\[
\left| (A^i V)_i - (AV)(x_{i-1/2}) \right| = O\left(\exp\left(-\frac{\gamma x_{i-1}}{\epsilon}\right)\right).
\]

Similarly, for a certain \( \xi, \tilde{\xi} \in (x_{i-1/2}, x_{i+1/2}) \), we have

\[
\frac{1}{h_i} \int_{x_{i-1}}^{x_{i+1}} q(x) V(x) dx - q(x_i) V(x_i) = (q V)(\xi) - (q V)(x_i) = O\left(\exp\left(-\frac{\gamma x_{i-1}}{\epsilon}\right)\right),
\]

and thus,

\[
\Psi_i(V) = \tilde{\Psi}_i + O\left(\min\left[\frac{h_i^2}{\epsilon}, 1\right] \exp\left(-\frac{\gamma x_{i-1}}{\epsilon}\right)\right), \quad i < M.
\]
where

\[ \tilde{\eta} = O\left( \min\left( \frac{1}{\epsilon}, 1 \right) \exp\left( -\frac{\gamma x_{i-1}}{\epsilon} \right) \right), \quad i \leq M. \] (4.9)

Now consider the uniform grid \( (i \geq M) \). Using the Taylor expansion and (4.3), we obtain

\[ [ (A \psi)(x_i) - (A \psi)(\tilde{x}_i) ] = O\left( \frac{H^2}{\epsilon} \exp\left( -\frac{\gamma x_{i-1}}{\epsilon} \right) \right), \quad i \geq M. \]

\[ \frac{1}{H} \int_{x_{i-1}}^{x_i} q(x) V(x) dx \cdot q(x_i) V(x_i) = O\left( \frac{H^2}{\epsilon} \exp\left( -\frac{\gamma x_{i-1}}{\epsilon} \right) \right) = O\left( \frac{H^2}{\epsilon} \exp\left( -\frac{\gamma x_{i-1}}{\epsilon} \right) \right), \quad i \geq M. \]

On the other hand, using (4.6), we have

\[ [ (A \psi)(x_i) - (A \psi)(x_{i-1/2}) ] = O\left( \frac{1}{H} \exp\left( -\frac{\gamma x_{i-1}}{\epsilon} \right) \right), \quad i \geq M. \]

and taking into account (4.7), we obtain the estimate

\[ \Psi(x_i) = O\left( \frac{1}{H} \min\left( \frac{H^2}{\epsilon}, 1 \right) \exp\left( -\frac{\gamma x_{i-1}}{\epsilon} \right) \right), \quad i \geq M. \] (4.10)

Thus, from estimates (4.4), (4.8), and (4.10), we obtain

\[ \Psi = \begin{cases} \eta - \tilde{\eta}, & i \leq M, \\ \frac{1}{H} \min\left( \frac{H^2}{\epsilon}, 1 \right) \exp\left( -\frac{\gamma x_{i-1}}{\epsilon} \right) + O(H^2), & i \geq M. \end{cases} \]

where \( \eta \) and \( \tilde{\eta} \) are functions (4.5) and (4.9), respectively. From this, by Theorem 3 and with regard to (2.3), we have

\[ ||z||_{L^2} \leq \text{const} \left( ||\eta||_{L^2} + ||\tilde{\eta}||_{L^2} + \max_{r \leq M} \min\left( \frac{H^2}{\epsilon}, 1 \right) \exp\left( -\frac{\gamma x_{i-1}}{\epsilon} \right) \right) \]

\[ + \frac{1}{H} \min\left( \frac{H^2}{\epsilon}, 1 \right) \exp\left( -\frac{\gamma x_{i-1}}{\epsilon} \right) + \max H_i^2 \] (4.11)

\[ \leq \text{const} \left( \max \left( \min\left( \frac{H^2}{\epsilon}, 1 \right) \exp\left( -\frac{\gamma x_{i-1}}{\epsilon} \right) + \sum_{i \geq M} \min\left( \frac{H^2}{\epsilon}, 1 \right) \exp\left( -\frac{\gamma x_{i-1}}{\epsilon} \right) + \max H_i^2 \right) \right). \]

