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An efficient collocation method for a Caputo two-point boundary value problem

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Abstract A two-point boundary value problem is considered on the interval $[0,1]$, where the leading term in the differential operator is a Caputo fractional-order derivative of order $2-\delta$ with $0<\delta<1$. The problem is reformulated as a Volterra integral equation of the second kind in terms of the quantity $u'(x)-u'(0)$, where u is the solution of the original problem. A collocation method that uses piecewise polynomials of arbitrary order is developed and analysed for this Volterra problem; then by postprocessing an approximate solution u_h of u is computed. Error bounds in the maximum norm are proved for $u-u_h$ and $u'-u_h'$. Numerical results are presented to demonstrate the sharpness of these bounds.

Keywords Caputo fractional derivative · collocation method · boundary value problem

Mathematics Subject Classification (2000) 65L10 · 65L60

1 Introduction

Fractional-order derivatives have recently risen to prominence in the modelling of various processes; see [7,9] for several applications. The mathematical analysis of problems involving these derivatives has also attracted much attention—a survey of recent activity is given in [8]. In the current paper, we contribute to these developments by describing and analysing a numerical method for a two-point boundary value problem whose leading term is a Caputo fractional derivative.

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For $r \in \mathbb{R}$ with $r > 0$, and all $g \in L_1[0,1]$, define the Riemann-Liouville fractional integral operator of order r by

$$(J^r g)(x) = \left[\frac{1}{\Gamma(r)} \int_{t=0}^{x} (x-t)^{r-1} g(t)\, dt\right] \quad \text{for } 0 \le x \le 1. \tag{1.1}$$

Let the parameter δ satisfy $0 < \delta < 1$. For $k \in \mathbb{N} := \{1, 2, \dots\}$, the Riemann-Liouville fractional derivative $D^{k-\delta}$ is defined by

$$D^{k-\delta} g(x) = \left(\frac{d}{dx}\right)^k (J^\delta g)(x) \quad \text{for } 0 < x \le 1, \tag{1.2}$$

for all functions g such that $D^{k-\delta} g(x)$ exists. Our interest centres on the *Caputo fractional derivative* $D_*^{k-\delta}$, which is defined [5, Definition 3.2] in terms of $D^{k-\delta}$ by

$$D_*^{k-\delta} g = D^{k-\delta}[g - T_{k-1}[g;0]], \tag{1.3}$$

where $T_{k-1}[g;0]$ denotes the Taylor polynomial of degree $k-1$ of the function g expanded around $x = 0$. If $g \in C^{k-1}[0,1]$ and $g^{(k-1)}$ is absolutely continuous on $[0,1]$, then [5, Theorem 3.1] one also has the equivalent formulation

$$D_*^{k-\delta} g(x) := \frac{1}{\Gamma(\delta)} \int_{t=0}^{x} (x-t)^{\delta-1} g^{(k)}(t)\, dt \quad \text{for } 0 < x \le 1. \tag{1.4}$$

Since the integrals in $D^{k-\delta} g(x)$ and $D_*^{k-\delta} g(x)$ are associated in a special way with the point $x = 0$, many authors write instead $D_0^{k-\delta} g(x)$ and $D_{*0}^{k-\delta} g(x)$, but for simplicity of notation we omit the subscript 0.

Throughout the paper we consider the *two-point boundary value problem*

$$-D_*^{2-\delta} u(x) + b(x)u'(x) + c(x)u(x) = f(x) \ \text{ for } x \in (0,1), \tag{1.5a}$$
$$u(0) - \alpha_0 u'(0) = \gamma_0, \quad u(1) + \alpha_1 u'(1) = \gamma_1, \tag{1.5b}$$

where the constants $\alpha_0, \alpha_1, \gamma_0, \gamma_1$ and the functions b, c and f are given. We assume that $c \ge 0$ and

$$\alpha_0 \ge \frac{1}{1-\delta} \ \text{ and } \ \alpha_1 \ge 0. \tag{1.6}$$

The conditions $c \ge 0$ and (1.6) guarantee that (1.5) satisfies a suitable comparison/maximum principle, from which existence and uniqueness of the solution u of (1.5) follows; see [12] and Theorem 2.1 below.

If the Robin boundary condition at $x = 0$ is replaced by a Dirichlet boundary condition, then the comparison/maximum principle may no longer be true; [12, Example 2.4] provides a counterexample. For δ near 1, condition (1.6) forces $u'(0) \approx 0$ in (1.5b); one might be suspicious that this requirement for a differential equation that is "almost first-order" is unnatural, but this interpretation is flawed since δ acts as a singular perturbation parameter as it approaches the value 1 (cf. [11]), so in the limit $\delta = 1$ one should not expect that all boundary conditions of (1.5b) will be satisfied.

