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ERROR ANALYSIS OF A 2D SINGULARLY PERTURBED SEMILINEAR REACTION-DIFFUSION PROBLEM[1]

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Abstract. A semilinear reaction-diffusion equation with multiple solutions is considered in a smooth two-dimensional domain. Its diffusion parameter ε^2 is arbitrarily small, which induces boundary layers. Constructing discrete sub- and super-solutions, we prove existence and investigate the accuracy of multiple discrete solutions on layer-adapted meshes of Bakhvalov and Shishkin types. It is shown that one gets second-order convergence (with, in the case of the Shishkin mesh, a logarithmic factor) in the discrete maximum norm, uniformly in ε for $\varepsilon \le Ch$. Here $h > 0$ is the maximum side length of mesh elements, while the number of mesh nodes does not exceed Ch^{-2}. Numerical experiments are performed to support the theoretical results.

Key words: singular perturbation, semilinear reaction-diffusion, maximum norm error estimate, Z-field, Bakhvalov mesh, Shishkin mesh, second order

1. Introduction

Consider the singularly perturbed semilinear reaction-diffusion boundary-value problem

$$Fu \equiv -\varepsilon^2 \triangle u + b(x,u) = 0, \qquad x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2, \tag{1.1a}$$

$$u(x) = g(x), \qquad x \in \partial\Omega, \tag{1.1b}$$

where ε is a small positive parameter, $\triangle = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ is the Laplace operator, and Ω is a bounded two-dimensional domain whose boundary $\partial\Omega$ is sufficiently smooth. Assume also that the functions b and g are sufficiently

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smooth. We shall examine solutions of (1.1) that exhibit boundary layer behaviour.

The reduced problem of (1.1) is defined by formally setting $\varepsilon = 0$ in (1.1a):

$$b(x, u_0(x)) = 0 \quad \text{for } x \in \Omega. \tag{1.2}$$

Note that any solution u_0 of (1.2) does not in general satisfy the boundary condition in (1.1b).

In the numerical analysis literature it is often assumed—see [1, 5]—that $b_u(x,u) > \gamma^2 > 0$ for all $(x,u) \in \Omega \times \mathbb{R}^1$, for some positive constant γ. Under this condition the reduced problem has a unique solution u_0 that is sufficiently smooth in $\bar{\Omega}$, as can be seen by using the implicit function theorem and the compactness of $\bar{\Omega}$. This global condition is nevertheless rather restrictive. E.g., mathematical models of biological and chemical processes frequently involve problems related to (1.1) with $b(x,u)$ that is *non-monotone* with respect to u. Hence we consider problem (1.1) under the following weaker *assumptions* from [4]:

- it has a *stable reduced solution*, i.e., there exists a sufficiently smooth solution u_0 of (1.2) such that

$$b_u(x, u_0) > \gamma^2 > 0 \quad \text{for all } x \in \Omega; \tag{$A1$}$$

- the boundary condition satisfies

$$\int_{u_0(x)}^{v} b(x,s)\, ds > 0 \qquad \text{for all } \; v \in \big(u_0(x), g(x)\big]', \qquad x \in \partial\Omega. \tag{$A2$}$$

  Here the notation $(a,b]'$ is defined to be $(a,b]$ when $a < b$ and $[b,a)$ when $a > b$, while $(a,b]' = \emptyset$ when $a = b$.

Note that if $g(x) \approx u_0(x)$, then (A2) follows from (A1) combined with (1.2), while if $g(x) = u_0(x)$ at some point $x \in \partial\Omega$, then (A2) does not impose any restriction on g at this point.

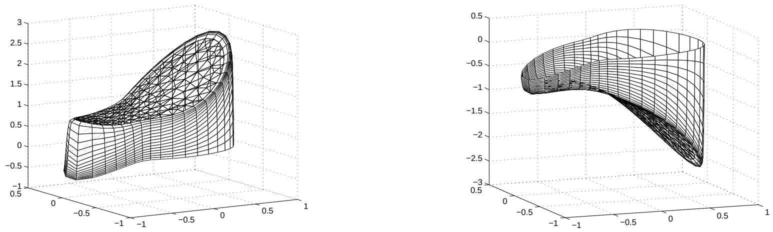
Conditions (A1), (A2) intrinsically arise from the asymptotic analysis of problem (1.1) and guarantee that there exists a boundary-layer solution u of (1.1) such that $u \approx u_0$ in the interior subdomain of Ω away from the boundary, while the boundary layer is of width $O(\varepsilon|\ln\varepsilon|)$ [4]; see Theorem 1 for a precise statement. Note that assumption (A1) is local, i.e., the reduced problem (1.2) is permitted to have more than one solution. Furthermore, if multiple stable solutions of the reduced problem satisfy (A2), problem (1.1) has *multiple* boundary-layer solutions; see Figure 1.

