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Maximum norm a posteriori error estimates for a one-dimensional singularly perturbed semilinear reaction-diffusion problem*

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A singularly perturbed semilinear two-point boundary value problem is discretized on arbitrary nonuniform meshes. We present second-order maximum norm a posteriori error estimates that hold true uniformly in the small parameter. Their application to monitor-function equidistribution and a posteriori mesh refinement is discussed. Numerical results are presented that support our theoretical estimates.

Keywords: reaction-diffusion, singular perturbation, finite differences, maximum norm, a posteriori error estimate, layer-adapted mesh, grid equidistribution.

1. Introduction

It is well-known that standard numerical methods applied to singularly perturbed problems, such as (1.1) below, may yield inaccurate results when the exact solution has sharp layers. One effective approach to obtaining accurate numerical solutions is to use suitable layer-adapted meshes that are very fine where layers appear in the solution. Such meshes can be divided into two classes. If a priori information is available about the layers, a suitable mesh can be constructed *a priori*; see, e.g. Bakhvalov (1969), Miller *et al.* (1996), Roos *et al.* (1996). Since this is often not the case, it is desirable to construct a suitable mesh *a posteriori*, i.e. an algorithm starts from an initial unsophisticated mesh and then detects the layers and generates a layer-adapted mesh using only the intermediate computed solutions and meshes.

This paper is concerned with a posteriori error estimates, which underlie any reliable a posteriori mesh construction. In grid equidistribution algorithms, they indicate a suitable monitor function to be equidistributed; in mesh refinement algorithms, they may suggest suitable refinement indicators; see §4 for more details. Note that our a posteriori error estimates are in the maximum norm, which is sufficiently strong to capture layers and hence seems most appropriate for singularly perturbed problems.

We consider the singularly perturbed semilinear reaction-diffusion problem

$$Tu = -\varepsilon^2 u''(x) + b(x,u) = 0 \quad \text{for } x \in (0,1), \qquad u(0) = u(1) = 0, \tag{1.1}$$

where $\varepsilon \in (0,1]$ is a small positive parameter, the function b is sufficiently smooth and

$$0 < \beta < b_u(x,u) \leqslant \bar{\beta} \qquad \text{for all } (x,u) \in [0,1] \times \mathbb{R}. \tag{1.2}$$

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Under condition (1.2), problem (1.1) has a unique solution, which exhibits sharp boundary layers of width $O(\varepsilon|\ln\varepsilon|)$ at $x=0$ and $x=1$; see Vulanović (2004). By (1.2), our problem is only weakly semilinear. This assumption enables us to linearize (1.1) and then invoke Green's functions in our analysis.

The present paper follows Kopteva (2001), where certain maximum norm a posteriori error estimates were derived for a singularly perturbed convection-diffusion problem. Here, similarly, the error in the maximum norm will be shown to be bounded by the residual (understood in the sense of distributions) in some weak negative norm. Then the latter will be estimated using simple a posteriori quantities, such as the right-hand side in (1.3). Since our differential operator T is very different from the convection-diffusion operator in Kopteva (2001)—see §2—the residual will be measured and then estimated in a somewhat different way from Kopteva (2001).

We discretize (1.1) using the standard second-order three-point difference scheme—see (3.2) for details—on an arbitrary mesh $\{x_i\}_{i=0}^N$, where $0=x_0<x_1<\cdots<x_N=1$, and $h_i=x_i-x_{i-1}$ is the local mesh size. Our main result is the following maximum norm *a posteriori error estimate*:

$$\big\|u^N(\cdot)-u(\cdot)\big\|_\infty \leqslant C\max_i\Big\{h_i\Big[\frac{|D^2u_{i-1}^N|^{1/2}+|D^2u_i^N|^{1/2}}{2}+1\Big]\Big\}^2; \tag{1.3}$$

see Theorem 3.6. Here $\{u_i^N\}_{i=0}^N$ is the computed solution, $u^N(x)$ is its piecewise linear interpolant (3.5), and $D^2u_i^N$ is the standard discrete approximation of $u''(x_i)$ defined in (3.1). Note that (1.3) follows from a more general a posteriori error estimate (3.7). Both of these estimates hold ε-uniformly, where $\varepsilon\in(0,1]$, and under no restrictions on the mesh.

The paper is organized as follows. In §2 we prove some stability properties of the differential operator T from (1.1). In §3 we describe the numerical method and present maximum norm a posteriori error estimates, which are the main results of the paper. In §4 we discuss the application of our a posteriori error estimates to monitor-function equidistribution and a posteriori mesh refinement. Numerical results are presented in §5 that support our theoretical estimates. Finally, in §6 we prove Lemma 3.5.

Notation: Throughout the paper, C denotes a generic positive constant that is independent of ε and of any mesh used. It may take different values in different places. When C is subscripted, it is a fixed constant that is independent of ε and the mesh.

In our estimates we use the L_∞, L_1, and $W^{-1,\infty}=\big(W_0^{1,1}\big)'$ norms defined, respectively, by

$$\big\|v(\cdot)\big\|_\infty=\operatorname*{ess\ sup}_{x\in[0,1]}\big|v(x)\big|,\qquad \big\|v(\cdot)\big\|_1=\int_0^1\big|v(x)\big|\,\mathrm{d}x,\qquad \big\|v(\cdot)\big\|_{-1,\infty}=\min_{V:V'=v}\big\|V(\cdot)\big\|_\infty. \tag{1.4}$$

Note that $\|F'\|_{-1,\infty}\leqslant\|F\|_\infty$.

REMARK 1.1 The assumption $b_u(x,u)\leqslant\bar\beta$ in (1.2) can be omitted since it follows, for some constant $\bar\beta$, from $0<\beta<b_u(x,u)$ and u being a unique and bounded solution of (1.1); see, e.g., Schatz & Wahlbin (1983), §12, or Vulanović (2004), §2.1.

2. Stability properties of differential operators

2.1 *Linear reaction-diffusion*

We start with a linear case of (1.1), where $b(x,u):=p(x)u-f(x)$:

$$Lu=-\varepsilon^2u''(x)+p(x)u(x)=f(x)\ \text{ for }\ x\in(0,1),\quad u(0)=u(1)=0. \tag{2.1}$$

Here $p \in C([0,1])$ and, in accordance with (1.2),

$$0 < \beta \leqslant p(x) \leqslant \bar{\beta}. \tag{2.2}$$

Introduce the Green's function of the operator L that for each $\xi \in (0,1)$ satisfies

$$\begin{aligned} LG(x,\xi) = -\varepsilon^2 G_{xx}(x,\xi) + p(x)G(x,\xi) &= \delta(x-\xi), \qquad x \in (0,1), \\ G(0,\xi) = G(1,\xi) &= 0, \end{aligned} \tag{2.3}$$

where $\delta(\cdot)$ is the Dirac δ-distribution. Then the solution u of (2.1) is given by

$$u(x) = \int_0^1 G(x,\xi)\, f(\xi)\, \mathrm{d}\xi. \tag{2.4}$$

Furthermore, since L is self-adjoint, $G(x,\xi)$ also solves the following problem for each $x \in (0,1)$:

$$\begin{aligned} -\varepsilon^2 G_{\xi\xi}(x,\xi) + p(\xi)G(x,\xi) &= \delta(x-\xi), \qquad x \in (0,1), \\ G(x,0) = G(x,1) &= 0. \end{aligned} \tag{2.5}$$

Note that $G_{xx}(\cdot,\xi)$ for each fixed ξ and $G_{\xi\xi}(x,\cdot)$ for each fixed x are distributions.