The problem (1.5) models superdiffusion of particle motion when convection is present; see the discussion and references in [7, Section 1]. It is a member of the general class of boundary value problems that is analysed in [10]. It is also discussed in [1]. Numerical methods for its solution are presented in [6,12] and their references. We assume that $b,c,f \in C[0,1]$; further hypotheses will be placed later on the regularity of these functions.

Our paper is structured as follows. In Section 2 we present a novel reformulation of the boundary value problem as a Volterra integral equation of the second kind whose unknown solution is $y(x) := u'(x) - u'(0)$, and discuss the properties of y. Section 3 presents and analyses a collocation method for this integral equation by extending the standard approach of [2] to problems whose right-hand sides are less smooth at $x = 0$. Then Section 4 explains how one post-processes the collocation solution to obtain an approximation of $u(x)$, and derives error bounds for this approximation and for the error in a computed approximation of $u'(x)$. The analysis in this section is completely original. Numerical results for our method are are presented in Section 5.

Notation. Set $\mathbb{N}_0 = \{0,1,2,\dots\}$. We use the standard notation $C^k(I)$ to denote the space of real-valued functions whose derivatives up to order k are continuous on an interval I, and write $C(I)$ for $C^0(I)$. For each $g \in C[0,1]$, set $\|g\|_\infty = \max_{x\in[0,1]} |g(x)|$.

In several inequalities C denotes a generic constant that depends on the data of the boundary value problem (1.5) and possibly on the mesh grading but is independent of the mesh diameter when (1.5) is solved numerically; note that C can take different values in different places.

2 Reformulation of the boundary value problem

When investigating solutions of (1.5), the natural setting is a weighted normed linear space $C^{q,\delta}$ with $q \in \mathbb{N}$, which we now define and which is a particular case of the more general Banach spaces $C^{q,\nu}$ for $-\infty < \nu < 1$ that are considered in [4,13]. Let $C^{q,\delta}(0,1]$ be the space of all q-times continuously differentiable functions $y : [0,1] \to \mathbb{R}$ such that

$$\|y\|_{q,\delta} := \sup_{0<x\le 1} |y(x)| + \sum_{k=1}^{q} \sup_{0<x\le 1} \left[x^{k-(1-\delta)} |y^{(k)}(x)| \right] < \infty.$$

In other words, $C^{q,\delta}(0,1]$ is the space of functions $y \in C[0,1] \cap C^q(0,1]$ such that $|y(x)| \le C$ and $|y^{(k)}(x)| \le C x^{(1-\delta)-k}$ for $k = 1,\dots,q$. By [13], $C^{q,\delta}(0,1]$ is a Banach space. Note that $C^q[0,1] \subset C^{q,\delta}(0,1]$.

Theorem 2.1 *[12, Corollary 3.1] Let $b,c,f \in C^{q,\delta}(0,1]$ for some integer $q \ge 2$. Assume that $c \ge 0$ and the condition (1.6) is satisfied. Then (1.5) has a unique solution u with $u \in C^1[0,1] \cap C^{q+1}(0,1]$ such that $u' \in C^{q,\delta}(0,1]$.*

Our numerical method for solving (1.5) is based on reformulating it as a Volterra integral equation of the second kind, to which we will apply a collocation method.

Invoking (1.4) twice, one sees that $(D_*^{2-\delta}u)(x) = (D_*^{1-\delta}u')(x)$. Now apply the Riemann-Liouville integral operator $J^{1-\delta}$ and appeal to [5, Theorem 3.8] to get

$$J^{1-\delta}(D_*^{2-\delta}u)(x) = J^{1-\delta}(D_*^{1-\delta}u')(x) = u'(x) - u'(0).$$

Hence, applying $J^{1-\delta}$ to (1.5a), one obtains

$$-u'(x)+u'(0)+J^{1-\delta}(bu'+cu)(x)=J^{1-\delta}(f)(x). \tag{2.1}$$

Set $\mu=u'(0)$, $y(x)=u'(x)-\mu$, and $Y(x)=\int_0^x y(s)\,ds$ for $0\le x\le 1$. Now

$$(cu)(x)=c(x)\big[Y(x)+\mu x+u(0)\big]=(cY)(x)+\mu(x+\alpha_0)c(x)+\gamma_0 c(x),$$

where we appealed to (1.5b) to replace $u(0)$ by $\alpha_0\mu+\gamma_0$. Then (2.1) can be rewritten as

$$y(x)-J^{1-\delta}(by+cY)(x)=J^{1-\delta}(\mu g_1+g_2)(x) \tag{2.2}$$

where

$$g_1(x):=b(x)+(x+\alpha_0)c(x), \qquad g_2(x):=\gamma_0 c(x)-f(x) \quad \text{for } 0\le x\le 1. \tag{2.3}$$

Equation (2.2) is a weakly singular Volterra integral equation of the second kind. The numerical solution of this type of equation is discussed at length in [2]. Our numerical method for solving (1.5) is based on discretizing (2.2). That is, we solve our original boundary value problem by applying m-point collocation on each of N mesh intervals to solve the initial value problem (2.2). This numerical method is cheap: a direct approach using (1.5) would entail solving a non-sparse linear system of equations with N unknowns, but when (2.2) is used instead, we solve $2N$ linear systems each having m unknowns, where $m\ll N$. The boundary conditions (1.5b) will be used subsequently to determine $u(0)$ and μ.