We make two further simplifying assumptions to facilitate our presentation. To avoid considering cases, assume that

$$u_0(x) < g(x) \quad \text{for all} \quad x \in \partial\Omega. \tag{$A3$}$$

Throughout our analysis take

$$\varepsilon \le Ch, \tag{$A4$}$$



a) b)

Figure 1. Multiple boundary-layer solutions of the model problem from [2, §5]: in the interior subdomain a) $u(x) \approx \bar{u}_0(x)$, b) $u(x) \approx -\bar{u}_0(x)$, where $\pm\bar{u}_0(x)$ are stable solutions of the reduced problem (1.2).

where $h > 0$ is the maximum side length of mesh elements, while the number of mesh nodes does not exceed Ch^{-2}. This is not a practical restriction, and from a theoretical viewpoint the analysis of a nonlinear problem such as (1.1) would be very different if ε were not small.

In this paper we present (almost) second-order nodal maximum norm error estimates for the computed solutions of problem (1.1) on layer-adapted meshes. A one-dimensional version of this problem was studied in [2]. We extend the analysis in [2] to two dimensions.

Notation. Throughout this paper we let C denote a generic positive constant that may take different values in different formulas, but is always independent of h and ε.

2. Asymptotic Expansion and its Generalization

Let the boundary $\partial\Omega$ be described by $x_1 = \varphi(l),\ x_2 = \psi(l),\ 0 \le l \le L$, where $(\varphi(0), \psi(0)) = (\varphi(L), \psi(L))$. We shall use the magnitude $\tau > 0$ of the tangent vector (φ', ψ') and the curvature κ of the boundary at $(\varphi(l), \psi(l))$ that are defined by

$$\tau = \sqrt{\varphi'^2 + \psi'^2}, \qquad \kappa = \kappa(l) = \frac{\varphi'\psi'' - \psi'\varphi''}{\tau^3}.$$

In a narrow neighbourhood of $\partial\Omega$ that will be specified later, introduce the curvilinear local coordinates (r, l) by

$$x_1 = \varphi(l) + r n_1(l), \qquad x_2 = \psi(l) + r n_2(l), \tag{2.1}$$

where (n_1, n_2) is the unit normal to $\partial\Omega$ at $(\varphi(l), \psi(l))$, i.e. it is orthogonal to the tangent vector (φ', ψ') and defined by $n_1 = -\psi'/\tau,\ n_2 = \varphi'/\tau$. Since $\partial\Omega$ is smooth, there exists a sufficiently small constant C_1 such that in the subdomain $\bar{\Omega}_{C_1} = \{0 \le r \le C_1\}$ the new coordinates are well-defined and the mapping $(r, l) \mapsto (x_1, x_2)$ is a one-to-one and invertible. Throughout the paper we shall use a smooth positive cut-off function $\omega(x)$ that equals 1 for $r \le C_1/2$ and vanishes in $\bar{\Omega} \backslash \bar{\Omega}_{C_1}$.

Lemma 1. *For the Laplace operator we have*

$$\triangle u = \eta^{-1}\frac{\partial}{\partial r}\Big(\eta\frac{\partial u}{\partial r}\Big) + \zeta\frac{\partial}{\partial l}\Big(\zeta\frac{\partial u}{\partial l}\Big),$$

where $\eta := 1-\kappa r,\ \zeta := (\tau\eta)^{-1}$.

Now introduce the stretched variable $\xi := r/\varepsilon$ and the functions $v_0(\xi,l)$ and $v_1(\xi,l)$ defined by

$$-\frac{\partial^2 v_0}{\partial\xi^2} + b(\bar{x}, u_0(\bar{x}) + v_0) = 0,$$

$$-\frac{\partial^2 v_1}{\partial\xi^2} + v_1 b_u(\bar{x}, u_0(\bar{x}) + v_0) = -\xi\,\frac{d}{dr}b\big(x, u_0(x)+t\big)\Big|_{x=\bar{x},t=v_0} - \kappa\frac{\partial v_0}{\partial\xi},$$

for $\xi > 0$, with the boundary conditions

$$v_0(0,l) = g(\bar{x}) - u_0(\bar{x}),\quad v_1(0,l) = 0,\quad v_0(\infty,l) = v_1(\infty,l) = 0.$$

