Central difference scheme on uniform meshes: approximation of smooth solutions

(SUMMARY OF [1, CHAPTER 4])

Natalia Kopteva

References

[1] N. V. Kopteva, Numerical Methods for Non-Selfadjoint Singularly Pertubed Equations, Diploma Thesis, Moscow State University, Moscow, 1993 (in Russian).

1 Model problem: no boundary layer

Consider the problem

$$-\varepsilon u'' - u' = f \quad \text{for} \quad x \in (0,1), \qquad\qquad u(0) = u(1) = 0 \tag{1}$$

under the condition

$$\int_0^1 f(x)\,dx = 0. \tag{2}$$

The unique solution of problem (1) is given by

$$u(x) = -\int_0^x \big[1 - e^{(s-x)/\varepsilon}\big] f(s)\,ds + \frac{1-e^{-x/\varepsilon}}{1-e^{-1/\varepsilon}} \int_0^1 \big[1 - e^{(s-x)/\varepsilon}\big] f(s)\,ds.$$

Since

$$\Big|\int_0^x e^{(s-x)/\varepsilon} f(s)\,ds\Big| \le \varepsilon \|f\|_\infty,$$

we obtain

$$u(x) = -\int_0^x f(s)\,ds + \frac{1-e^{-x/\varepsilon}}{1-e^{-1/\varepsilon}} \int_0^1 f(s)\,ds + O(\varepsilon).$$

Finally, by (2), we get

$$u(x) = -\int_0^x f(s)\,ds + O(\varepsilon). \tag{3}$$

Thus, condition (2) implies that the solution has *no boundary layer*.

2 Central difference approximation on uniform meshes

Introduce the uniform mesh $\{x_i = iH \mid i = 0 \dots N,\ H = N^{-1}\}$.
The central difference scheme is given by

$$-\varepsilon\frac{u_{i+1}^N - 2u_i^N + u_{i-1}^N}{H^2} - \frac{u_{i+1}^N - u_{i-1}^N}{2H} = f_i \quad \text{for} \quad i = 1 \dots N-1, \tag{4}$$

where $u_0^N = u_N^N = 0,\ f_i := f(x_i)$.

The solution of this discrete problem is

$$u_i^N = -\sum_{j=1}^{i-1}\left(1 - q^{i-j}\right) f_j H + \frac{1-q^i}{1-q^N}\sum_{j=1}^{N-1}\left(1-q^{i-j}\right) f_j H, \tag{5}$$

where

$$q := -\frac{1 - 2\varepsilon/H}{1 + 2\varepsilon/H}.$$

Clearly, representation (5) might be rewritten as

$$u_i^N = -\frac{q^i - q^N}{1-q^N}\sum_{j=1}^{i-1}\left(1-q^{i-j}\right) f_j H + \frac{1-q^i}{1-q^N}\sum_{j=i}^{N-1}\left(1-q^{i-j}\right) f_j H. \tag{6}$$

We are interested in the extreme case of $\varepsilon \ll H$ and even $\varepsilon \ll H^2$.
Furthermore, we shall assume that H is fixed while $\varepsilon \to 0$. Then we have

$$\lim_{\varepsilon\to 0} q = -1, \qquad \lim_{\varepsilon\to 0} q^i = (-1)^i, \quad (i > 0), \qquad \lim_{\varepsilon\to 0} q^N = (-1)^N.$$

2.1 N is odd

We shall use (6). Consider the two cases separately:
(i) i is odd.
In this case we have $\lim_{\varepsilon\to 0} q^N = \lim_{\varepsilon\to 0} q^i = -1$, which, by (6), implies that

$$\lim_{\varepsilon\to 0} u_i^N = \sum_{j=i+1,\ j \text{ is even}}^{N-1} f_j\,(2H) = \int_{x_i}^1 f(x)\,dx + O(H^2).$$

Hence, by (2),(3), we have

$$\lim_{\varepsilon\to 0}\left|u_i^N - u(x_i)\right| = O(H^2).$$

(ii) i is even.
In this case we have $\lim_{\varepsilon\to 0} q^N = -1,\ \lim_{\varepsilon\to 0} q^i = 1$, which, by (6), implies that

$$\lim_{\varepsilon\to 0} u_i^N = -\sum_{j=1,\ j \text{ is odd}}^{i-1} f_j(2H) = -\int_0^{x_i} f(x)\,dx + O(H^2).$$

Now, by (3), we again have

$$\lim_{\varepsilon\to 0}\Big|u_i^N - u(x_i)\Big| = O(H^2).$$

2.2 N is even

We shall use (5). Consider the two cases separately:
(i) i is even.
In this case we have $\lim_{\varepsilon\to 0} q^N = \lim_{\varepsilon\to 0} q^i = 1$, which, by (5), implies that

$$\lim_{\varepsilon\to 0} u_i^N = -\sum_{j=1,\, j \text{ is odd}}^{i-1} f_j(2H) + \frac{i}{N}\sum_{j=1,\, j \text{ is odd}}^{N-1} f_j(2H).$$

Here we used

$$\lim_{\varepsilon\to 0}\frac{1-q^i}{1-q^N} = \frac{i}{N}.$$

Hence, using (2), we get

$$\lim_{\varepsilon\to 0} u_i^N = -\int_0^{x_i} f(x)\,dx + x_i\int_0^1 f(x)\,dx + O(H^2) = -\int_0^{x_i} f(x)\,dx + O(H^2).$$

Now, by (3), we again have

$$\lim_{\varepsilon\to 0}\Big|u_i^N - u(x_i)\Big| = O(H^2).$$

(ii) i is odd—the interesting case!
In this case we have $\lim_{\varepsilon\to 0} q^N = 1$, $\lim_{\varepsilon\to 0} q^i = -1$, which, by (5), implies that

$$\lim_{\varepsilon\to 0} u_i^N = O(1)+\lim_{\varepsilon\to 0}\frac{2}{1-q^N}\sum_{j=2,\, j \text{ is even}}^{N-2} f_j(2H) = O(1)+\lim_{\varepsilon\to 0}\frac{2}{1-q^N}\Big(R_1+R_2\Big),$$

where, by (2),

$$R_1 := \Big[f_{1/2}H + \sum_{j=2,\, j \text{ is even}}^{N-2} f_j(2H) + f_{N-1/2}H\Big] - \int_0^1 f(x)\,dx = O(H^2).$$

$$R_2 := -\big(f_{1/2} + f_{N-1/2}\big)H = O(H).$$

Since one can easily construct many functions $f(x)$ such that $|R_1 + R_2| > CH$, we arrive at

Corollary. *If N is even, there exist many functions $f(x)$ such that for odd i we have*

$$\lim_{\varepsilon\to 0}\Big|u_i^N - u(x_i)\Big| = \infty.$$

Remark. However, for certain functions we cannot see this interesting effect of oscillating computed solutions. E.g., if $f(x)$ is linear and satisfies $f(x) = -f(1-x)$, then $R_1 = R_2 = 0$.