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A POSTERIORI ERROR ESTIMATION FOR A DEFECT-CORRECTION METHOD APPLIED TO CONVECTION-DIFFUSION PROBLEMS

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Abstract. We consider a two-point boundary-value problem for a singularly perturbed convection-diffusion problem. The problem is solved using a defect-correction method based on a first-order upwind difference scheme and a second-order (unstabilized) central difference scheme.
A robust *a posteriori* error estimate in the maximum norm is derived. It provides computable and guaranteed upper bounds for the discretization error. Numerical examples are given that illustrate the theoretical findings and verify the efficiency of the error estimator on *a priori* adapted meshes and in an adaptive mesh movement algorithm.

Key Words. convection-diffusion problems, finite difference schemes, defect correction, a posteriori error estimation, singular perturbation

1. Introduction

Defect correction methods (DCM) have been advocated for the numerical solution of ordinary and partial differential equations since the early 1970s and 80s [27, 5]. The idea of DCMs is to combine the good stability properties of a low order upwinded discretization with the higher order accuracy of unstabilized discretizations. They have been successfully applied in computational fluid dynamics, for example to combustion problems [3] or when solving the Navier-Stokes equations [14, 19].

Hemker [12, 13] proposed the use of DCM for the numerical treatment of convection-diffusion and other singularly perturbed boundary-value problems. Most of the papers found in the literature deal with DCM on (quasi)uniform meshes. Only recently adaptivity and layer-adapted meshes have been used in combination with DCM, see [9, 10, 15, 22]. Of particular interest are parameter-uniform methods, i.e., methods that perform equally well no matter how small the perturbation parameter.

Let us consider the convection-diffusion problem

$$\mathcal{L}u := -\varepsilon u'' - (bu)' + cu = f \quad \text{in} \quad (0,1), \quad u(0) = \gamma_0, \ u(1) = \gamma_1, \tag{1}$$

where ε is a small positive parameter and $b \geq \beta > 0$ on $[0,1]$. It provides an excellent paradigm for numerical techniques in computational fluid dynamics for the treatment of problems with boundary layers, i. e., regions where the solution and its derivates change rapidly [26].

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In the present paper we shall investigate a DCM for (1) based on finite difference discretizations. Ervin and Layton [8] proved that this method is uniformly convergent of second in the maximum norm order outside layers. However, the crucial point in singularly perturbed problems is the resolution of layers. This can for example be achieved by the use of layer-adapted meshes, i.e., meshes that are significantly refined inside the layer regions. The resulting non-uniformity of the mesh results in difficulties both in the appropriate construction of the DCM and its analysis which must be overcome.

In [10] a DCM on a particular class of layer-adapted meshes, so called Shishkin-type meshes, is considered. The authors conduct an *a priori* error analysis and establish uniform nodal convergence of essentially second order in all mesh points. A theory for arbitrary meshes has been derived in [22, 23].

Let us describe the DCM from [10, 22, 23]. Given a mesh $\omega_N : 0 = x_0 < x_1 < \cdots < x_N = 1$ with mesh sizes $h_i := x_i - x_{i-1}$ and $\hbar_i = (h_i + h_{i+1})/2$ define the difference operators

$$v_{x,i} := \frac{v_{i+1} - v_i}{h_{i+1}}, \quad v_{\bar{x},i} := \frac{v_i - v_{i-1}}{h_i}, \quad v_{\hat{x},i} := \frac{v_{i+1} - v_i}{\hbar_i} \quad \text{and} \quad v_{\mathring{x},i} := \frac{v_{i+1} - v_{i-1}}{2\hbar_i}.$$

Then the central difference approximation on ω_N for (1) is

$$\left[L^c \bar{u}^N\right]_i := -\varepsilon \bar{u}^N_{\bar{x}\hat{x},i+1} - (b\bar{u}^N)_{\mathring{x},i} + c_i \bar{u}^N_i = f_i.$$

It is combined with the upwind scheme

$$\left[L^u \hat{u}^N\right]_i := -\varepsilon \hat{u}^N_{\bar{x}x,i+1} - (b\hat{u}^N)_{x,i} + c_i \hat{u}^N_i = f_i,$$

where for any function $g \in C[0,1]$ we have set $g_i := g(x_i)$.

With this notation the DCM is as follows:

1. Compute an initial first-order approximation $\hat{u}^N$ using simple upwinding:

(2a) $$\left[L^u \hat{u}^N\right]_i = f_i \quad \text{for} \quad i = 1, \dots, N-1, \quad \hat{u}^N_0 = \gamma_0, \quad \hat{u}^N_N = \gamma_1.$$

2. Estimate the defect τ in the differential equation by means of the central difference scheme:

(2b) $$\tau_i = \left[L^c \hat{u}^N\right]_i - f_i \quad \text{for} \quad i = 1, \dots, N-1.$$

3. Compute the defect correction Δ by solving

(2c) $$\left[L^u \Delta\right]_i = \kappa_i \tau_i, \quad \kappa_i := \frac{\hbar_i}{h_{i+1}} \quad \text{for} \quad i = 1, \dots, N-1, \quad \Delta_0 = \Delta_N = 0.$$

4. Then the final computed solution is

(2d) $$u^N_i = \hat{u}^N_i - \Delta_i \quad \text{for} \quad i = 0, \dots, N.$$

Remark 1. *At a first glance both the upwind discretization and the particular weighting of the residual in* (2c) *appear a bit non-standard. No justification for these choices are provided by* [10, 22, 23]*. An argument that suggests this particular choice is presented in Sect. 4. Furthermore, our weighting becomes the standard* $\kappa_i = 1$ *on uniform meshes; however, when used on non-uniform meshes,* $\kappa_i = 1$ *might reduce the order of convergence (see numerical results in Sect. 5.3).*

While the *a priori* results [10, 22, 23] establish the asymptotic behaviour of the error as the mesh is refined, it cannot give guaranteed upper bounds for the error on a particular mesh. The constant in the error bound, though independent of the perturbation parameter ε, depends on the exact solution u which in turn is unknown.

The main contribution of the present paper is in establishing an *a posteriori* error bound which provides an upper bound on the error:

$$\left\|u - u^N\right\|_\infty \le C^* \max_{i=1,\dots,N} \min\left\{\frac{h_i}{\|b\|_\infty}, \frac{h_i^2}{4\varepsilon}\right\} \left|\left(f - cu^N\right)_{i-1/2} + (bu^N)_{\bar{x},i}\right| + \frac{1}{\beta} \max_{i=1,\dots,N} h_i \left|(b\Delta)_{\bar{x},i}\right| + \text{higher order terms}$$

The constant C^* can be expressed explicitely in terms of the data of the problem.