Here $x = x(r,l)$ is defined by (2.1),

$$\bar{x} = \bar{x}(l) := (\varphi(l),\psi(l)),\qquad \eta^{-1}\frac{\partial\eta}{\partial r}\Big|_{r=0} = \frac{-\kappa}{1-\kappa r}\Big|_{r=0} = -\kappa.$$

Theorem 1. *[4, Theorem 3] Under hypotheses* (A1), (A2)*, for sufficiently small* ε *there exists a solution* $u(x)$ *of* (1.1) *in a neighbourhood of the zero-order asymptotic expansion* $u_0(x) + v_0(\xi,l)\omega(x)$*. Furthermore, for the first-order asymptotic expansion*

$$u_{\rm as}(x) := u_0(x) + [v_0(\xi,l) + \varepsilon v_1(\xi,l)]\omega(x)$$

we have

$$\big|Fu_{\rm as}(x)\big| \le C\varepsilon^2,\quad \big|u(x) - u_{\rm as}(x)\big| \le C\varepsilon^2\quad\text{for all } x\in\bar{\Omega}.$$

To construct discrete sub- and super-solutions, we shall use the auxiliary function $v(\xi,l;p)$, where p is a small real parameter, defined by

$$-\frac{\partial^2 v}{\partial\xi^2} + b(\bar{x}, u_0(\bar{x}) + v) = pv,$$

$$v(0,l;p) = g(\bar{x}) - u_0(\bar{x}),\quad v(\infty,l;p)) = 0.$$

Furthermore, set

$$\beta(x;p) := u_0(x) + \big[v(\xi,l;p) + \varepsilon v_1(\xi,l)\big]\omega(x) + C_0 p,$$

where C_0 is a sufficiently small positive constant.

Lemma 2. *There exists* $C_0 > 0$ *such that for all* $|p| \le p_0$ *we have*

$$F\beta \ge \quad C_0 p\,\gamma^2 + O(\varepsilon^2 + p^2),\quad \text{if}\quad p > 0,$$
$$F\beta \le -C_0|p|\,\gamma^2 + O(\varepsilon^2 + p^2),\quad \text{if}\quad p < 0.$$

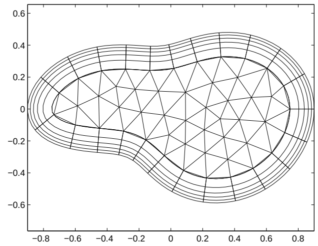


Figure 2. Layer-adapted mesh.

3. Layer-Adapted Meshes

Introduce a small positive parameter σ that will be specified later. Let $\sigma \le C_1$ so that the closed curve $\partial\Omega_\sigma$ that is defined by the equation $r = \sigma$ does not intersect itself. Furthermore, let Ω_σ be the interior of $\partial\Omega_\sigma$. Our problem will be discretized separately in Ω_σ and $\Omega \setminus \Omega_\sigma$, to which we shall refer as the interior region and the layer region respectively; see Figure 2.

The boundary-layer region $\Omega \setminus \Omega_\sigma$ is the rectangle $(0,\sigma) \times [0,L]$ in the coordinates (r,l). Hence in this subdomain introduce the tensor-product mesh $\{(r_i,l_j), i = 0,\ldots,N, j = 0,\ldots N_l\}$, where, as usual, $r_0 = 0$, $r_N = \sigma$, $l_0 = 0$, and $l_{N_l} = L$. Furthermore, let $\{l_j\}$ be a quasiuniform mesh on $[0,L]$, i.e., $C^{-1}h \le l_j - l_{j-1} \le Ch$. The choice of the *layer-adapted* mesh $\{r_i\}$ on $[0,\sigma]$ is crucial and will be discussed later; see (a),(b). Now we only assume that $r_i - r_{i-1} \le Ch$ and $C^{-1}h^{-1} < N \le Ch^{-1}$.

In the interior region Ω_σ introduce a quasiuniform Delaunay triangulation, i.e., the maximum side length of any triangle is at most h, the area of any triangle is bounded below by Ch^2, and the sum of the angles opposite to any edge is less than or equal to π. Then the piecewise linear finite element discretization of the operator $-\triangle$ yields an M-matrix. We also require that both the interior and layer meshes have the same sets of nodes on $\partial\Omega_\sigma$.