REMARK 2.1 The right-hand side of (2.1) will often be considered in the form $f(x) = -F'(x)$, where $F(x)$ is at least a bounded piecewise continuous function. This implies that f can have singularities similar to the Dirac δ-distribution. Thus problem (1.1) is solved in the sense of distributions; see, e.g., Griffel (1981). To be precise, under our assumptions there exists a unique weak solution $u \in C^{0,1}\big([0,1]\big) \subset W^{1,2}(0,1) \subset C\big([0,1]\big)$, where $C^{0,1}\big([0,1]\big)$ is a standard Hölder space and $W^{1,2}(0,1)$ is a standard Sobolev space.

LEMMA 2.2 Let $f = -F'$, where $F(x)$ is a bounded piecewise continuous function; then there exists a unique solution $u \in C([0,1])$ of (2.1),(2.2). Furthermore,

$$\|u\|_\infty \leqslant C_1 \|Lu\|_\infty = C_1 \|f\|_\infty, \tag{2.6a}$$

$$\|u\|_\infty \leqslant C_2 \varepsilon^{-1} \|Lu\|_{-1,\infty} \leqslant C_2 \varepsilon^{-1} \|F\|_\infty, \tag{2.6b}$$

where $C_1 := \beta^{-1}$ and $C_2 := \beta^{-1/2}$.

Proof. Estimate (2.6a) is well-known and follows from the maximum/comparison principle extended to functions in the Sobolev space $W^{1,2}(0,1)$ (see Gilbarg & Trudinger (1998), §8.1). The second desired estimate (2.6b) follows from (2.4) combined with

$$\max_x \|G_\xi(x,\cdot)\|_1 \leqslant C_2 \varepsilon^{-1}. \tag{2.7}$$

Thus to complete the proof it suffices to obtain (2.7).

Note that, by (2.5), for each fixed x the function $G(x,\xi) \geqslant 0$ is increasing for $\xi \in (0,x)$ and decreasing for $\xi \in (x,1)$. Hence,

$$\|G_\xi(x,\cdot)\|_{L_1} = 2G(x,x). \tag{2.8}$$

Fix ξ and denote $v(x) := G(x,\xi)$. Set $v_0 := v(\xi) = G(\xi,\xi)$ and use this as a boundary condition to augment (2.3), i.e., treat $v(x)$ as a solution of the two corresponding boundary problems on $(0,\xi)$ and $(\xi,1)$. By the maximum/comparison principle applied to these boundary problems, we have

$$v(x) \leqslant B_1(x) := v_0 B(\xi - x) \ \text{ for } \ x \in [0,\xi]; \qquad v(x) \leqslant B_2(x) := v_0 B(x-\xi) \ \text{ for } \ x \in [\xi,1].$$

Here $B(x) := e^{-\sqrt{\beta}x/\varepsilon}$ so that, by (2.2), $LB = [-\beta + p]B \geqslant 0$. Now, one can easily check that

$$v'(\xi^-) - v'(\xi^+) \geqslant v_0\big[B_1'(\xi) - B_2'(\xi)\big] = 2v_0\frac{\sqrt{\beta}}{\varepsilon}.$$

Combining this with the standard Green's function derivative condition $v'(\xi^-) - v'(\xi^+) = 1/\varepsilon^2$, we get $G(\xi,\xi) = v_0 \leqslant (2\sqrt{\beta}\varepsilon)^{-1} = C_2(2\varepsilon)^{-1}$. Now, by (2.8), we have (2.7) and complete the proof.

Note that a similar estimate of $G(\xi,\xi)$ follows from Theorem 4.2 by O'Riordan & Stynes (1986), where $p(x)$ is piecewise constant and the constant C_2 is not specified. A discrete analogue of (2.7) is given in Lemma 2.2 of Andreev (2004) with $C_2 = 2\beta^{-1/2}$. □

COROLLARY 2.3 For the Green's function $G(x,\xi)$ from (2.3) we have

$$\max_x \|G(x,\cdot)\|_1 \leqslant C_1, \qquad \max_x \int_0^1 |G_{\xi\xi}(x,\xi)|\,\mathrm{d}\xi \leqslant C_3/\varepsilon^2, \tag{2.9}$$

where $C_1 = \beta^{-1}$ is from Lemma 2.2 and $C_3 := \bar{\beta}/\beta$.

Proof. The first relation follows from (2.4) with $f = Lu$ combined with (2.6a). To get the second desired relation, obtain $G_{\xi\xi}$ from (2.5) and note that $\int_0^1 \delta(x-\xi)\,\mathrm{d}\xi = 1$ and $G(x,\xi) \geqslant 0$. Hence $C_3 = \max\{\bar{\beta}C_1, 1\} = \bar{\beta}/\beta$. □

LEMMA 2.4 Let $F(x) = A_i(x - x_{i-1/2})$ for $x \in (x_{i-1}, x_i)$, $i = 1, \ldots, N$, where A_i are fixed constants. Then under the conditions of Lemma 2.2 we have

$$\|u\|_\infty \leqslant \frac{C_3}{4}\max_{1\leqslant i\leqslant N}\left\{\frac{|A_i|\,h_i^2}{\varepsilon^2}\right\}. \tag{2.10}$$

Proof. Fix x and denote $v(\xi) := G(x,\xi)$. Now, by (2.4) with $f = -F'$, we have

$$u(x) = \int_0^1 f(\xi)v(\xi)\,\mathrm{d}\xi = \sum_{i=1}^N A_i\int_{x_{i-1}}^{x_i}(\xi - x_{i-1/2})\,v'(\xi)\,\mathrm{d}\xi.$$

Note that

$$v'(\xi) = v'(x_{i-1}) + \int_{x_{i-1}}^{\xi} v''(s)\,\mathrm{d}s$$

and $\int_{x_{i-1}}^{x_i}(\xi - x_{i-1/2})\,\mathrm{d}\xi = 0$. Hence

$$u(x) = \sum_{i=1}^N A_i\int_{x_{i-1}}^{x_i}(\xi - x_{i-1/2})\int_{x_{i-1}}^{\xi} v''(s)\,\mathrm{d}s\ \mathrm{d}\xi.$$

Furthermore,

$$|u(x)| \leqslant \sum_{i=1}^N |A_i|\frac{h_i^2}{4}\int_{x_{i-1}}^{x_i}|v''(s)|\,\mathrm{d}s \leqslant \max_{1\leqslant i\leqslant N}\left\{\frac{|A_i|\,h_i^2}{4}\right\}\int_0^1 |v''(\xi)|\,\mathrm{d}\xi.$$

Combine this with $\int_0^1 |v''(\xi)|\,\mathrm{d}\xi \leqslant C_3/\varepsilon^2$, which follows from the second estimate in (2.9). □

REMARK 2.5 A similar result for a singularly perturbed convection-diffusion problem is given in Lemma 2.2 of Kopteva (2001), where $u(x)$ was estimated in the nodal maximum norm. This yielded certain second-order a posteriori error estimates also in the nodal maximum norm. Note that our present estimate (2.10) and a posteriori error estimates in §3 are in the full maximum norm $C([0,1])$.