The paper is organized as follows. Sect. 2 quotes some *a priori* results for the DCM from the literature. Our novel *a posteriori* estimates are presented in Sect. 3, while in Sect. 4 we consider particular issues in the construction of the DCM, namely the choice of a suitable upwind scheme. Finally in Sect. 5 results of numerical experiments for a test problem are given that illustrate the theoretical findings and verify the efficiency of the error estimator on *a priori* adapted meshes and in an adaptive mesh movement algorithm.

Notation. Throughout we shall use C to denote a generic positive constant that is independent of both the perturbation parameter ε and of N, the number of mesh intervals.

2. A priori error analysis

This type of analysis is based on derivative bounds for the exact solution and on stability properties of the discrete operator L^u. These will be provided first. Next a sketch of the analysis in [22, 23] for arbitrary meshes is given. This general result is then applied to two special layer-adapted meshes.

Lemma 1. *Assume* $b, c, f \in C^1[0,1]$. *Then* (1) *possesses a unique solution* $u \in C^3[0,1]$ *with*

$$\left|u^{(k)}(x)\right| \le C\left\{\varepsilon^{\min\{0,2-k\}} + \varepsilon^{-k} e^{-\beta x/\varepsilon}\right\} \quad \textit{for } k = 0,1,2,3 \quad \textit{and } x \in [0,1],$$

Proof. A first proof was given in [16]. The precise smoothness requirement are stated in [21]. □

Introduce the discrete norms

$$|||v|||_{\varepsilon,\infty,\omega} := \max\left\{\frac{\varepsilon}{2} \max_{i=1,\dots,N} |v_{\bar{x},i}|\,, \frac{\beta}{2} \max_{i=0,\dots,N} |v_i|\right\}$$

and

$$\|v\|_{-1,\infty,\omega} := \min_{\gamma\in\mathbb{R}} \max_{i=0,\dots,N-1} \left|\sum_{j=i}^{N-1} h_{j+1} v_j - \gamma\right|$$

Theorem 1. *Suppose the coefficients in* (1) *satisfy*

$$c \ge 0 \quad \textit{and} \quad c - b' \ge 0 \quad \textit{on} \quad [0,1]. \tag{3}$$

Then the operator L^u *enjoys the stability inequality*

$$|||v|||_{\varepsilon,\infty,\omega} \le \|L^u v\|_{-1,\infty,\omega} \quad \textit{for all} \;\; v \in \mathbb{R}_0^{N+1} := \left\{w \in \mathbb{R}^{N+1} : w_0 = w_N = 0\right\}.$$

Proof. See [23] for a proof based on comparison principles. The proof in [1] uses the Green's function associated with L^u. □

Remark 2. *The results hold also without the restrictions imposed by* (3)*, see* [1]*, however the arguments become more complicated and the constants in the estimates will differ.*

Furthermore, note that when $\beta > 0$ *then* (3) *can always be ensured for* ε *smaller than some threshold value* ε_0 *by a simple transformation* $u = \tilde{u}e^{\chi x}$ *with* χ *chosen appropriately.*

Set

$$(\mathcal{A}v)(x) := \varepsilon v'(x) + (bv)(x) + \int_x^1 (cv)(s)ds, \quad \mathcal{F}(x) := \int_x^1 f(s)ds,$$

$$[A^u v]_i := \varepsilon v_{\bar{x},i} + (bv)_i + \sum_{k=i}^{N-1} h_{k+1}(cv)_k, \quad F_i^u := \sum_{k=i}^{N-1} h_{k+1} f_k,$$

and

$$[A^c v]_i := \varepsilon v_{\bar{x},i} + \frac{(bv)_i + (bv)_{i-1}}{2} + \sum_{k=i}^{N} \hbar_k (cv)_k, \quad F_i^c := \sum_{k=i}^{N} \hbar_k f_k,$$

where we have formally set $\hbar_N = h_N/2$. The differential equation (1) yields

$$(\mathcal{A}u - \mathcal{F})(x) = \alpha \quad \text{for all } x \in [0,1], \tag{4a}$$

while (2) implies

$$\left[A^c u^N - (A^u - A^c)\Delta - F^c\right]_i = a \quad \text{for } i = 1, \dots, N. \tag{4b}$$

A direct calculation gives

$$\left[A^u\left(u - u^N\right)\right]_i = \left[(A^c - A^u)\left(\hat{u}^N - u\right)\right]_i + [A^c u - F^c]_i - (\mathcal{A}u - \mathcal{F})(x_{i-1/2}) - a - \alpha.$$

The first term on the right-hand side, the so-called relative truncation error, constituted the main difficulty in [10, 22]. It is related to the approximation of derivatives by the upwind scheme. In [22] an error expansion for the error of the upwind scheme is constructed that addresses this issue. Then using Theorem 1 and Lemma 1, one obtains

$$\left|\left|\left|u^N - u\right|\right|\right|_{\varepsilon,\infty,\omega} \le C\vartheta(\omega_N)^2 \tag{5}$$

with

$$\vartheta(\omega_N) := \max_{i=1,\dots,N} \int_{x_{i-1}}^{x_i} \left(1 + \varepsilon^{-1} e^{-\beta s/2\varepsilon}\right) ds,$$

see [22, 23].

The general convergence result (5) can be used to conclude the uniform order of convergence for various layer-adapted meshes. We shall restrict ourselve to the two most commonly used meshes.

Bakhvalov meshes [4] for (1) are generated by equidistributing the function

$$M_{Ba}(x) = \max\left\{1,\ K\varepsilon^{-1} e^{-\beta x/(\varepsilon\sigma)}\right\}$$

with positive constants K and σ, i.e. the mesh points x_i are chosen such that

$$\int_0^{x_i} M_{Ba}(s)ds = \frac{i}{N}\int_0^1 M_{Ba}(s)ds.$$

The parameter K determines the number of mesh points used to resolve the layer. For the Bakhvalov mesh ω_N^B it can be shown [20, 23] that

$$\vartheta(\omega_N^B) \le CN^{-1} \quad \text{if } \sigma \ge 2.$$

Shishkin meshes [24] are constructed as follows. Let $q \in (0,1)$ and $\sigma > 0$ be mesh parameters. Set

$$\tau = \min\left\{ q,\ \frac{\sigma\varepsilon}{\beta} \ln N \right\}.$$

Assume that qN is an integer. Then the Shishkin mesh ω_N^S for problem (1) divides the interval $[0,\tau]$ into qN equidistant subintervals, while $[\tau,1]$ is divided into $(1-q)N$ equidistant subintervals. For this mesh we have

$$\vartheta(\omega_N^S) \le CN^{-1} \ln N \quad \text{if } \sigma \ge 2.$$

3. A posteriori error analysis

This type of error analysis employs stability properties of the differential operator. We use results from [2, 23] and a generalization of Lemma 2.2 in [17].