Using the fact that

\[ \sum_{i \geq M} \exp\left( -\frac{\gamma x_{i-1}}{\epsilon} \right) \leq \exp\left( -\frac{\gamma x_{M-1}}{\epsilon} \right) \sum_{j = 0}^{\infty} \exp\left( -\frac{\gamma H_j}{\epsilon} \right) = \exp\left( -\frac{\gamma x_{M-1}}{\epsilon} \right) \left[ 1 - \exp\left( -\frac{\gamma H}{\epsilon} \right) \right]^{-1} \]

and that \( 0 < \min\{t, 1\} [1 - \exp(-\gamma)]^{-1} \leq \text{const} \) for \( t > 0 \), we have

\[ \sum_{i \geq M} \min\left( \frac{H^2}{\epsilon}, 1 \right) \exp\left( -\frac{\gamma x_{i-1}}{\epsilon} \right) \leq \min\left( \frac{H^2}{\epsilon}, 1 \right) \exp\left( -\frac{\gamma x_{i-1}}{\epsilon} \right). \]

Upon substitution in (4.11), this inequality proves Theorem 4.
5. CONVERGENCE

Theorem 5. Let $u(x)$ be the solution to problem (1.1), (1.2) with sufficiently smooth coefficients and the right-hand side, and let $u^k_i$ be the solution to problem (1.10), (1.11). Let, in addition, the grid $\omega$ be one of the following grids: (a) the Bakhvalov grid (1.3), (1.4), (1.7) or (b) the Shishkin grid (1.9).

If the parameter $C$ of grids (1.4) and (1.9), respectively, satisfies the condition

$$C > 2p(0)\ell r(0),$$

then the following estimates hold uniformly with respect to $\varepsilon$: (a) $\|u^k_i - u(x_i)\|_{L^2} = O(N^{-\frac{1}{2}})$, (b) $\|u_i^k - u(x_i)\|_{L^2} = O(N^{-1} \ln^2 N)$.

To prove this theorem, we need two lemmas.

**Lemma 2.** For certain $\varepsilon_0$ and $h_0$ from (1.12), the Bakhvalov grid (1.3), (1.4), (1.7) and the Shishkin grid (1.9) satisfy condition (iii) in Theorem 4, and

$$x_m \geq 24\varepsilon\beta\ell r_0.$$  

**Proof.** The Bakhvalov grid satisfies condition (iii) with the following natural $M$ and $m$:

$$\theta N \leq M - 1 < \theta N + 1, \quad M - 3 - C\ell r/2p_0 < m \leq M - 2 - C\ell r/2p_0.$$  

Indeed, for these numbers, $M - m < 3 + C\ell r/2p_0$, and for $i \geq M$ this grid is uniform by virtue of the inequality $x_{M-i} < x(\theta) \leq x_{M-i+1}$. For $0 \leq \tau < \tau_2$, we have

$$x(\tau_2) - x(\tau_1) \leq \max_{\tau \in [\tau_1, \tau_2]} x'(\tau)(\tau_2 - \tau_1) \leq \frac{\varepsilon C}{b - \tau_2}(\tau_2 - \tau_1),$$

and by virtue of $m/N \leq (M - 2)/\ell r < 1$, we have, for $i \leq m$, that

$$h_i \leq h_m = x(m/N) - x((m - 1)/N) \leq [\varepsilon C]/(b - m/N).$$

To complete the proof of condition (iii), it is sufficient to prove that $\varepsilon/h_m \geq \ell r/2p_0$:

$$\frac{\varepsilon}{h_m} \geq \frac{bN - m}{C} \geq \frac{\theta N - (M - 2 - C\ell r/2p_0)}{C} \geq \frac{\theta N - (M - 2)}{C} + \frac{\varepsilon C}{2p_0} \geq \frac{\varepsilon C}{2p_0}.$$  

The validity of (5.2) for sufficiently small $\varepsilon_0$ and $h_0$ from (1.12) follows from the estimate

$$x_m = x(\theta - \text{const}/N) = \frac{b}{(b - \theta) + \text{const}/N} \geq \frac{\varepsilon \ln \text{const}}{\varepsilon_0 + h_0},$$

that is obtained with regard to (1.7).