2.2 *Semilinear reaction-diffusion*

Now we discuss properties of the semilinear differential operator T from (1.1) applied to a function u that satisfies $u(0) = u(1) = 0$.

THEOREM 2.6 Let $b \in C^1([0,1] \times \mathbb{R})$ satisfy (1.2). Let also $F(x) = A_i(x - x_{i-1/2})$ for $x \in (x_{i-1}, x_i)$, $i = 1,\dots,N$, where A_i are fixed constants, $\bar{F}$ a bounded piecewise continuous function, and f in $L_\infty(0,1)$. Then, for any v, $w \in W^{1,2}(0,1)$ such that $v(0) = w(0)$, $v(1) = w(1)$ and

$$Tv(x) - Tw(x) = -\big[F(x) + \bar{F}(x)\big]' + f(x),$$

we have

$$\|v - w\|_\infty \leqslant \frac{C_3}{4} \max_{1\leqslant i\leqslant N} \left\{ \frac{|A_i|\, h_i^2}{\varepsilon^2} \right\} + C_2\varepsilon^{-1}\|\bar{F}\|_\infty + C_1\|f\|_\infty.$$

Proof. Using the standard linearization technique, we have $Tv(x) - Tw(x) = L\big[v(x) - w(x)\big]$, where the operator L is linear and defined by (2.1) with $p(x) = \int_0^1 b_u(x, w(x) + s[v(x) - w(x)])\,ds$. Since, by (1.2), condition (2.2) is satisfied, the desired estimate follows from Lemmas 2.2 and 2.4. □

3. Analysis of the numerical method. Maximum norm a posteriori error estimates

3.1 *Numerical method: preliminary results*

Given a discrete function $\{v_i\}$, define the discrete difference operators

$$\begin{aligned} &D^- v_i = \frac{v_i - v_{i-1}}{h_i}, \qquad D v_i = \frac{v_{i+1} - v_i}{\hbar_i}, \qquad D^2 v_i = DD^- v_i = \frac{D^- v_{i+1} - D^- v_i}{\hbar_i}, \\ &h_i = x_i - x_{i-1}, \qquad \hbar_i = (h_i + h_{i+1})/2, \qquad x_{i-1/2} = x_i - h_i/2, \qquad h = \max_i h_i. \end{aligned} \tag{3.1}$$

We require the computed solution u^N to satisfy the standard finite difference discretization of (1.1):

$$T^N u_i^N := -\varepsilon^2 D^2 u_i^N + b(x_i, u_i^N) = 0 \quad \text{for } i = 1,\dots,N-1, \qquad u_0^N = u_N^N = 0. \tag{3.2}$$

LEMMA 3.1 Under condition (1.2), on an arbitrary mesh $\{x_i\}_{i=0}^N$ there exists a unique solution $\{u_i^N\}_{i=0}^N$ of (3.2). Furthermore,

$$\max_{0\leqslant i\leqslant N} |u_i^N| \leqslant C_4, \qquad \text{where} \quad C_4 := \beta^{-1}\|b(\cdot,0)\|_\infty. \tag{3.3}$$

Proof. Existence and uniqueness can be shown by imitating the argument in §2.1 of Vulanović (2004) for the continuous problem (1.1). To get (3.3), note that $T^N u_i^N - T^N(0)_i = -b(x_i,0)$ and use the standard linearization $L^N u_i^N = -\varepsilon^2 D^2 u_i^N + p_i^N u_i^N = -b(x_i,0)$. Here $p_i^N := \int_0^1 b_u(s, s u_i^N)\,\mathrm{d}s$, and, by (1.2), we have $0 < \beta < p_i^N \leqslant \bar{\beta}$. Hence the discrete operator L^N satisfies the discrete maximum principle, which yields (3.3). □

Now we recall the following maximum norm *a priori error estimate* from Kopteva *et al.* (2005).

LEMMA 3.2 Let u be a solution of (1.1),(1.2) and $\{u_i^N\}$ be a solution of (3.2) on an arbitrary mesh $\{x_i\}$. Then

$$\max_i \big|u_i^N - u(x_i)\big| \leqslant C\left[\max_{1\leqslant i\leqslant N} \max_{x\in[x_{i-1},x_i]} \varepsilon h_i^2 |u'''(x)| + \max_{1\leqslant i\leqslant N-1} \max_{x\in[x_{i-1/2},x_{i+1/2}]} \hbar_i^2 \left|\frac{\mathrm{d}^2}{\mathrm{d}x^2} b(x,u(x))\right| \right]. \tag{3.4}$$

Proof. This result is given in Theorem 3.1 of Kopteva *et al.* (2005) for the linear case (2.1), (2.2) under the additional assumption that $|p'| \leqslant C_5$ for some constant C_5. This assumption is made in Kopteva *et al.* (2005) indirectly by quoting Theorem 3.2 from Andreev (2004) as Theorem 2.1 in Kopteva *et al.* (2005), whose constant involves C_5. It is important to note that, by Lemma 2.2 in Andreev (2004), the constant in question can be replaced by $2\beta^{-1/2}$. Hence Theorem 3.1 in Kopteva *et al.* (2005) holds true for problem (2.1), (2.2) under no additional assumptions on p.

Now, the desired error estimate follows after we linearize by $T^N u^N - T^N u = L^N[u^N - u]$, where L^N is as in the proof of Lemma 3.1, but with $p^N := \int_0^1 b_u(x, u + [u^N - u]s)\,\mathrm{d}s$, which satisfies (2.2). □

Note that one of our main results, the a posteriori error estimate (3.7) in §3.2, is an a posteriori analogue of the a priori error estimate (3.4) of Lemma 3.2.

3.2 *Maximum norm a posteriori error estimate*

Denote by u^N the piecewise linear interpolant of the discrete computed solution u_i^N, i.e., u^N is continuous on $[0,1]$, linear on each $[x_{i-1}, x_i]$, and equal to u_i^N at the mesh nodes:

$$u^N(x_i) = u_i^N \quad \text{for } i = 0, 1, \ldots N. \tag{3.5}$$

Furthermore, we extend $D^2 u_i^N$ to the mesh nodes $i = 0, N$ obtaining it from the discrete equation (3.2) that is formally extended to $i = 0, 1$:

$$D^2 u_0^N := \varepsilon^{-2} b(x_0, u_0^N) = b(0,0), \qquad D^2 u_N^N := \varepsilon^{-2} b(x_N, u_N^N) = b(1,0). \tag{3.6}$$

Now we state our first maximum norm *a posteriori error estimate*.

THEOREM 3.3 Let u be a solution of (1.1),(1.2), $\{u_i^N\}$ be a solution of (3.2) on an arbitrary mesh $\{x_i\}$, and $u^N(x)$ be its piecewise linear interpolant (3.5). Then

$$\left\|u^N(\cdot) - u(\cdot)\right\|_\infty \leqslant \max_{1 \leqslant i \leqslant N} \left\{ h_i^2 \left[\frac{C_3}{4} |D^2 u_i^N| + \frac{C_2}{2} \varepsilon |D^- D^2 u_i^N| + \frac{C_1 C_6}{4} \left(|D^- u_i^N|^2 + 1 \right) \right] \right\}, \tag{3.7}$$

where C_1, C_2, C_3, C_4 are defined in Lemma 2.2, Corollary 2.3, and Lemma 3.1, while $C_6 := \|b(\cdot,\cdot)\|_{C^2(\Omega)}$ and $\Omega := \{(x,u) \in [0,1] \times [-C_4, C_4]\}$.