(a) Set $\sigma := 2\gamma^{-1}\varepsilon|\ln\varepsilon|$ and define a *Bakhvalov*-type mesh by

$$r_i := r(i/N),\ i = 0\ldots,N,\ \text{where } r(t) = -2\gamma^{-1}\varepsilon\ln(1-t) \text{ for } t \in [0,1-\varepsilon].$$

(b) Define a *Shishkin* mesh as follows. Set $\sigma = 2\gamma^{-1}\varepsilon\ln N$ and introduce a uniform mesh $\{r_i\}_{i=0}^N$ on $[0,\sigma]$, i.e. $r_i - r_{i-1} = \sigma/N = 2\gamma^{-1}\varepsilon N^{-1}\ln N$.

4. *Z*-Field Discretization

Definition 1. An operator $H : \mathbb{R}^n \to \mathbb{R}^n$ is a Z-field if for all $i \ne j$ the mapping $x_j \mapsto \big(H(x_1,x_2,\ldots,x_n)\big)_i$ is a monotonically decreasing function from $\mathbb{R}$ to $\mathbb{R}$ when $x_1,\ldots,x_{j-1},x_{j+1},\ldots x_n$ are fixed.

Lemma 3. *[2, 3] Let $H : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and a Z-field. Let $r \in \mathbb{R}^n$ be given. Assume that there exist $\alpha, \beta \in \mathbb{R}^n$ such that $\alpha \le \beta$ and $H\alpha \le r \le H\beta$. (The inequalities are understood to hold true component-wise.) Then the equation $Hy = r$ has a solution $y \in \mathbb{R}^n$ with $\alpha \le y \le \beta$.*

Remark 1. Let $X_1, X_2 \ldots, X_n$ be interior points of Ω, while $\bar{X}_1, \bar{X}_2 \ldots, \bar{X}_m$ be on $\partial\Omega$, and $U \in \mathbb{R}^{n+m}$ be a discrete function defined at these points. Suppose that $F^h : \mathbb{R}^{n+m} \to \mathbb{R}^n$ has the form

$$F^h U = \varepsilon^2 \Lambda U + \big[b(X_i, U_i)\big]_{i=1}^n,$$

where Λ is an M-matrix discretization of the operator $-\triangle$. Then the mapping

$$(X_1 \ldots, X_n, \bar{X}_1, \ldots, \bar{X}_m) \mapsto \big(F^h U_1, \ldots, F^h U_n, g(\bar{X}_1), \ldots, g(\bar{X}_m)\big)$$

is a Z-field.

Thus to invoke the theory of Z-fields we require the following:

(i) an M-matrix discretization of $-\triangle$;
(ii) the discretization of $b(x, u)$ at any interior mesh point X_i involves only U_k with $k = i$. Hence we use finite differences in the layer region and the lumped mass finite elements on Delaunay triangulations in the interior region.

Theorem 2. *There exists a discrete solution U on the Bakhvalov/Shishkin meshes (a)/(b) that satisfies*

$$|U(X_i) - u(X_i)| \le Ch^2 |\ln h|^m \quad \textit{for all mesh nodes } X_i.$$

where $m = 0$ for the Bakhvalov mesh and $m = 2$ for the Shishkin mesh.

5. Numerical Results

Our model problem is (1.1) in the domain Ω as in Figure 2 with

$$b(x, u) = \big(u - \bar{u}_0(x)\big) u \big(u + \bar{u}_0(x)\big), \quad \bar{u}_0(x) = x_1^2 + x_1 + 1.$$

Here $\pm\bar{u}_0(x)$ are two stable solutions and 0 is an unstable solution of the corresponding reduced problem. The boundary condition $g(x) = (x_1 - x_1^2)/3$ satisfies (A2) for both $\pm\bar{u}_0$; see Figure 1. Table 1 presents numerical results for computed solutions close to $\bar{u}_0$ on the Bakhvalov mesh. We give rates of convergence and maximum nodal errors computed as described in [2, §4].

Table 1. Bakhvalov mesh. Rates of convergence and maximum nodal errors.

N	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-8}$
32	2.010	2.011	2.011
64	1.995	1.997	1.997
128	1.995	2.001	2.001
32	3.745e-3	3.842e-3	3.843e-3
64	9.296e-4	9.534e-4	9.536e-4
128	2.333e-4	2.388e-4	2.388e-4
256	5.854e-5	5.967e-5	5.968e-5

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