Given an arbitrary function v with $v(0) = v(1) = 0$, we have

$$v(x) = \int_0^1 \mathcal{G}(x,\xi)\big(\mathcal{L}v\big)(\xi) d\xi \quad \text{for} \quad x \in [0,1],$$

where $\mathcal{G}$, the Green's function associated with $\mathcal{L}$ and Dirichlet boundary conditions, solves for fixed $\xi \in [0,1]$

$$(\mathcal{L}\mathcal{G}(\cdot,\xi))\,(x) = \delta(x-\xi) \quad \text{for} \quad x \in (0,1), \quad \mathcal{G}(0,\xi) = \mathcal{G}(1,\xi) = 0, \tag{6}$$

with δ denoting the Dirac-δ function. Therefore (6) has to be read in the context of distributions. $\mathcal{G}$ can also be defined using the adjoint operator $\mathcal{L}^*v := -\varepsilon v'' + bv' + cv$. For fixed $x \in [0,1]$ it solves

$$(\mathcal{L}^*\mathcal{G}(x,\cdot))\,(\xi) = \delta(\xi - x) \quad \text{for} \quad \xi \in (0,1), \quad \mathcal{G}(x,0) = \mathcal{G}(x,1) = 0. \tag{7}$$

For our further investigations, let us introduce the supremum and the L_1 norms

$$\|v\|_\infty := \operatorname{ess} \sup_{x\in[0,1]} |v(x)| \qquad \text{and} \qquad \|v\|_1 := \int_0^1 |v(x)| dx,$$

and the $W^{-1,\infty}$ norm

$$\|v\|_{-1,\infty} := \min_{V: V'=v} \|V\|_\infty = \min_{C\in\mathbb{R}} \left\| \int_\cdot^1 v(s) ds + C \right\|_\infty = \sup_{u \in W_0^{1,1}} \frac{\langle u, v\rangle}{|u|_{1,1}}.$$

For a detailed discussion of this norm the reader is referred to [2, §2.2].

For arbitrary $x \in (0,1)$ we have the following bounds on various semi-norms of $\mathcal{G}$, see [2, 23]:

$$\|\mathcal{G}(x,\cdot)\|_1 \le \|\mathcal{G}(x,\cdot)\|_\infty \le \beta^{-1}, \quad \|\mathcal{G}_\xi(x,\cdot)\|_1 \le 2\beta^{-1} \quad \text{and} \quad \|\mathcal{G}_{x\xi}(x,\cdot)\|_1 = 2\varepsilon^{-1}. \tag{8}$$

These norms are used in [2, 23] to establish stabily properties for the operator $\mathcal{L}$.

Theorem 2. *Suppose* (3) *holds true. Then the operator* $\mathcal{L}$ *satisfies*

$$|||v|||_{\varepsilon,\infty} := \max\left\{ \frac{\beta}{2}\|v\|_\infty, \frac{\varepsilon}{2}\|v'\|_\infty \right\} \le \|\mathcal{L}v\|_{-1,\infty} \quad \textit{for all} \quad v \in W_0^{1,\infty}(0,1).$$

Remark 3. *Again the conditions* (3) *can be relaxed though the stability constant will change, see* [2].

The next result is an extension of Lemma 2.2 in [17] which gave bounds in the mesh points only. It is a crucial ingredient for the analysis of second-order approximations.

Theorem 3. *Let u be the solution of the boundary value problem*

$$\mathcal{L}u = -F' \quad \text{on} \quad (0,1), \; u(0) = u(1) = 0$$

with

$$F(x) = A_{i-1/2}(x - x_{i-1/2}) \quad \text{for} \quad x \in (x_{i-1}, x_i).$$

Then

$$\|u\|_\infty \le C^* \max_{i=1,\ldots,N} \left\{ \left|A_{i-1/2}\right| \min\left[\frac{h_i}{\|b\|_\infty}, \frac{h_i^2}{4\varepsilon}\right] \right\}, \tag{9}$$

where

$$C^* = \frac{2\|b\|_\infty + \|c\|_\infty + \beta}{2\beta} \tag{10}$$

Proof. Let $x \in (0,1)$ be arbitrary, but fixed. The Green's function representation gives

$$u(x) = \int_0^1 \mathcal{G}_\xi(x,\xi) F(\xi) d\xi = \sum_{i=1}^N A_{i-1/2} I_i \tag{11}$$

with

$$I_i := \int_{x_{i-1}}^{x_i} \mathcal{G}_\xi(x,\xi)(\xi - x_{i-1/2}) d\xi = \int_{x_{i-1}}^{x_i} \mathcal{G}_{\xi\xi}(x,\xi) \left[\frac{h_i^2}{8} - \frac{(\xi - x_{i-1/2})^2}{2}\right] d\xi.$$

Therefore we have two bounds for the I_i's:

$$|I_i| \le \frac{h_i}{2} \int_{x_{i-1}}^{x_i} |\mathcal{G}_\xi(x,\xi)| \, d\xi$$

and

$$|I_i| \le \frac{h_i^2}{8\varepsilon} \int_{x_{i-1}}^{x_i} \varepsilon \, |\mathcal{G}_{\xi\xi}(x,\xi)| \, d\xi$$

Using (7), we get the combined estimate

$$|I_i| \le \min\left\{\frac{h_i^2}{8\varepsilon}, \frac{h_i}{2\|b\|_\infty}\right\} \left\{ \int_{x_{i-1}}^{x_i} \delta(\xi - x) d\xi + \|b\|_\infty \int_{x_{i-1}}^{x_i} |\mathcal{G}_\xi(x,\xi)| \, d\xi \right.$$
$$\left. + \|c\|_\infty \int_{x_{i-1}}^{x_i} |\mathcal{G}(x,\xi)| \, d\xi \right\}.$$

The I_i's in (11) have been bounded and application of a discrete Hölder inequality completes the proof. □

With these stability results at hand we can now derive our *a posteriori* error bounds. We shall identify any mesh function v with its piecewise linear nodal interpolant.