Proof. By (1.1) and (3.2), we have

$$Tu^N(x) - Tu(x) = Tu^N(x) = -\varepsilon^2 \big(u^N(x)\big)'' + b\big(x, u^N(x)\big),$$

where $(u^N(x))''$ is understood in the sense of distributions. Define an auxiliary function q by

$$q(x) := b\big(x, u^N(x)\big) \tag{3.8}$$

and let q^I be its piecewise linear interpolant on the mesh $\{x_i\}$, i.e., q^I is continuous on $[0,1]$, linear on each $[x_{i-1}, x_i]$, and equal to $q(x_i)$ at each mesh node x_i. Now, we have

$$Tu^N(x) - Tu(x) = -\varepsilon^2 (u^N)'' + q^I + \big[q - q^I\big] = -\Big[\varepsilon^2 (u^N)' + \int_x^1 q^I(s)\,\mathrm{d}s\Big]' + \big[q - q^I\big]. \tag{3.9}$$

Note that $q_i := q(x_i) = b(x_i, u_i^N)$. Hence the discrete equation (3.2) combined with (3.6) yields

$$\varepsilon^2 D^2 u_i^N = q_i, \qquad i = 0, \ldots, N. \tag{3.10}$$

Combining this with $D^2u_i^N = (D^-u_{i+1}^N - D^-u_i^N)/\hbar_i$, where $i=1,\dots,N-1$, we get

$$\varepsilon^2(u^N)' = \varepsilon^2 D^- u_i^N = \varepsilon^2 D^- u_N^N - \sum_{j=i}^{N-1} q_j \hbar_j\,, \qquad x\in(x_{i-1},x_i),\quad i=1,\dots,N.$$

Here we also recalled that u^N is the piecewise linear interpolant of u_i^N. Now, substitute the representation that we obtained for $\varepsilon^2(u^N)'$ in (3.9), omitting the derivative of the constant $\varepsilon^2 D^- u_N^N$, and get

$$Tu^N(x) - Tu(x) = -\big[-\sum_{j=i}^{N-1} q_j\hbar_j + \int_x^1 q^I(s)\,\mathrm{d}s\big]' + \big[q-q^I\big], \qquad x\in(x_{i-1},x_i),\quad i=1,\dots,N.$$

A calculation shows that

$$\sum_{j=i}^{N-1} q_j\hbar_j = q_i\frac{h_i}{2} + \int_{x_i}^1 q^I(s)\,\mathrm{d}s - q_N\frac{h_N}{2},$$

and, omitting the derivative of another constant $q_N h_N/2$, we obtain

$$Tu^N(x) - Tu(x) = \big[q_i\frac{h_i}{2} - \int_x^{x_i} q^I(s)\,\mathrm{d}s\big]' + \big[q-q^I\big], \qquad x\in(x_{i-1},x_i),\quad i=1,\dots,N.$$

To compute $\int_x^{x_i} q^I(s)\,\mathrm{d}s$, note that $q^I(x) = q_i - (x_i - x)\,D^-q_i$. This implies that

$$Tu^N(x) - Tu(x) = \big[q_i\,(x - x_{i-1/2}) + \frac{D^-q_i}{2}\,(x_i-x)^2\big]' + \big[q-q^I\big], \qquad x\in(x_{i-1},x_i),\quad i=1,\dots,N.$$

Now apply Theorem 2.6 and obtain

$$\big\|u^N - u\big\|_\infty \leqslant \frac{C_3}{4}\max_{1\leqslant i\leqslant N}\Big\{\frac{h_i^2|q_i|}{\varepsilon^2}\Big\} + \frac{C_2}{2}\max_{1\leqslant i\leqslant N}\Big\{\frac{h_i^2|D^-q_i|}{\varepsilon}\Big\} + C_1\big\|q-q^I\big\|_\infty. \tag{3.11}$$

By (3.10), this yields

$$\big\|u^N - u\big\|_\infty \leqslant \max_{1\leqslant i\leqslant N}\Big\{h_i^2\big(\frac{C_3}{4}|D^2u_i^N| + \frac{C_2}{2}\varepsilon|D^-D^2u_i^N|\big)\Big\} + C_1\big\|q-q^I\big\|_\infty. \tag{3.12}$$

For the interpolation error $q-q^I$ of the function q from (3.8) we have

$$\big\|q-q^I\big\|_\infty \leqslant \max_{1\leqslant i\leqslant N}\Big\{\frac{h_i^2}{8}\sup_{(x_{i-1},x_i)}|q''|\Big\} \leqslant \frac{C_6}{4}\max_{1\leqslant i\leqslant N}\Big\{h_i^2\big(|D^-u_i^N|^2+1\big)\Big\}. \tag{3.13}$$

Here we used $q''(x) = b_{xx}(x,u^N) + 2\,(u^N)'\,b_{xu}(x,u^N) + [(u^N)']^2\,b_{uu}(x,u^N)$ and (3.3). Finally, combining (3.12) with (3.13), we get (3.7). □

REMARK 3.4 In (3.11) the quantity $|q_i|$ can be replaced by $|q_{i-1}|$ or $|q_{i-1}+q_i|/2$. Hence in (3.12) the quantity $|D^2u_i^N|$ can be replaced by $|D^2u_{i-1}^N|$, or $\min\{|D^2u_{i-1}^N|, |D^2u_i^N|\}$, or $(|D^2u_{i-1}^N| + |D^2u_i^N|)/2$. Combining a version of (3.12) with (3.13), we get the following alternative formulation of (3.7):

$$\big\|u^N(\cdot) - u(\cdot)\big\|_\infty \leqslant C\Big\{\max_{0\leqslant i\leqslant N}\hbar_i^2\,|D^2u_i^N| + \max_{1\leqslant i\leqslant N} h_i^2\,\big(\varepsilon|D^-D^2u_i^N| + \sup_{x\in(x_{i-1},x_i)}\big|\tfrac{\mathrm{d}^2}{\mathrm{d}x^2}b(x,u^N(x))\big|\big)\Big\}, \tag{3.14}$$

where $\hbar_1 := h_1/2$ and $\hbar_{N-1} := h_N/2$; compare (3.14) with (3.4).

3.3 *Further a posteriori error estimate*

LEMMA 3.5 Under the conditions of Theorem 3.3 we have

$$h_i^2\varepsilon|D^-D^2u_i^N| + h_i^2|D^-u_i^N|^2 \leqslant C \max_{0\leqslant i\leqslant N} \left\{ h_i^2\big(|D^2u_{i-1}^N| + |D^2u_i^N| + 1\big) \right\}, \qquad i = 1,\ldots,N. \tag{3.15}$$

Proof. We defer the proof of this lemma to §6. □

Note that (3.15) gives discrete analogues of the bounds $\varepsilon|u'''| \leqslant C(|u''|+1)$ and $|u'|^2 \leqslant C(|u''|+1)$ (see Kopteva *et al.* (2005)). Now we can simplify the statement of Theorem 3.3 as follows.