It is easily seen that the Shishkin grid satisfies condition (iii) for $M = n + 1$ and $m = n$ for sufficiently small $h_0$ from (1.12). Indeed, $h_i = H$ for $i \geq M$ and $\varepsilon/h_i \geq N/(C \ln N)$ for $i \leq n = m$. Since

$$x_n - x_{n-1} = \left\{ \begin{array}{ll}
\varepsilon C \ln [N(1 - 1/n)], & \varepsilon C \ln N \leq A, \\
A (1 - 1/n), & \varepsilon C \ln N \geq A,
\end{array} \right.$$

(5.2) holds for sufficiently small $\varepsilon_0$ and $h_0$ from (1.12), which completes the proof of the lemma.

**Lemma 3.** The Bakhvalov grid (1.3), (1.4), (1.7) and the Shishkin grid (1.9) satisfy both conditions (i) and (ii).

**Proof.** It is clear that both grids satisfy condition (i). As for condition (ii), taking into account notation (2.16) and estimate (2.17) from Lemma 1, it is sufficient to prove that

$$\prod_{i=1}^{m} q_i \leq 1/4,$$

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where \( m \) is the number from condition (iii). For \( i \leq m \), taking into account (2.14), (2.21), and the inequality \( e^{p_i}/(R_i R_{i-1}) \geq 1/2 \), we have

\[
0 \leq q_i = 1 - (1 - \alpha_i) \left( \frac{e^{p_i}}{h_i R_{i-1}} + \frac{1}{2} - \alpha_i \right)^{-1} \leq 1 - \frac{h_i R_{i-1}}{6 e^{p_i}} \leq 1 - \frac{r_i}{6p_i e},
\]

therefore,

\[
0 \leq \prod_{i=1}^{m} q_i \leq \exp\left( -\frac{r_i x_i}{6p_i e} \right),
\]

which, by virtue of (5.2), entails the assertion of the lemma.

**Proof of Theorem 5.** Notice that for both grids

\[
\max(x_i - x_{i-1}) = O(N^{-1}).
\]

By virtue of (5.1), we have \( 2/C < \alpha(0)/\rho(0) \). Therefore, for \( \gamma = 2/C \), we have from Theorem 3 that

\[
\left\| u_i^n - u(x_i) \right\|_{L^2} = O\left( \max_i \left[ \min\left( \frac{h_i}{\varepsilon}, 1 \right) \exp\left( \frac{x_i}{\varepsilon C} \right) \right] + O(N^{-2}) \right).
\]

Consider cases (a) and (b) separately.

(a) For \( \varepsilon < \varepsilon_0 \), we have \( h_i/\varepsilon = O(N^{-1}) \) and the assertion of the theorem is obvious. Let \( \varepsilon \leq \varepsilon_0 \). Define \( t_i = i/N \) (i.e., \( x_i = x(t_i) \) by virtue of (1.3)), and consider a number \( m \) such that \( t_m < \theta \) and \( \theta - t_m = \text{const}N^{-1} \). By virtue of (1.4), we have for \( t_i \leq \theta \) that

\[
\exp\left( \frac{x_i}{\varepsilon C} \right) = \frac{b - t_i}{b},
\]

Taking into account (5.3), we have that for \( i \leq m \)

\[
\frac{h_i}{\varepsilon} \exp\left( \frac{x_{i-1}}{\varepsilon C} \right) \leq C \left( \frac{t_i}{b} - \frac{t_{i-1}}{b} \right) \left( \frac{b - t_{i-1}}{b} - \frac{t_i}{b} \right) = \text{const}(t_i - t_{i-1}) \left( 1 + \frac{t_i - t_{i-1}}{b - t_i} \right).
\]