Theorem 4. *Suppose* (3) *holds true. Set* $\psi := f - cu^N$. *Then the error of the defect-correction method satisfies*

$$\left\|u - u^N\right\|_\infty \le \eta := \eta_1 + \eta_2 + \eta_3 + \eta_4 + \eta_5$$

with

$$\eta_1 := C^* \max_{i=1,\dots,N} \min\left\{\frac{h_i}{\|b\|_\infty}, \frac{h_i^2}{4\varepsilon}\right\} \left|\psi_{i-1/2} + \left(bu^N\right)_{\bar{x},i}\right|,$$

$$\eta_2 := \frac{1}{\beta} \max_{i=1,\dots,N} h_i \left|(b\Delta)_{\bar{x},i}\right|, \quad \eta_3 := \frac{1}{\beta} \max_{i=1,\dots,N-1} \left|\sum_{k=i}^{N-1} \frac{h_{k+1} - h_k}{2} c_k \Delta_k\right|,$$

$$\eta_4 := \frac{1}{6\beta} \sum_{i=1}^{N} h_i^3 \|\psi''\|_{\infty,[x_{i-1},x_i]},$$

and

$$\eta_5 := \frac{3}{4\beta} \max_{i=1,\dots,N} h_i^2 \left\{2\|\psi'\|_{\infty,[x_{i-1},x_i]} + \left\|(bu^N)''\right\|_{\infty,[x_{i-1},x_i]}\right\}.$$

Proof. By (4) we have, for $x \in (x_{i-1}, x_i)$,

$$\mathcal{A}\left(u - u^N\right)(x) = \mathcal{F}(x) - F_i^c + \left[A^c u^N\right]_i - \left(\mathcal{A}u^N\right)(x) - \left[\left(A^u - A^c\right)\Delta\right]_i + \alpha - a.$$

Recalling the definitions of $\mathcal{F}$, F^c, $\mathcal{A}$, A^c and A^u, we obtain the representation

$$\begin{aligned} \mathcal{A}\left(u - u^N\right)(x) = &\int_x^1 \psi(s)ds - \sum_{k=i}^{N} \hbar_k \psi_k + \frac{\left(bu^N\right)_i + \left(bu^N\right)_{i-1}}{2} - \left(bu^N\right)(x) \\ &- \frac{h_i}{2}(b\Delta)_{\bar{x},i} - \sum_{k=i}^{N-1} \frac{h_{k+1} - h_k}{2} c_k \delta_k + \alpha - a. \end{aligned} \tag{12}$$

Taylor expansions yield

$$\int_x^1 \psi(s)ds - \sum_{k=i}^{N} \hbar_k \psi_k = \int_{x_i}^1 (\psi - \psi^I)(s)ds + (x_{i-1/2} - x)\psi_{i-1/2} + \mu_i(x),$$

and

$$\frac{\left(bu^N\right)_i + \left(bu^N\right)_{i-1}}{2} - \left(bu^N\right)(x) = (x_{i-1/2} - x)\left(bu^N\right)_{\bar{x},i} + \tilde{\mu}_i(x)$$

with

$$\|\mu_i\|_{\infty,[x_{i-1},x_i]} \le \frac{3h_i^2}{4} \|\psi'\|_{\infty,[x_{i-1},x_i]}$$

and

$$\|\tilde{\mu}_i\|_{\infty,[x_{i-1},x_i]} \le \frac{3h_i^2}{8} \|(bU)''\|_{\infty,[x_{i-1},x_i]}$$

Substite the above two equations into (12).

$$\begin{aligned} \mathcal{A}(u - U)(x) = &\int_{x_i}^1 (\psi - \psi^I)(s)ds + \left(x_{i-1/2} - x\right)\left(\psi_{i-1/2} + \left(bu^N\right)_{\bar{x},i}\right) \\ &- \frac{h_i}{2}(b\Delta)_{\bar{x},i} - \sum_{k=i}^{N-1} \frac{h_{k+1} - h_k}{2} c_k \delta_k + (\mu_i + \tilde{\mu}_i)(x) + \alpha - a. \end{aligned}$$

Furthermore

$$\left|\int_{x_i}^{1} (\psi - \psi^I)(s)ds\right| \le \frac{1}{12}\sum_{i=1}^{N} h_i^3 \, \|\psi''\|_{\infty,[x_{i-1},x_i]} \, .$$

Finally, note that $(\mathcal{A}v)' = -\mathcal{L}v$. Use Theorems 2 and 3 to complete the proof. □

Remark 4. *Theorem 4 gives an error estimate of a second-order method. Therefore, all components in η, explicitly or implicitly, involve h_i^2. In particular, for η_3 an integration-by-parts calculation shows that $\eta_3 = (2\beta)^{-1}\max_i\big|h_N(c\Delta)_{N-1} - h_i(c\Delta)_i - \sum_{k=i+1}^{N-1} h_k^2 (c\Delta)_{\bar{x},k}\big|$, i.e. η_3 is also a second-order term.*

Remark 5. *The error estimate of Theorem 4 contains terms, namely η_5 and η_6, that in general have to be approximated, for example*

$$\psi' \approx \frac{\psi_i - \psi_{i-1}}{h_i}, \quad \psi'' \approx 4\frac{\psi_i - 2\psi_{i-1/2} + \psi_{i-1}}{h_i^2}$$

and

$$(bu^N)'' \approx 4\frac{(bu^N)_i - 2(bu^N)_{i-1/2} + (bu^N)_{i-1}}{h_i^2}.$$

The additional errors introduced this way are of third order and therefore decay rapidly when the mesh is refined.

4. Construction of the defect correction method

The crucial point in the design of the DCM is the choice of a suitable upwind operator L^u and of appropriate weights κ_i.

Let $\xi, \zeta \in [0,1]$ be arbitrary, but fixed. Define the weighted step sizes

$$h_i' = \xi h_i + (1-\xi)h_{i+1} \quad \text{and} \quad h_i'' = \zeta h_i + (1-\zeta)h_{i+1}.$$

With this notation a general first-order upwinded scheme for (1) takes the form

$$\left[L^u \hat{u}^N\right]_i := -\frac{\varepsilon h_{i+1}}{h_i'}\hat{u}^N_{\bar{x}x,i} - \frac{h_{i+1}}{h_i''}\left(b\hat{u}^N\right)_{x,i} + c_i\hat{u}_i^N = f_i. \tag{13}$$

On a uniform mesh $h_i = \hbar_i = h_i' = h_i'' = h_{i+1}$ and there is no variation in the upwind scheme and the choice $\kappa \equiv 1$ is successful; see the analysis in [8]. For non-uniform meshes the situation is different and will be discussed in detail now.

For the error $\chi := u^N - u$ of the defect correction method we have

$$L^u(u^N - u) = \left(L^u - \kappa L^c\right)\left(u^N - u\right) + \kappa\left(f - L^c u\right).$$

The second term on the right-hand side is the truncation error of the central difference scheme. It is of second order and does not need any further discussion at the moment.

For the first term, the so-called relative truncation error, we have

$$\begin{aligned}\left[\left(L^u - \kappa L^c\right)\hat{\chi}\right]_i = -\varepsilon\left(\frac{\hbar_i}{h_i'} - \kappa_i\right)\hat{\chi}_{\bar{x}\hat{x},i} - \frac{(b\hat{\chi})_{i+1} - (b\hat{\chi})_i}{h_i''} \\ + \kappa_i\frac{(b\hat{\chi})_{i+1} - (b\hat{\chi})_{i-1}}{2\hbar_i} + (1-\kappa_i)c_i\hat{\chi}_i,\end{aligned}$$

where $\hat{\chi} := \hat{u}^N - u$ is the error of the upwind scheme. It is of first order, while $\hat{\chi}_{i+1} - \hat{\chi}_i$ can be of second order at best.