THEOREM 3.6 Let u be a solution of (1.1), (1.2), $\{u_i^N\}$ a solution of (3.2) on an arbitrary mesh $\{x_i\}$, and u^N its piecewise linear interpolant (3.5). Then

$$\big\|u^N(\cdot) - u(\cdot)\big\|_\infty \leqslant C \max_{0\leqslant i\leqslant N} \left\{ h_i^2\big(|D^2u_{i-1}^N| + |D^2u_i^N| + 1\big) \right\}. \tag{3.16}$$

Furthermore, (3.16) allows the following equivalent formulation:

$$\big\|u^N(\cdot) - u(\cdot)\big\|_\infty \leqslant C \max_{1\leqslant i\leqslant N} \left\{ h_i \left[\frac{|D^2u_{i-1}^N|^{1/2} + |D^2u_i^N|^{1/2}}{2} + 1 \right] \right\}^2. \tag{3.17}$$

Proof. Combine Theorem 3.3 with Lemma 3.5. □

We prefer the a posteriori error estimates (3.16) and (3.17) since, by (3.15), they are as sharp as (3.7) (up to a constant multiplier) and they are much simpler. In a posteriori error estimation, much attention focuses on specifying the error constants as we did in (3.7) (but not in (3.16) and (3.17)). Note that for singularly perturbed problems, the error constant might blow up as ε becomes small, and hence the existence of an ε-uniform error constant is more significant than its precise value.

Furthermore, the a posteriori error estimate (3.16) is *optimal* in the following sense.

LEMMA 3.7 Under the conditions of Theorem 3.6, we have

$$\max_i \left\{ \frac{h_i^2}{16} \min_{x\in[x_{i-1},x_i]} |u''(x)| \right\} \leqslant \|u^N(\cdot) - u(\cdot)\|_\infty .$$

Proof. Consider $e(x) := u^N(x) - u(x)$ on each $[x_{i-1}, x_i]$, $i = 1,\ldots,N$. It suffices to prove that

$$\max_{x\in[x_{i-1},x_i]} |e(x)| \geqslant \frac{h_i^2}{16} \min_{x\in[x_{i-1},x_i]} |u''(x)| . \tag{3.18}$$

Clearly,

$$\max_{x\in[x_{i-1},x_i]} |e(x)| \geqslant \max\left\{ |e_{i-1}|, |e_i|, |e_{i-1/2}| \right\} \geqslant \frac{1}{4}\big(|e_{i-1}+e_i| + 2|e_{i-1/2}|\big),$$

where we use the notation $e_i := e(x_i)$. Let u^I and e^I be the piecewise linear interpolants of $u(x)$ and $e(x)$, respectively, on the mesh $\{x_i\}$. Then $e_{i-1/2} = e^I_{i-1/2} + (e-e^I)_{i-1/2}$, where $e^I_{i-1/2} = (e_{i-1}+e_i)/2$ and $(e-e^I)_{i-1/2} = (u^I-u)_{i-1/2}$, yields

$$\max_{x\in[x_{i-1},x_i]} |e(x)| \geqslant \frac{1}{4}\big(|e_{i-1}+e_i| + |(e_{i-1}+e_i) + 2(u^I-u)_{i-1/2}|\big) \geqslant \frac{1}{2}|(u^I-u)_{i-1/2}| .$$

Since

$$|(u^I-u)_{i-1/2}| \geqslant \frac{h_i^2}{8} \min_{x\in[x_{i-1},x_i]} |u''(x)| ,$$

we get (3.18) and thus complete the proof. □

4. Application to a posteriori mesh construction

In this section we present some corollaries of Theorem 3.6 that provide a theoretical framework for a posteriori mesh construction. We invoke the a posteriori error estimate (3.17).

COROLLARY 4.1 Suppose that the conditions of Theorem 3.6 are satisfied and

$$h_i M_i \leqslant CN^{-1}, \quad i = 1, \dots, N, \qquad \text{where} \quad M_i := \frac{|D^2 u_{i-1}^N|^{1/2} + |D^2 u_i^N|^{1/2}}{2} + 1; \tag{4.1}$$

then

$$\left\| u^N(\cdot) - u(\cdot) \right\|_\infty \leqslant CN^{-2}. \tag{4.2}$$

4.1 *Grid equidistribution of a monitor function*

Given the a posteriori error estimate (3.17) and Corollary 4.1, we face the problem of finding $\{x_i\,;u_i^N\}$ that satisfy (3.2) and (4.1). Then the computed solution $\{u_i^N\}$ on the mesh $\{x_i\}$ is guaranteed to be second-order accurate ε-uniformly.

One approach to solving this problem is to use $\{M_i\}_{i=1}^N$ from (4.1) as a monitor function to be equidistributed; see, e g., Beckett & Mackenzie (2001), Kopteva & Stynes (2001), Kopteva *et al.* (2005) and references there for more details on grid equidistribution.

- The monitor function $\{M_i\}_{i=1}^N$ implies the following *equidistribution problem*:
  Find a mesh $\{x_i\}$ and the computed solution $\{u_i^N\}$ on this mesh such that

$$h_i M_i = \frac{1}{N} \sum_{i=1}^{N} h_j M_j, \qquad i = 1, \dots, N. \tag{4.3}$$

- This formulation may be weakened to the following *quasi-equidistribution problem*:
  Find a mesh $\{x_i\}$ and the computed solution $\{u_i^N\}$ on this mesh such that

$$h_i M_i \leqslant \frac{C^*}{N} \sum_{i=1}^{N} h_j M_j, \qquad i = 1, \dots, N. \tag{4.4}$$

  for some constant $C^* \geqslant 1$.

COROLLARY 4.2 Suppose that the conditions of Theorem 3.6 are satisfied. If $\{x_i\,;u_i^N\}$ solve the equidistribution problem (4.3) or the quasi-equidistribution problem (4.4), and for some constant C we have

$$\sum_{i=1}^{N} h_j M_j \leqslant C, \tag{4.5}$$

then $\left\| u^N(\cdot) - u(\cdot) \right\|_\infty \leqslant CN^{-2}$.

Proof. (4.5) combined with (4.3) or (4.4) implies (4.1) and hence (4.2). □

REMARK 4.3 The monitor function M_i from (4.1) is a discrete analogue of the continuous function $M(x) := |u''|^{1/2} + 1$, which is a particular case of the monitor function $\tilde{M}(x) := |u''|^{1/m} + \alpha$, whose equidistribution was investigated by Beckett & Mackenzie (2001). Note also that the same continuous monitor function $M(x)$ is the one-dimensional version of the monitor function that was suggested by Chen *et al.* (2006) based on the piecewise-linear-interpolation error analysis in the L_p norm in the case of $p = \infty$.

In Kopteva *et al.* (2005) a variant of de Boor's algorithm is implemented to solve the quasi-equidistribution problem (4.4). Numerical experiments show that the method is quite successful: the errors are robust with respect to ε, and the convergence rates are close to 2. In summary, these numerical results confirm our theoretical error estimates.