Since \( t_i - t_{i-1} = N^{-1}, b - t_i \geq b - t_m \), and by virtue of (1.7), we obtain

\[
\text{const}N^{-1} \leq b - t_m = (b - \theta) + (\theta - t_m) \leq \varepsilon + \text{const}N^{-1}.
\]

Therefore, for \( i \leq m \), we have

\[
\min\left( \frac{h_i}{\varepsilon}, 1 \right) \exp\left( \frac{x_{i-1}}{\varepsilon C} \right) = O(N^{-1}).
\]

For \( i > m \), by virtue of (5.4) and (5.6), we have

\[
\min\left( \frac{h_i}{\varepsilon}, 1 \right) \exp\left( \frac{x_{i-1}}{\varepsilon C} \right) \leq \text{const} \cdot \max\left( \frac{N^{-1}}{\varepsilon}, 1 \right) \exp\left( \frac{x_m}{\varepsilon C} \right) \leq \text{const} \cdot \frac{N^{-1} b - t_m}{\varepsilon + N^{-1} b}
\]

and, taking into account (5.7), we again obtain (5.8). Substitute it in (5.5) to obtain the assertion of the theorem.

(b) Let \( \varepsilon C N < A \). If \( i > n \), then \( x_{i-1} \geq \delta \) and \( \exp(-x_{i-1}/(\varepsilon C)) \leq \exp(-\delta/(\varepsilon C)) = N^{-1} \). Therefore,

\[
\min\left( \frac{h_i}{\varepsilon}, 1 \right) \exp\left( \frac{x_{i-1}}{\varepsilon C} \right) \leq \text{const} N^{-1} \ln N.
\]

If \( i \leq n \), then \( h_i/\varepsilon = O(N^{-1} \ln N) \), and we again obtain (5.9).

Now, let \( \varepsilon C N \geq A \). Then \( h_i = O(N^{-1}), h_i/\varepsilon = O(N^{-1} \ln N) \), and we again have (5.9). Substitute (5.9) in (5.5) to complete the proof of the theorem.
Remarks. 1. Note that the smooth grid (1.3), (1.4) can be constructed without using an iterative process. As a rule, for an arbitrary value of $b$ from $(0, 1)$ (which characterizes the proportion of grid points in a neighborhood of the boundary layer), $\theta = \theta(\varepsilon, b)$ and $d = d(\varepsilon, b)$ are determined from the linear equation (1.5). An alternative way is to take an arbitrary $B = bd$ from $(0, 1)$. Then, the constants $b$, $d$, and $\theta$ are determined explicitly as

$$b = b(\varepsilon, B) = B(1 + B - \varepsilon C[1 + \ln|B/(\varepsilon C)|])^{-1},$$

$$d = B/b(\varepsilon, B), \quad \theta = b(\varepsilon, B)[1 - \varepsilon C/B].$$

We can estimate the proportion of grid points in the boundary layer, $b = b(\varepsilon, B)$, taking into account the fact that, for $\varepsilon \leq \varepsilon_0 \leq B/C$, the function $b(\varepsilon, B)$ is increasing with respect to $\varepsilon$ and

$$1 - (1 + B)^{-1} \geq \lim_{\varepsilon \to 0} b(\varepsilon, B) \leq b(\varepsilon, B) \leq b(\varepsilon = B/C, B) = B.$$

2. The assertion of Theorem 1 remains true for any value of the constant $0 < c < 1$ in (i).

3. The assertion of Theorem 1 remains true if, instead of condition (ii), the inequality $h_i \geq \text{const} h_j$ is satisfied for $i \leq j$.