First, look at the term arising from the discretization of the diffusion term. This is a discrete second order derivative of $\hat{\chi}$. Hence it is of of first order too. Therefore, second order convergence can only be achieved when this term vanishes, i.e., when $\kappa_i = \hbar_i/h'_i$. This determines the weights of the defect in (2c).

Next, with κ fixed the error contribution form the convection term is

$$-\frac{(b\hat{\chi})_{i+1} - (b\hat{\chi})_i}{h''_i} + \frac{(b\hat{\chi})_{i+1} - (b\hat{\chi})_{i-1}}{2h'_i}.$$

These are second order terms divided by h, i.e. will in general give an error contribution of first order. Only when $h'_i = h''_i$ this becomes

$$-\frac{(b\hat{\chi})_{i+1} - 2(b\hat{\chi})_i + (b\hat{\chi})_{i-1}}{2h'_i}. \tag{14}$$

This is roughly h times a discrete second-order derivative of a term of first order. Thus is of second order.

Similar to Theorem 1 the upwind operator L^u defined by (13) with $h'_i = h''_i$ satisfies the stability inequality

$$\|v\|_{\infty,\omega} \le \min_{\gamma\in\mathbb{R}} \max_{i=0,\dots,N-1} \left| \sum_{j=i}^{N-1} h'_j \left[L^u v\right]_j - \gamma \right| \quad \text{for all } v \in \mathbb{R}_0^{N+1}.$$

Thus, (14) gives the error contribution

$$\max_{i=1,\dots,N} \left|(b\hat{\chi})_i - (b\hat{\chi})_{i-1}\right|$$

which can be of second order only when $h'_i = h_{i+1}$. This issue is related to the approximation of derivative in the upwind scheme and was discussed in [18], see Remark 1 therein.

The numerical experiments in Sect. 5.3 also show that other choices than $h'_i = h''_i = h_{i+1}$ and $\kappa_i = \hbar_i/h'_i$ reduce the order of convergence of the DCM to first order.

An alternative construction of this DCM is using a Galerkin method with piecewise linear test and trial functions and special quadrature rules: the trapezium rule for the central difference approximation and the left-sided rectangle rule for the upwind scheme.

5. Numerical experiments

We now consider the test problem

$$-\varepsilon u''(x) - (2+x)u'(x) + \cos x\, u(x) = e^{1-x} \quad \text{for } x \in (0,1),\ u(0) = u(1) = 0 \tag{15}$$

in order to illustrate the results of our theoretical findings and to study numerically the magnitude of the various components η_i of our error estimator. We shall also verify how sharp the results are. For this problem we have $\beta = 2$.

5.1. Numerical results for *a priori* adapted meshes. The exact solution to (15) is not available. Instead the maximum-norm errors are estimated by comparing the numerical solution u^N with $\tilde{u}^{16N}$, the solution of the DCM on the mesh obtained by bisecting the original mesh four times, i.e. a mesh that is 16 times finer:

$$\left\|u - u^N\right\|_\infty \approx \chi^N := \left\|\tilde{u}^{16N} - u^N\right\|_\infty.$$

N	χ^N	η_1^N	η_2^N	η_3^N	η_4^N	η_5^N	η^N	r^N
2^{10}	5.78e-06	8.69e-06	5.88e-06	3.88e-07	4.62e-07	3.15e-07	1.57e-05	2.72
2^{11}	1.44e-06	2.17e-06	1.48e-06	9.79e-08	1.16e-07	7.86e-08	3.93e-06	2.72
2^{12}	3.60e-07	5.39e-07	3.70e-07	2.46e-08	2.89e-08	1.96e-08	9.82e-07	2.73
2^{13}	8.98e-08	1.34e-07	9.24e-08	6.15e-09	7.23e-09	4.91e-09	2.45e-07	2.73
2^{14}	2.24e-08	3.33e-08	2.31e-08	1.54e-09	1.81e-09	1.23e-09	6.10e-08	2.73
2^{15}	5.56e-09	8.26e-09	5.77e-09	3.85e-10	4.52e-10	3.07e-10	1.52e-08	2.73
2^{16}	1.38e-09	2.05e-09	1.44e-09	9.63e-11	1.13e-10	7.67e-11	3.78e-09	2.73
2^{17}	3.44e-10	5.07e-10	3.60e-10	2.41e-11	2.82e-11	1.92e-11	9.39e-10	2.73
2^{18}	8.53e-11	1.25e-10	9.00e-11	6.02e-12	7.06e-12	4.79e-12	2.33e-10	2.73

TABLE 1. Bakhvalov mesh with $\sigma = 2$, $K = \sigma/\beta$, $\varepsilon = 10^{-8}$.

N	χ^N	η_1^N	η_2^N	η_3^N	η_4^N	η_5^N	η^N	r^N
2^{10}	6.72e-05	2.96e-04	2.42e-04	3.93e-07	4.62e-07	3.15e-07	5.39e-04	8.02
2^{11}	2.04e-05	9.01e-05	7.54e-05	9.85e-08	1.16e-07	7.86e-08	1.66e-04	8.13
2^{12}	6.07e-06	2.69e-05	2.28e-05	2.46e-08	2.89e-08	1.96e-08	4.98e-05	8.19
2^{13}	1.78e-06	7.90e-06	6.76e-06	6.16e-09	7.23e-09	4.91e-09	1.47e-05	8.23
2^{14}	5.17e-07	2.29e-06	1.97e-06	1.54e-09	1.81e-09	1.23e-09	4.27e-06	8.25
2^{15}	1.49e-07	6.59e-07	5.68e-07	3.85e-10	4.52e-10	3.07e-10	1.23e-06	8.27
2^{16}	4.23e-08	1.87e-07	1.62e-07	9.63e-11	1.13e-10	7.67e-11	3.50e-07	8.27
2^{17}	1.19e-08	5.29e-08	4.59e-08	2.41e-11	2.82e-11	1.92e-11	9.88e-08	8.28
2^{18}	3.35e-09	1.48e-08	1.29e-08	6.02e-12	7.06e-12	4.79e-12	2.77e-08	8.28

TABLE 2. Shishkin mesh with $\sigma = 2$, $q = 1/2$, $\varepsilon = 10^{-8}$.

The efficiency constant of the error estimator is evaluated computing the quantities $r^N := \eta^N/\chi^N$. In our experiments we take $\varepsilon = 10^{-8}$ which is a sufficiently small value to bring out the singularly perturbed nature of the problem.

Table 1 contains the results of our test computations for a Bakhvalov mesh with parameters $\sigma = 2$ and $K = \sigma/\beta$. With this choice approximately half of the mesh points are used to resolve the layer. The table lists, from left to right, the number N of mesh intervals, the maximum-norm error, the five components of the error estimator, the upper error bound of Theorem 4 and, finally, the efficiency constant r^N.