4.2 *Remark on a posteriori mesh refinement*

Here we make no attempt to suggest or discuss any specific mesh refinement algorithm. Note only that possible applications of any a posteriori error estimate are not restricted to mesh movement algorithms. E.g., the a posteriori error estimate (3.17) immediately suggests a possible mesh refinement principle:

- Use $h_i M_i$ as a *mesh refinement indicator* performing mesh refinement/coarsening to satisfy

$$C_*^{-1}\rho \leqslant h_i M_i \leqslant C_*\rho \qquad \forall\ i, \tag{4.6}$$

  where ρ^2 is the desired accuracy (up to a constant ε-independent multiplier), and $C_* > 1$ is some sufficiently large constant.

Indeed, if (4.6) is satisfied, then by Corollary 4.1, $\left\|u^N - u\right\|_\infty \leqslant C\rho^2$. A mesh refinement algorithm can possibly keep the number of mesh nodes N constant with $\rho = O(N^{-1})$; or new mesh nodes can be inserted/removed to achieve the prescribed accuracy.

5. Numerical results on a priori chosen meshes

In addition to the numerical results in Kopteva *et al.* (2005) for the quasi-equidistribution problem (4.4), we present numerical results on a priori chosen meshes to illustrate the efficiency of the *upper maximum norm error estimator*

$$\eta^N := \max_{1\leqslant i\leqslant N}[h_i M_i]^2, \qquad \text{where} \quad M_i = \frac{|D^2u_{i-1}^N|^{1/2} + |D^2u_i^N|^{1/2}}{2} + 1,$$

that follows from the a posteriori error estimate (3.17) of Theorem 3.6. Since (3.17) can be rewritten as

$$\|u^N(\cdot) - u(\cdot)\|_\infty \leqslant C\eta^N,$$

to investigate the constant C in this estimate and its dependence on ε, N, and any particular mesh choice, we compute the quantities

$$e^N := \|u^N(\cdot) - u(\cdot)\|_\infty, \qquad C^N = e^N/\eta^N \qquad \text{and} \qquad C_{\max}^N = \max_{\varepsilon=10^{-k}:\,k=0,1,\dots 10} C^N.$$

Furthermore, the even rows of Tables 1 and 2 give the rates of convergence of e^N and η^N computed using the standard formulae $r(e^N) = \log_2\left(e^N/e^{2N}\right)$ and $r(\eta^N) = \log_2\left(\eta^N/\eta^{2N}\right)$. Provided that C^N stabilizes as $\varepsilon \to 0$, the weaker the dependence of $C_{\max}^N$ on N and the mesh choice, the sharper (3.17) is and the more efficient the ε-uniform upper maximum norm error estimator η^N that is suggested by (3.17).

Our test problem is the linear equation from (2.1) with the boundary conditions $u(0) = 2, u(1) = -1$, in which $p(x) = 4(1+x)^{-4}[1+\varepsilon(1+x)]$, and whose exact solution

$$u(x) = -\cos(2\pi t) + 3(e^{-t/\varepsilon} - e^{-1/\varepsilon})[1 - e^{-1/\varepsilon}]^{-1}, \qquad t := 2x/(x+1),$$

Table 1. Bakhvalov mesh: maximum norm error e^N, upper maximum norm error estimator η^N, and its efficiency constant $C^N := e^N/\eta^N$ (odd rows), their rates of convergence $r(e^N)$ and $r(\eta^N)$ (even rows).

	$\varepsilon = 1$			$\varepsilon = 10^{-4}$			$\varepsilon = 10^{-8}$			
N	e^N	η^N	C^N	e^N	η^N	C^N	e^N	η^N	C^N	$C^N_{\max}$
32	2.27e-2	1.97e-1	1.16e-1	1.20e-1	9.32e-1	1.29e-1	1.20e-1	9.32e-1	1.29e-1	1.29e-1
	1.96	1.91		1.85	1.84		1.85	1.84		
64	5.83e-3	5.23e-2	1.12e-1	3.33e-2	2.60e-1	1.28e-1	3.33e-2	2.60e-1	1.28e-1	1.28e-1
	1.98	1.96		1.93	1.92		1.93	1.92		
128	1.48e-3	1.34e-2	1.10e-1	8.74e-3	6.89e-2	1.27e-1	8.74e-3	6.89e-2	1.27e-1	1.27e-1
	1.99	1.98		1.97	1.96		1.97	1.96		
256	3.71e-4	3.39e-3	1.09e-1	2.24e-3	1.78e-2	1.26e-1	2.24e-3	1.78e-2	1.26e-1	1.26e-1
	2.00	1.99		1.98	1.98		1.98	1.98		
512	9.31e-5	8.53e-4	1.09e-1	5.66e-4	4.51e-3	1.26e-1	5.66e-4	4.51e-3	1.26e-1	1.26e-1
	2.00	1.20		1.99	1.99		1.99	1.99		
1024	2.33e-5	2.14e-4	1.09e-1	1.42e-4	1.14e-3	1.25e-1	1.42e-4	1.14e-3	1.25e-1	1.25e-1

Table 2. Uniform mesh: maximum norm error e^N, upper maximum norm error estimator $\tilde{\eta}^N$, and its efficiency constant $C^N := e^N/\tilde{\eta}^N$ (odd rows), their rates of convergence $r(e^N)$ and $r(\tilde{\eta}^N)$ (even rows).

	$\varepsilon = 1$			$\varepsilon = 10^{-4}$			$\varepsilon = 10^{-8}$			
N	e^N	$\tilde{\eta}^N$	C^N	e^N	$\tilde{\eta}^N$	C^N	e^N	$\tilde{\eta}^N$	C^N	$C^N_{\max}$
32	2.27e-2	1.75e-1	1.30e-1	2.97e+0	3.23e+0	9.18e-1	3.00e+0	3.23e+0	9.29e-1	9.29e-1
	1.96	1.82		0.01	0.06		0.00	0.06		
64	5.83e-3	4.97e-2	1.17e-1	2.94e+0	3.09e+0	9.50e-1	3.00e+0	3.09e+0	9.71e-1	9.71e-1
	1.98	1.92		0.03	0.03		0.00	0.03		
128	1.48e-3	1.31e-2	1.13e-1	2.88e+0	3.04e+0	9.50e-1	3.00e+0	3.04e+0	9.88e-1	9.88e-1
	1.99	1.97		0.05	0.01		0.00	0.01		
256	3.71e-4	3.36e-3	1.11e-1	2.79e+0	3.02e+0	9.27e-1	3.00e+0	3.02e+0	9.95e-1	9.95e-1
	2.00	1.98		0.08	0.01		0.00	0.00		
512	9.31e-5	8.48e-4	1.10e-1	2.64e+0	3.00e+0	8.80e-1	3.00e+0	3.01e+0	9.98e-1	9.98e-1
	2.00	1.99		0.14	0.01		0.00	0.00		
1024	2.33e-5	2.13e-4	1.09e-1	2.39e+0	2.99e+0	8.00e-1	3.00e+0	3.00e+0	9.99e-1	9.99e-1

exhibits a boundary layer near $x = 0$; the right-hand side f and the boundary conditions are chosen so that (2.1) is satisfied; see Schatz & Wahlbin (1983), O'Riordan & Stynes (1986), Kopteva *et al.* (2005).

