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UNIFORM CONVERGENCE WITH RESPECT TO A SMALL PARAMETER OF A FOUR-POINT SCHEME FOR THE ONE-DIMENSIONAL STATIONARY CONVECTION-DIFFUSION EQUATION

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For a second-order ordinary differential equation with a small parameter in the coefficient of the second derivative, we study a four-point finite-difference scheme with a one-sided three-point second-order approximation of the first derivative. We show that on a piecewise uniform Shishkin grid condensing in the boundary layer, the scheme converges at the rate $O\left(N^{-2}\ln^2 N\right)$ uniformly with respect to the small parameter, where N is the number of grid points.

1. Introduction

We consider the simplest boundary value problem

$$Lu \equiv -\varepsilon\,(p(x)u')' - (r(x)u)' = f(x), \quad 0 < x < 1; \qquad u(0) = g_0, \quad u(1) = g_1 \tag{1.1}$$

for a singularly perturbed second-order ordinary differential equation. Here

$$p(x) \geq p_0 = \text{const} > 0, \qquad r(x) \geq r_0 = \text{const} > 0, \tag{1.2}$$

and $\varepsilon \in (0,1]$ is a small parameter.

The boundary layer phenomenon [1] is observed for small ε in a neighborhood of the point $x = 0$: the solution $u(x)$ varies rapidly and the derivatives of $u(x)$ are not uniformly bounded with respect to ε. Because of this, the accuracy of classical finite-difference methods depends not only on the grid increment but also on the small parameter [2, 3].

To achieve uniform convergence with respect to the small parameter, one has to develop special numerical methods [2, 3]. One of the possible approaches is to use classical finite-difference approximations on grids concentrated in the boundary layer in some special way [4, 5].

Shishkin [5] suggested the piecewise uniform grid

$$\begin{aligned}\Omega = \Big\{x_i \;\Big|\; & x_i = ih,\ i = 1,\dots,n;\ x_i = x_n + (i-n)H,\ i = n+1,\dots,N-1;\\ & h = \delta/n,\ H = (1-\delta)/(N-n),\ \delta = \min(C\varepsilon\ln N, A)\Big\},\end{aligned} \tag{1.3}$$

which is concentrated in the vicinity of the boundary layer. The numbers n and $N-n$ of small and large increments are assumed to be of equal order of magnitude. The grid parameter C is determined by the coefficients in Eq. (1.1), and the number $A \in (0,1)$ is arbitrary. It was shown in [5] that the well-known three-point scheme with the approximation of the first derivative by a one-sided difference on Ω converges at the rate $O\left(N^{-1}\ln^2 N\right)$ uniformly with respect to ε in $L^h_\infty(\Omega)$.

It was shown in [6] that the three-point scheme with the approximation of the first derivative by the central difference on Ω converges at the rate $O\left(N^{-2}\ln^2 N\right)$ uniformly with respect to ε.

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The *monotonicity* of a finite-difference scheme (in the sense that the roots of the characteristic polynomial of the scheme on the uniform grid are nonnegative) is a desirable property. It is known that the scheme with the central difference does not have this property, whereas the three-point scheme with a one-sided difference is monotone but provides only a first-order approximation to Eq. (1.1) on smooth solutions. A modification of the monotone Samarskii scheme was constructed and shown to converge at the rate $O\left(N^{-2}\ln^2 N\right)$ uniformly with respect to the small parameter on the Shishkin grid Ω. Note that the monotonicity of this three-point scheme is due to the special choice of the coefficient of the higher derivative.

The present paper deals with a four-point scheme with a one-sided three-point second-order approximation of the term with the first derivative. This scheme has the form

$$-\varepsilon u^h_{\bar{x}x,i} - \left(u^h - 0.5hu^h_x\right)_{x,i} = f^h_i \tag{1.4}$$

for $p(x) = r(x) = 1$. It was first suggested in [7] for the convection-diffusion equation. The roots of the characteristic polynomial of this scheme are nonnegative [8]. In [9], this scheme was considered from the viewpoint of the method of adaptive grids. Note that the scheme in question can be represented on the uniform grid in the form

$$-\varepsilon u^h_{\bar{x}x,i} - \frac{u^h_{i+1} - u^h_{i-1}}{2h} + 0.5h^2 u^h_{\bar{x}xx,i} = f^h_i$$

and hence, the truncation error is $O\left(h^2\right)$ on smooth solutions.

The main result of the present paper is Theorem 2, asserting that on the piecewise uniform Shishkin grid Ω, a four-point scheme similar to (1.4) converges at the rate $O\left(N^{-2}\ln^2 N\right)$ uniformly with respect to ε in $L^h_\infty(\Omega)$.

2. The Difference Green Function

We introduce the grid $\bar{\omega} = \left\{x_i \middle| 0 = x_0 < x_1 < \dots < x_{N-1} < x_N = 1\right\}$ on the closed interval $[0,1]$. As usual, we write $h_i = x_i - x_{i-1}$, $\hbar_i = (h_i + h_{i+1})/2$, $v_{x,i} = (v_{i+1} - v_i)/h_{i+1}$, $v_{\bar{x},i} = (v_i - v_{i-1})/h_i$, $v_{\hat{x},i} = (v_{i+1} - v_i)/\hbar_i$, and $v_{\check{x},i} = (v_i - v_{i-1})/\hbar_i$.

For grid functions defined on $\bar{\omega}$ and vanishing for $i = 0$ and $i = N$, we introduce the inner product

$$(u, v) = \sum_{j=1}^{N-1} u_j v_j \hbar_j \tag{2.1}$$

and the norms $\|v\|_{L^h_2} = \|v\| = \sqrt{(v,v)}$, $\|v\|_{L^h_1} = (|v|, 1)$, and $\|v\|_{L^h_\infty} = \max_i |v_i|$.

On the grid $\bar{\omega}$, we consider the problem

$$\left(L^h v\right)_i \equiv -\varepsilon\left(p^h v_{\bar{x}}\right)_{\hat{x},i} - \left(A\left[r^h v\right]\right)_{\hat{x},i} = f^h_i, \quad i = 1,\dots,N-1, \quad v_0 = v_N = 0, \tag{2.2}$$

where

$$p^h_i = p\left(x_i - h_i/2\right), \qquad r^h_i = r\left(x_i\right), \qquad f^h_i = f\left(x_i\right), \tag{2.3}$$

and the operator A is defined by the relations

$$(Aw)_i = \begin{cases} w_1 - 0.5\left(w_2 - w_0\right) & \text{for } i = 1,\\ w_i - 0.5\left(w_{i+1} - w_i\right) & \text{for } i = 2,\dots,N-1,\\ 0.5\left(w_{N-1} + w_N\right) & \text{for } i = N.\end{cases} \tag{2.4}$$

The solution of problem (2.2) can be represented in the form

$$v_i = \left(G\left(x_i, \xi_j\right), f^h_j\right). \tag{2.5}$$

Here $G\left(x_i, \xi_j\right)$ is the *Green function* of the finite-difference problem (2.2), that is, the function of x_i determined for each fixed ξ_j by the relations

$$L^h G\left(x_i, \xi_j\right) = \delta^h\left(x_i, \xi_j\right), \quad i, j = 1,\dots,N-1, \qquad G\left(0, \xi_j\right) = G\left(1, \xi_j\right) = 0, \quad j = 1,\dots,N-1, \tag{2.6}$$