Second-order convergence is observed as the mesh is refined. Also there is strong correlation between the error and any of η_1^N, η_2^N and η^N. The remaining η_i^N, $i = 3, 4, 5$ a smaller and contribute little to the final error estimator. The efficiency constant is close to 3.

In Table 2 we present results for the Shishkin mesh. The errors behave like $N^{-2}\ln^2 N$. Again there is a strong correlation between the actual error and η_1 and η_2. The other η_i contribute much less to the *a posteriori* error bound. They are of order N^{-2}. This time the efficiency constant is approx. 8.

Our last experiment, documented in Table 3, uses a Bakhvalov mesh with $\sigma = 1$. This is a deliberately bad choice because the convergence is only of first order. However the terms η_i, $i = 3, 4, 5$, in the error bound are of order N^{-2}. This clearly illustrates that η_1 and η_2 are the dominating terms and a potential adaptive algorithm should aim at minimizing these two.

Remark 6. *Ideally, one would like to establish the efficiency of the error estimator of Theorem 4 theoretically, i.e. to prove a lower estimate* $\|u - u^N\|_\infty \geq C_0\eta$ *for some positive efficiency constant* $C_0 < 1$*. However, the authors are not aware of any such error estimate for singularly perturbed problems in the maximum norm. For our particular problem, such an analysis would require a lower-bound version*

N	χ^N	η_1^N	η_2^N	η_3^N	η_4^N	η_5^N	η^N	r^N
2^{10}	8.78e-04	3.27e-03	8.83e-04	5.36e-07	4.62e-07	3.15e-07	4.15e-03	4.73
2^{11}	4.32e-04	1.63e-03	4.38e-04	1.33e-07	1.16e-07	7.86e-08	2.06e-03	4.78
2^{12}	2.12e-04	8.09e-04	2.17e-04	3.30e-08	2.89e-08	1.96e-08	1.03e-03	4.85
2^{13}	1.03e-04	4.02e-04	1.07e-04	8.18e-09	7.23e-09	4.91e-09	5.09e-04	4.93
2^{14}	5.04e-05	2.00e-04	5.31e-05	2.03e-09	1.81e-09	1.23e-09	2.53e-04	5.02
2^{15}	2.44e-05	9.90e-05	2.62e-05	5.01e-10	4.52e-10	3.07e-10	1.25e-04	5.12
2^{16}	1.18e-05	4.91e-05	1.29e-05	1.24e-10	1.13e-10	7.67e-11	6.20e-05	5.25
2^{17}	5.67e-06	2.43e-05	6.33e-06	3.04e-11	2.82e-11	1.92e-11	3.06e-05	5.41
2^{18}	2.70e-06	1.20e-05	3.10e-06	7.47e-12	7.06e-12	4.79e-12	1.51e-05	5.59

TABLE 3. Bakhvalov mesh with $\sigma = 1$, $K = \sigma/\beta$, $\varepsilon = 10^{-8}$.

of the stability estimate (9)*, which cannot be obtained due to the non-symmetric nature of the Green's function for the convection-diffusion operator.*

5.2. Numerical results using adaptive mesh movement. We shall now consider a simple mesh movement algorithm, originally due to de Boor [6], which starts with a uniform mesh and aims to construct a mesh that solves the following equidistribution problem

$$M_i h_i = \frac{1}{N} \sum_{j=1}^{N} M_j h_j \quad \text{for} \quad i = 1, \dots, N, \tag{16}$$

where we choose the monitor function $M = M(u^N, \omega_N)$ in the algorithm from the *a posteriori* error estimate of Theorem 4:

$$M_i := \sqrt{\varrho_0 + \varrho_1 \eta_{1;i} + \varrho_2 \eta_{2;i} + \varrho_3 \eta_{3;i} + \varrho_4 \eta_{4;i} + \varrho_5 \eta_{5;i}}$$

with

$$\eta_{1;i} := C^* \min\left\{ \frac{h_i}{\|b\|_\infty}, \frac{h_i^2}{4\varepsilon} \right\} \left| \psi_{i-1/2} + \left(bu^N\right)_{\bar{x},i} \right|, \quad \eta_{2;i} := \frac{h_i}{\beta} \left| (b\Delta)_{\bar{x},i} \right|,$$

$$\eta_{3;i} := \frac{1}{\beta} \left| \sum_{k=i}^{N-1} \frac{h_{k+1} - h_k}{2} c_k \Delta_k \right|, \quad \eta_{4;i} := \frac{2}{3\beta} \left| \psi_i - 2\psi_{i-1/2} + \psi_{i-1} \right|,$$

$$\eta_{5;i} := \frac{3}{2\beta} h_i \left| \psi_i - \psi_{i-1} \right| + 3 \left| (bu^N)_i - 2(bu^N)_{i-1/2} + (bu^N)_{i-1} \right|$$

and non-negative weights ϱ_ℓ. Because all η's explicitely or implicitely involve h_i^2—see also Remark 4—we have to take the square root in the definition of the monitor function M.

The equidistribution principle (16) does not need to be enforced strictly. The de Boor algorithm we are going to describe now can be stopped when the weakend equidistribution principle

$$M_i h_i \le \frac{C_0}{N} \sum_{j=1}^{N} M_j h_j \quad \text{for} \quad i = 1, \dots, N,$$

with a user-chosen constant $C_0 > 1$ is satisfied. We will see that $C_0 = 2$ produces suitable layer-adapted meshes and requires quite few iterations.

Algorithm:

1. Initialization: Fix N and choose the weights ϱ_ℓ, $\ell = 0, \dots, 5$, and the constant $C_0 > 1$. The initial mesh $\omega_N^{(0)}$ is uniform with mesh size $1/N$.
2. For $k = 0, 1, \dots$, given the mesh $\omega^{(k)}$, compute the discrete solution $u^{N,(k)}$ by means of the defect-correction method (2) on this mesh. Set $h_i^{(k)} = x_i^{(k)} - x_{i-1}^{(k)}$ for each i. Let the piecewise-constant monitor function $\tilde{M}^{(k)}$ be defined by
$$\tilde{M}^{(k)}(x) := M_i^{(k)} := M_i\left(u^{N,(k)}, \omega_N^{(k)}\right) \quad \text{for} \quad x \in \left(x_{i-1}^{(k)}, x_i^{(k)}\right)$$
Then the total integral of the monitor function $M^{(k)}$ is
$$I^{(k)} := \int_0^1 \tilde{M}^{(k)}(x)dx = \sum_{i=1}^{N} M_i^{(k)} h_i^{(k)}.$$
3. Test mesh: If
$$\max_{i=1,\dots,N} M_i^{(k)} h_i^{(k)} \le C_0 I^{(k)} N^{-1}, \tag{17}$$
then go to Step 5. Otherwise, continue to Step 4.
4. Generate a new mesh by equidistributing the monitor function $\tilde{M}^{(k)}$ of the current computed solution: Choose the new mesh $\omega_N^{(k+1)}$ such that
$$\int_{x_{i-1}^{(k+1)}}^{x_i^{(k+1)}} M^{(k)}(x)dx = I^{(k)}/N, \quad i = 0, \dots, N.$$
(Since $\int_0^x M^{(k)}(t)\,\mathrm{d}t$ is increasing in x, the above relation clearly determines the $x_i^{(k+1)}$ uniquely.) Return to Step 2.
5. Set $\omega_N^* = \omega_N^{(k)}$ and $u^{N,*} = u^{N,(k)}$ then stop.