We present numerical results for two a priori chosen meshes: a variant of the mesh by Bakhvalov (1969), which guarantees second-order ε-uniform accuracy and a simple uniform mesh; see Tables 1,2. To be precise, if $\varepsilon \leqslant \bar{\varepsilon}$, our Bakhvalov-type mesh is given by $x_i = x(i/N)$ for $i = 0, 1, \ldots, N$, where $x(\xi) := \varepsilon\lambda \ln[b/(b-\xi)]$ for $\xi \in [0, \theta]$, $x(1) := 1$, and $x(\xi)$ is continuous on $[0, 1]$ and linear on $[\theta, 1]$. We use the constants $\lambda = 5$, $b = 1/2$, $\bar{\varepsilon} = b/\lambda$, and $\theta = b - \varepsilon\lambda$. For $\varepsilon > \bar{\varepsilon}$, the Bakhvalov mesh is defined to be a simple uniform mesh.

We observe that the rates of convergence of both $e^N$and η^N are very close to second order on the Bakhvalov meshes; see Table 1. Here not only does C^N stabilize, but it becomes very close to the linear interpolation error constant $1/8 = 1.25\text{e-}1$. When uniform meshes are used—see Table 2—the boundary layer is not resolved and $e^N = O(1)$. This is indicated by η^N blowing up even more significantly than e^N. Hence in Table 2 instead of η^N we give the related quantity $\tilde{\eta}^N \leqslant \eta^N$ that is defined similarly to η^N but with M_i replaced by $\tilde{M}_i := \min\{|D^2 u^N_{i-1}|^{1/2}, |D^2 u^N_i|^{1/2}\} + 1 \leqslant M_i$. Unlike η^N, the quantity $\tilde{\eta}^N$ remains bounded. Here $\tilde{\eta}^N$ not being small implies that η^N is not small, and they both indicate correctly that the method is inaccurate.

In summary, for both meshes considered the error estimators η^N indicate correctly whether the method is ε-uniformly accurate or not. Furthermore, when the errors are small, the efficiency constants C^N stabilize as $\varepsilon \to 0$ and $C^N_{\max}$ stabilizes as N increases. This implies that on the meshes considered, the upper error estimator $\eta^N = \max_i [h_i M_i]^2$ is efficient ε-uniformly.

6. Proof of Lemma 3.5

We decompose the discrete solution u^N into a smooth component w^N and a singular component v^N similarly to Shishkin-type decompositions; see, e.g., §6 of Miller *et al.* (1996) for a linear reaction-diffusion problem. Let $u_0(x)$ be the solution of the reduced problem $b(x,u_0) = 0$. Define w^N by

$$T^N w_i^N = 0, \quad i = 1,\dots,N-1, \qquad w_0^N = u_0(0), \quad w_N^N = u_0(1).$$

Now if $u^N = w^N + v^N$, then v^N satisfies

$$-\varepsilon^2 D^2 v_i^N + b(x_i, w_i^N + v_i^N) - b(x_i, w_i^N) = 0, \quad i = 1,\dots,N-1, \quad v_0^N = -u_0(0), \quad v_N^N = -u_0(1).$$

LEMMA 6.1 Let $\tilde{w}_i^N := w_i^N - u_0(x_i)$. Then

$$|\tilde{w}_i^N| \leqslant C\varepsilon^2, \quad i = 0,\dots,N; \qquad |D^- \tilde{w}_i^N| \leqslant C\frac{\varepsilon h}{h_i}, \quad i = 1,\dots,N. \tag{6.1}$$

Proof. Note that $\tilde{L}^N \tilde{w}^N := -\varepsilon^2 D^2 \tilde{w}^N + \tilde{p}\tilde{w}^N = \varepsilon^2 D^2 u_0$ with $\tilde{p} := \int_0^1 b_u(x, u_0 + \tilde{w}^N s)\, ds$, $\tilde{w}_0^N = \tilde{w}_N^N = 0$. Since $0 < \beta < \tilde{p} \leqslant \bar{\beta}$, the operator $\tilde{L}^N$ satisfies the discrete maximum/comparison principle. Hence $|\tilde{w}^N| \leqslant \varepsilon^2 \beta^{-1} \|u_0''\|_\infty$, which yields the first desired estimate.

To prove the second desired estimate, introduce the continuous analogue $\tilde{w} := w - u_0$ of $\tilde{w}^N$, where w satisfies $-\varepsilon^2 w'' + b(x,w) = 0$ and the same boundary conditions as w^N. We invoke the standard bounds $|w'''| \leqslant C\varepsilon^{-1}$ and $|\tilde{w}'| \leqslant C\varepsilon$ (see, e.g., Vulanović (2004)). The first of them yields $|\tilde{w}_i^N - \tilde{w}(x_i)| = |w_i^N - w(x_i)| \leqslant C\varepsilon h$ and hence $|D^-[\tilde{w}_i^N - \tilde{w}(x_i)]| \leqslant C\varepsilon h/h_i$. The second one implies $|D^- \tilde{w}(x_i)| \leqslant C\varepsilon \leqslant C\varepsilon h/h_i$. Combining these, we get the second bound in (6.1). □

Now we consider v^N, which satisfies

$$\tilde{L}^N v_i^N = -\varepsilon^2 D^2 v_i^N + \tilde{p}_i v_i^N = 0, \quad i = 1,\dots,N-1, \qquad \tilde{p}_i := \int_0^1 b_u(x_i, w_i^N + s v_i^N)\,\mathrm{d}s. \tag{6.2}$$

Since $0 < \beta < \tilde{p}_i \leqslant \bar{\beta}$, by the discrete maximum/comparison principle applied to (6.2) restricted to $i = 1,\dots,m-1$, we get

$$|v_i^N| \leqslant \max\{|v_0^N|, |v_m^N|\} \leqslant \max\{|v_0^N|, |v_N^N|\}, \qquad i = 0,\dots,m. \tag{6.3}$$

LEMMA 6.2 We have

$$h_i^2 \varepsilon |D^- v_i^N| \leqslant C\Big[\max_{1\leqslant j\leqslant N} h_j^2\big(|v_{j-1}^N| + |v_j^N|\big) + \varepsilon^2 h^2\Big], \qquad i = 1,\dots,N; \tag{6.4a}$$

$$h_i \varepsilon |D^- v_i^N| \leqslant C\Big[\max_{1\leqslant j\leqslant N} h_j\big(|v_{j-1}^N| + |v_j^N|\big) + \varepsilon^2 h\Big], \qquad i = 1,\dots,N. \tag{6.4b}$$

Proof. If $h_i \geqslant C_0\varepsilon$ for some constant C_0, then $\varepsilon|D^- v_i^N| \leqslant C_0[|v_{i-1}| + |v_i|]$ implies the desired bounds.

Now consider $h_i \leqslant C_0\varepsilon$ such that $x_i \leqslant 1/2$ (the other case of $x_i > 1/2$ is similar). We shall assume that $h_1 \leqslant C_0\varepsilon$ and prove (6.4) for $i = 1$. If $h_m \leqslant C_0\varepsilon$ for some $m > 1$ and $x_m \leqslant 1/2$, a similar argument applies to $\{v_j^N\}_{i=m-1}^N$ and yields (6.4) for $i = m$.