4. If we omit condition (5.1) in Theorem 5, more general estimates hold uniformly with respect to $\varepsilon$:

$$\|u^n_h - u(x_i)\|_{L_2} = O(N^{-2}[(1 + (N^{-1})^{C_1/2}]) \leq O(N^{-2} + N^{-C_1}),$$

and

$$\|u^n_h - u(x_i)\|_{L_4} = O(N^{-2}\ln^2 N + N^{-C_1}),$$

where $\gamma$ is an arbitrary constant satisfying condition (4.1) (roughly speaking, this is equivalent to the relation $C_1 = C(0)/p(0)$). In addition, if $C$ does not satisfy condition (5.1), then $C_1 < 2$ and the order of the uniform convergence decreases. Note that the analysis of the equation with constant coefficients and the zero right-hand side shows that the oscillating component, which is involved in the solution of the scheme with the central difference for $\varepsilon \leq \text{const} N^{-1}$, has the amplitude of $O(N^{-C_1})$. Hence, it is reasonable to choose $C$ such that the order of magnitude of $O(N^{-C_1})$ is less than two (e.g., $C(0)/p(0) \geq 2.5$ or $3$).

5. The assertions of Theorems 4 and 5 remain true if, for $i > M$ (see condition (iii)), the grid is piecewise uniform with a finite number of transition points from one grid step to another.

6. Consider the equation $L^+u = -\varepsilon(p(x)u')' - r(x)u' + q(x)u = f(x)$, which is adjoint to equation (1.1), and approximate it using the operator $L^+\theta$ defined as $(L^+\theta u^n_h)_i = -\varepsilon(p^n_h u^n_h)'_i - r^n_h u^n_h + q^n_h u^n_h = f^n_i$, which is adjoint to $L^+$ from (1.10) in the sense of (2.1), on the grids that refine similar to (1.3), (1.4), (1.7) and (1.9) near the right side of the interval $[0, 1]$. If we replace condition (5.1) by the condition $C > 2p(1)/r(1)$, then the assertions of Theorem 1, Corollary 1, and Theorem 5 remain true. Indeed, by Remark to Theorem 1, Green's function (2.7) of the operator $L^+$ is uniformly bounded in the small parameter on grids that refine near the right side of the interval $[0, 1]$. Therefore, Green's function of the operator $L^+\theta$ is bounded, and the a priori estimate of Corollary 1 is true. This estimate entails the assertion of Theorem 5 (see the estimates of the approximation error in the grid norm $L^+\theta$ in [7] for the Bakhvalov grid and in [12] for the Shishkin grid).

7. If the operator $L$ (see (1.1)) is approximated using one of the nonuniform monotone three-point difference schemes [8] or the monotone four-point difference operator with one-sided three-point second-order approximation of the term involving the first-order derivative [11], a theorem similar to Theorem 5 holds.

6. NUMERICAL RESULTS

In this section we present numerical results that illustrate the accuracy of the scheme considered above. It is easily seen that the function

$$u(x) = \frac{e^{-x/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}} + 2x\cos\frac{\pi x}{2}$$

is the solution to the problem

$$\varepsilon u'' + u = -f(x), \quad 0 < x < 1, \quad u(0) = 1, \quad u(1) = 0.$$
with the right-hand side

\[ f(x) = \left( \frac{\epsilon \pi^2 x}{2} - 2 \right) \cos \frac{\pi x}{2} + \pi(2\epsilon + x) \sin \frac{\pi x}{2}. \]

This problem was solved using scheme (1.10), (1.11) on the Bakhvalov grid (1.3), (1.4), (1.8) for \( C = 3, b = 0.5, \) and \( \epsilon_0 = b/C. \) The table presents the values of the \( L^\infty \)-norm of the approximation error of the solution for various \( \epsilon \) and \( N \) and the rate of decrease of the error as the number of grid points doubles. The last column of the table shows the maximum value of the error over different \( \epsilon \) for various \( N. \) Similar results for the Shishkin grid are presented in [10]. The analysis of the table in terms of rows shows that for each \( N \) the error becomes stable as \( \epsilon \rightarrow 0. \) This reflects the uniform convergence. The analysis of the table in terms of columns shows that the scheme has the second-order convergence rate (the error decreases by a factor of 4 as the number of grid points doubles (whereas, for the Shishkin grid, it decreases by a factor of 3.2); i.e., the convergence rate is not less than that obtained in theory.

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