Our first test of the adaptive algorithm is with $\varrho_0 = 0$, $\varrho_1 = \varrho_2 = \dots = \varrho_5 = 1$, i.e., with the full error estimator of Theorem 4. We choose $C_0 = 1.001$. Thus the equidistribution principle is rather strongly enforced. Table 4 contains the results of our test computations. It contains, in dependence on N and ε, the actual errors χ^N and the error estimator η^N with the corresponding rates, the efficiency constant r^N and the number of iterations K of the de Boor algorithm. The last line contains the averaged rates.

The table illustrates how the adaptive algorithm works on minimizing the error estimator rather than the error. When the number of mesh points is doubled the theoretical error bound is reduced by a factor of 4. The rates for the actual errors are more irregular, in particular when ε is small. However, on average, i.e., over a number of refinements the rates of convergence are close to two.

The number of iterations of the algorithm can be significantly reduced by chosing C_0 larger. The effects of this choice is illustrated by Table 5 where we have taken $C_0 = 2$. The number of iterations are reduced by a factor of 10 to 20. The rates observed become more irregular, but their average over a sequence of refinements are nonetheless close to 2.

Computing the full error estimator in step 2 of the algorithm is rather expensive. In view of our observations for *a priori* chosen meshes we may simplify this step by considering the leading terms of the error estimator only. We consider the monitor function defined by $\varrho_0 = \varrho_1 = 1$ and $\varrho_2 = \dots = \varrho_5 = 0$. Chosing $\varrho_0 \neq 0$ avoids mesh starvation and ensures that the maximum step size if of order N^{-1}. Results

	$\epsilon = 10^{-2}$			$\epsilon = 10^{-4}$			$\epsilon = 10^{-8}$		
N	χ^N	η^N	r^N	χ^N	η^N	r^N	χ^N	η^N	r^N
	rate	rate	K	rate	rate	K	rate	rate	K
2^{10}	3.84e-06	1.11e-05	2.90	4.93e-06	1.12e-05	2.28	5.34e-06	1.12e-05	2.09
	1.99	1.98	5	2.02	2.01	7	2.09	1.98	50
2^{11}	9.63e-07	2.82e-06	2.93	1.21e-06	2.78e-06	2.29	1.26e-06	2.83e-06	2.25
	1.99	1.99	5	2.03	2.03	7	2.00	2.00	9
2^{12}	2.42e-07	7.10e-07	2.94	2.97e-07	6.83e-07	2.30	3.13e-07	7.07e-07	2.26
	2.00	2.00	5	2.04	2.01	17	2.01	2.01	9
2^{13}	6.05e-08	1.78e-07	2.93	7.21e-08	1.69e-07	2.35	7.77e-08	1.75e-07	2.25
	2.00	2.00	5	2.05	1.89	12	2.00	2.00	10
2^{14}	1.51e-08	4.45e-08	2.94	1.74e-08	4.56e-08	2.62	1.94e-08	4.39e-08	2.26
	2.00	2.00	5	2.08	2.00	13	2.00	2.01	9
2^{15}	3.79e-09	1.11e-08	2.94	4.11e-09	1.14e-08	2.78	4.85e-09	1.09e-08	2.25
	2.00	2.00	5	2.01	2.00	10	1.98	1.98	82
2^{16}	9.47e-10	2.79e-09	2.94	1.02e-09	2.85e-09	2.78	1.22e-09	2.76e-09	2.25
	2.00	2.00	5	2.00	1.96	5	2.00	2.00	8
2^{17}	2.37e-10	6.97e-10	2.94	2.57e-10	7.31e-10	2.85	3.06e-10	6.90e-10	2.25
	2.00	2.00	5	2.00	1.98	5	2.00	2.00	8
2^{18}	5.92e-11	1.74e-10	2.94	6.44e-11	1.85e-10	2.87	7.65e-11	1.72e-10	2.25
	—	—	5	—	—	5	—	—	8
	2.00	2.00		2.03	1.99		2.01	2.00	

TABLE 4. Adaptive algorithm with full error estimator and strongly enforced equidistribution

for this monitor and $C_0 = 2$ are given in Table 6. They compare to those for the monitor function based on the full error estimator.

In Sect. 5.1 it was observed that $\eta_{2;i} = \beta^{-1}\left|(b\Delta)_{\bar{x},i}\right|$ is the other dominant term in the error estimator. However the adaptive algorithm based on this part of the error estimator only, i.e., with $\varrho_0 = \varrho_2 = 1$ and $\varrho_1 = \varrho_3 = \varrho_4 = \varrho_5 = 0$ fails.

	$\epsilon = 10^{-2}$			$\epsilon = 10^{-4}$			$\epsilon = 10^{-8}$		
N	χ^N	η^N	r^N	χ^N	η^N	r^N	χ^N	η^N	r^N
	rate	rate	K	rate	rate	K	rate	rate	K
2^{10}	4.00e-06	1.32e-05	3.29	8.60e-06	2.29e-05	2.67	1.40e-05	4.41e-05	3.15
	2.01	2.05	2	2.18	2.17	3	2.18	2.43	4
2^{11}	9.91e-07	3.18e-06	3.21	1.90e-06	5.09e-06	2.68	3.09e-06	8.19e-06	2.65
	2.01	2.03	2	2.30	2.28	3	2.09	2.10	4
2^{12}	2.46e-07	7.81e-07	3.17	3.84e-07	1.05e-06	2.73	7.24e-07	1.91e-06	2.64
	2.00	2.00	2	2.21	2.31	3	2.27	2.26	4
2^{13}	6.14e-08	1.95e-07	3.18	8.33e-08	2.12e-07	2.55	1.50e-07	3.98e-07	2.66
	2.00	2.01	2	2.03	1.21	3	2.33	2.46	4
2^{14}	1.53e-08	4.86e-08	3.17	2.04e-08	9.19e-08	4.51	2.97e-08	7.22e-08	2.43
	2.00	2.00	2	2.15	2.48	2	1.54	1.39	4
2^{15}	3.83e-09	1.21e-08	3.17	4.59e-09	1.65e-08	3.59	1.02e-08	2.76e-08	2.69
	2.00	2.00	2	2.07	2.19	2	2.04	2.07	3
2^{16}	9.58e-10	3.03e-09	3.17	1.10e-09	3.61e-09	3.29	2.48e-09	6.56e-09	2.64
	2.00	2.00	2	2.03	2.10	2	2.01	2.01	3
2^{17}	2.39e-10	7.58e-10	3.16	2.69e-10	8.42e-10	3.13	6.16e-10	1.62e-09	2.64
	2.00	2.00	2	2.01	2.00	2	2.00	2.01	3
2^{18}	5.99e-11	1.89e-10	3.16	6.65e-11	2.10e-10	3.16	1.54e-10	4.05e-10	2.63
	—	—	2	—	—	2	—	—	3
	2.00	2.01		2.12	2.09		2.06	2.09	