If $h_1 \leqslant C_0\varepsilon$, clearly there exists $k \geqslant 1$ such that $h_i \leqslant C_0\varepsilon$, $i = 1,\ldots,k$, while either $h_{k+1} > C_0\varepsilon$ or $x_k = 1$. We now consider two cases.
Case A. $x_k \leqslant \bar{C}\varepsilon$ for some constant $\bar{C}$ that will be specified later. By (6.2),(6.3), we get

$$\varepsilon\big|D^- v_1^N\big| = \big|\varepsilon D^- v_{k+1}^N - \varepsilon^{-1}\sum_{j=1}^{k} \hbar_j \tilde{p}_j v_j^N\big| \leqslant \varepsilon\big|D^- v_{k+1}^N\big| + \varepsilon^{-1}\bar{\beta}\Big(x_k \max\{|v_0|,|v_{k+1}|\} + 0.5h_{k+1}|v_k^N|\Big).$$

Note that $h_{k+1} > C_0\varepsilon$ implies $\varepsilon|D^- v_{k+1}^N| \leqslant C_0^{-1}(|v_k^N| + |v_{k+1}^N|)$. Combining this with $x_k \leqslant \bar{C}\varepsilon$ and again using (6.3), we get

$$\varepsilon|D^- v_1^N| \leqslant C\Big[|v_0| + |v_{k+1}| + \varepsilon^{-1}h_{k+1}|v_k^N|\Big]. \tag{6.5}$$

Finally, multiply (6.5) by h_1^2 and note that $h_1 \leqslant C_0\varepsilon < h_{k+1}$ implies $h_1^2\varepsilon^{-1}h_{k+1} \leqslant C_0 h_{k+1}^2$ to obtain (6.4a) for $i = 1$. Similarly, multiplying (6.5) by h_1, we obtain (6.4b) for $i = 1$ and complete Case 1.
Case B. $x_k \geqslant \bar{C}\varepsilon$, where the constant $\bar{C}$ is sufficiently large so that $e^{-(\gamma+\bar{\gamma})x_k/\varepsilon} \leqslant e^{-(\gamma+\bar{\gamma})\bar{C}} \leqslant 1/2$. Here $\bar{\gamma}^2 := 2\bar{\beta}$ and $\gamma^2 := \beta/2$ so that if C_0 is sufficiently small, then $\tilde{L}^N e^{\pm\gamma x_i/\varepsilon} \geqslant 0$ and $\tilde{L}^N e^{\pm\bar{\gamma}x_i/\varepsilon} \leqslant 0$. Then $B_i^- \leqslant v_i^N \leqslant B_i^+$ for $i = 0,\ldots,k$, where the barrier functions $B_i^\pm = B^\pm(x_i)$ are defined by

$$B^\pm(x) := v_0^N e^{-\tilde{\gamma}x/\varepsilon} \pm \big(|v_0^N| + |v_k^N|\big)\frac{e^{-\gamma(x_k-x)/\varepsilon} - e^{-(\gamma x_k+\bar{\gamma}x)/\varepsilon}}{1 - e^{-(\gamma+\bar{\gamma})x_k/\varepsilon}},$$

where $\pm$ is understood as $+$ in B^+ and $-$ in B^-, while $\tilde{\gamma} = \gamma$ in B^+ and $\tilde{\gamma} = \bar{\gamma}$ in B^- provided that $v_0^N \geqslant 0$ (the other case is similar). Indeed, $B_0^- = v_0^N = B_0^+$ and $B_k^- \leqslant -|v_k^N|$, $|v_k^N| \leqslant B_k^+$, while $\tilde{L}^N B^- \leqslant 0 \leqslant \tilde{L}^N B^+$. Combining $B_0^- = v_0^N = B_0^+$ and $B_1^- \leqslant v_1^N \leqslant B_1^+$, we get

$$\varepsilon|D^- v_1^N| \leqslant \max\{\varepsilon|D^- B_1^-|, \varepsilon|D^- B_1^+|\} \leqslant C\big(|v_0^N| + |v_k^N|e^{-\gamma x_k/\varepsilon}\big).$$

Now if $x_k = 1$, we get $\varepsilon|D^- v_1^N| \leqslant C[|v_0^N| + \varepsilon^2]$, which implies (6.4) for $i = 1$. Otherwise, $h_{k+1} > C_0\varepsilon \geqslant h_1$, and we have $h_1^l\varepsilon|D^- v_1^N| \leqslant C[h_1^l|v_0^N| + h_{k+1}^l|v_k^N|]$, $l = 1,2$, which again yields (6.4) for $i = 1$. □

Proof of Lemma 3.5. First we estimate $h_i^2\varepsilon|D^-D^2u_i^N| = h_i^2\varepsilon^{-1}|D^- b(x_i,u_i^N)|$ for $i = 1,\ldots,N$. Recall the decomposition $u_i^N = w_i^N + v_i^N = [u_0(x_i) + \tilde{w}_i^N] + v_i^N$. Then, by (6.1), we get

$$|D^- b(x_i,u_i^N)| \leqslant \bar{\beta}\big(|D^- v_i^N| + |D^- \tilde{w}_i^N|\big) + \max_{t\in[0,1]}\Big|\frac{\mathrm{d}}{\mathrm{d}t}b(t,u_0(t) + v_i^N + \tilde{w}_i^N)\Big| \leqslant C\big(|D^- v_i^N| + \frac{\varepsilon h}{h_i} + |v_i^N| + \varepsilon^2\big),$$

where we used $b(t,u_0(t)) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}b(t,u_0(t)) = 0$, and also $|\tilde{w}_i^N| \leqslant C\varepsilon^2$. Hence

$$h_i^2\varepsilon|D^-D^2u_i^N| \leqslant C\frac{h_i^2}{\varepsilon}\big(|D^- v_i^N| + \frac{\varepsilon h}{h_i} + |v_i^N| + \varepsilon^2\big) \leqslant C\Big[\varepsilon^{-2}\max_{0\leqslant j\leqslant N} h_j^2\big(|v_{j-1}^N| + |v_j^N|\big) + h^2\Big], \tag{6.6}$$

where we used (6.4a). To estimate $h_i^2|D^- u_i^N|^2$, note that $|D^- u_0(x_i)| \leqslant C$ and, by (6.1), $h_i|D^-\tilde{w}_i| \leqslant C\varepsilon h$. Combining these with (6.4b) and $|v_i^N| \leqslant C$, we get

$$h_i^2|D^- u_i^N|^2 \leqslant h_i^2|D^- v_i|^2 + Ch^2 \leqslant C\Big[\varepsilon^{-2}\max_{0\leqslant j\leqslant N} h_j^2\big(|v_{j-1}^N| + |v_j^N|\big) + h^2\Big]. \tag{6.7}$$

Finally, note that (3.2),(3.6) imply $\varepsilon^2|D^2u_i^N| = |b(x_i,u_0(x_i) + \tilde{w}_i^N + v_i^N)| \geqslant \beta|v_i^N| - C\varepsilon^2$ for $i = 0,\ldots,N$, where we used (1.2), $b(x,u_0(x)) = 0$, and $|\tilde{w}_i^N| \leqslant C\varepsilon^2$. Now combining $|v_i^N| \leqslant C\varepsilon^2(|D^2u_i^N| + 1)$ with (6.6),(6.7) we get the desired estimate (3.15) and complete the proof. □

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