TABLE 5. Adaptive algorithm with full error estimator and weak equidistribution

	$\epsilon = 10^{-2}$			$\epsilon = 10^{-4}$			$\epsilon = 10^{-8}$		
N	χ^N	η^N	r^N	χ^N	η^N	r^N	χ^N	η^N	r^N
	rate	rate	K	rate	rate	K	rate	rate	K
2^{10}	4.67e-06	1.42e-05	3.05	4.75e-06	1.18e-05	2.48	5.81e-06	2.84e-05	4.89
	2.12	2.09	2	2.06	2.03	3	2.08	2.71	4
2^{11}	1.07e-06	3.34e-06	3.11	1.14e-06	2.88e-06	2.53	1.38e-06	4.34e-06	3.15
	2.02	2.03	2	2.02	1.98	3	2.33	2.57	4
2^{12}	2.65e-07	8.18e-07	3.09	2.80e-07	7.31e-07	2.61	2.74e-07	7.29e-07	2.66
	2.00	2.00	2	1.98	1.96	3	2.00	2.00	4
2^{13}	6.61e-08	2.04e-07	3.08	7.11e-08	1.88e-07	2.64	6.86e-08	1.82e-07	2.65
	2.00	2.00	2	1.92	1.87	3	1.99	1.98	4
2^{14}	1.65e-08	5.09e-08	3.08	1.88e-08	5.16e-08	2.74	1.72e-08	4.61e-08	2.67
	2.00	2.00	2	2.00	1.96	3	1.98	1.88	4
2^{15}	4.13e-09	1.27e-08	3.08	4.70e-09	1.33e-08	2.82	4.38e-09	1.25e-08	2.85
	2.00	2.00	2	2.00	1.97	3	1.93	1.73	4
2^{16}	1.03e-09	3.18e-09	3.08	1.18e-09	3.37e-09	2.87	1.15e-09	3.76e-09	3.28
	2.00	2.00	2	2.00	1.95	3	2.06	2.33	6
2^{17}	2.58e-10	7.96e-10	3.08	2.94e-10	8.75e-10	2.98	2.76e-10	7.49e-10	2.71
	2.00	2.00	2	1.79	1.78	3	1.90	1.96	3
2^{18}	6.45e-11	1.99e-10	3.08	8.47e-11	2.55e-10	3.02	7.41e-11	1.92e-10	2.59
	—	—	2	—	—	2	—	—	3
	2.02	2.02		1.98	1.94		2.03	2.14	

TABLE 6. Adaptive algorithm with reduced error estimator and weak equidistribution

Remark 7. *Many successful algorithms in higher dimensions simply assume that the error is related to the interpolation error. Thus the generated, possibly, anisotropic mesh is supposed to be (quasi-)uniform under a certain metric induced, e.g., by the positive definite Hessian matrix of the solution* [7, 11, 25]. *Note that the principal part of our error estimator is* $\eta_1 \lesssim C^* \max_i\{h_i^2|u_{x\hat{x}}^N|\}$*, so our adaptive algorithm may be viewed as similar to interpolation-error-based algorithms.*

5.3. Results for variants of the defect correction method. In Sect. 4 the precise design of the DCM has been studied. We shall confirm this by numerical experiments now. To this end consider (1) with $\varepsilon = 10^{-8}$ again. We use a modified Shishkin mesh which is constructed as follows. Pick the transition point $\tau = 2\varepsilon\beta^{-1}\ln N$ as usual. Set $h = 2\tau/N$ and $H = 2(1-\tau)/N$. Then the mesh is defined by

$$h_i = \begin{cases} 7h/6 & \text{if } i \text{ is odd and } i \le N/2, \\ 5h/6 & \text{if } i \text{ is even and } i \le N/2, \\ 7H/6 & \text{if } i \text{ is odd and } i > N/2, \\ 5H/6 & \text{if } i \text{ is even and } i > N/2. \end{cases}$$

Thus instead of a uniform mesh on each of the two subdomains $[0,\tau]$ and $[\tau,1]$ we use non-uniform, though very regular sub meshes.

Table 7 displays results for three variants of the defect correction method:

- $\kappa_i = \hbar_i/h_{i+1}$, $h_i' = h_i'' = h_{i+1}$, i.e. the method analysed in Sect. 3,
- $\kappa_i = 1$, $h_i' = h_i'' = h_{i+1}$ and
- $\kappa_i = 1$, $h_i' = \hbar_i$, $h_i'' = h_{i+1}$.

The table gives the errors and the estimated "Shishkin" rates of convergence, i.e. p in the error bound $C\left(N^{-1}\ln N\right)^p$ computed from the numerical solution.

	$\kappa_i = \hbar_i/h_{i+1}$,		$\kappa_i = 1$		$\kappa_i = 1$	
	$h'_i = h''_i = h_{i+1}$		$h'_i = h''_i = h_{i+1}$		$h'_i = \hbar_i,\ h''_i = h_{i+1}$	
N	χ^N	p^N	χ^N	p^N	χ^N	p^N
2^{10}	9.13e-05	2.00	6.93e-03	0.00	4.83e-02	0.96
2^{11}	2.76e-05	2.00	6.95e-03	0.00	2.72e-02	0.98
2^{12}	8.21e-06	2.00	6.97e-03	0.00	1.51e-02	0.99
2^{13}	2.41e-06	2.00	6.98e-03	0.00	8.21e-03	0.99
2^{14}	6.98e-07	2.00	6.99e-03	0.00	4.44e-03	1.00
2^{15}	2.00e-07	2.00	6.99e-03	0.00	2.38e-03	1.00
2^{16}	5.69e-08	2.00	6.99e-03	0.00	1.27e-03	1.00
2^{17}	1.61e-08	2.00	6.99e-03	0.00	6.77e-04	1.00
2^{18}	4.50e-09	—	6.99e-03	—	3.58e-04	—

TABLE 7. Variants of the DCM on a modified Shishkin mesh

In full agreement with the theoretical conclusions of Sect. 4, the latter two variants fail to achieve second-order convergence aimed at by the defect correction approach.

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