2.1 Structure of the solution of (2.2)

For the later analysis of our numerical method, further information about the structure of the solution y of (2.2) is needed. We shall split y as a sum of functions corresponding to terms on the right-hand side of (2.2). Consequently, recalling the definition (1.1), we now consider the integral equation

$$\begin{aligned} z(x)-\frac{1}{\Gamma(1-\delta)}\int_{t=0}^{x}(x-t)^{-\delta}\left[b(t)z(t)+c(t)\int_0^t z(s)\,ds\right]dt \\ =\frac{1}{\Gamma(1-\delta)}\int_{t=0}^{x}(x-t)^{-\delta}g(t)\,dt \quad \text{for } 0\le x\le 1, \end{aligned} \tag{2.4}$$

where g can be g_1 or g_2 (both defined in (2.3)), and whose solution is $z(x)$.

Lemma 2.1 *Assume that $b,c,g\in C^q[0,1]$ for some $q\in\mathbb{N}$. Then (2.4) has a unique solution $z\in C^{q,\delta}(0,1]$, and $|z(x)-z(0)|\le Cx^{1-\delta}$ for $0\le x\le 1$.*

Proof Set

$$G(x)=\int_{t=0}^{x}(x-t)^{-\delta}g(t)\,dt, \quad \text{for } x\in[0,1].$$

Via integration by parts it is easy to check that

$$G\in C^{q,\delta}(0,1]. \tag{2.5}$$

Define the operator $S: C^{q,\delta}(0,1] \to C^{q,\delta}(0,1]$ by

$$(S\phi)(x) = \frac{1}{\Gamma(1-\delta)} \int_{t=0}^{x} (x-t)^{-\delta} b(t)\phi(t)\,dt, \quad \text{for } x \in [0,1];$$

then S satisfies the hypothesis of [4, Lemma 2.2] and thus S is a compact operator. Next, it is easy to see that $\phi \mapsto \int_0^t \phi$ is a continuous mapping from $C^{q,\delta}(0,1]$ to $C^{q,\delta}(0,1]$. Hence the composition $\phi \mapsto \int_{t=0}^{x}(x-t)^{-\delta}c(t)(\int_0^t \phi)\,dt$ is a compact mapping from $C^{q,\delta}(0,1]$ to itself. Finally, as the sum of two compact operators is a compact operator, the integral operator $\phi \mapsto \int_{t=0}^{x}(x-t)^{-\delta}\left[b(t)\phi(t) + c(t)(\int_0^t \phi)\right] dt$ is a compact operator from $C^{q,\delta}(0,1]$ to $C^{q,\delta}(0,1]$. This property and (2.5) together imply, via a Fredholm alternative argument (for (2.4) one can prove a Gronwall-type inequality for $\bar{z}(x) := \max_{[0,x]} |z(t)|$), that (2.4) has a unique solution $z \in C^{q,\delta}(0,1]$; cf. [4, Remark 3, p.964].

Finally, for $0 \le x \le 1$ we have

$$|z(x) - z(0)| = \left| \int_{s=0}^{x} z'(s)\,ds \right| \le C \int_0^x s^{-\delta}\,ds = Cx^{1-\delta}.$$

Remark 2.1 The proof of Lemma 2.1 is included for completeness because (2.4) does not fit exactly into the standard Volterra integral equation framework, owing to the presence of the term $c(t)\int_0^t z(s)\,ds$.

3 Collocation method for (2.4)

We use collocation to solve (2.4) numerically for $g = g_1$ and $g = g_2$, then we take a linear combination of these computed solutions to find the solution y of (2.2), where the boundary conditions (1.5b) will be used to determine the value of μ.

In this section we rely heavily on the convergence analysis for collocation methods applied to weakly singular Volterra equations of the second kind that is presented in [2, Chapter 6]. If $c \equiv 0$ then the integral operator defined by the left-hand side of (2.4) fits instantly into the framework of [2]; when $c \not\equiv 0$, an inspection of the arguments in [2] shows that the terms corresponding to c are a lower-order perturbation that does not disturb the collocation convergence analysis. The crucial properties of z that are needed in this analysis were derived in Lemma 2.1.

Let $N \in \mathbb{N}$. Subdivide $[0,1]$ by the mesh $0 = x_0 < x_1 < \cdots < x_N = 1$, where $x_i = (i/N)^r$ for $i = 0,1,\ldots,N$. The parameter $r \in [1,\infty)$ determines the grading of the mesh; when $r = 1$ the mesh is uniform.

Set $h_i = x_{i+1} - x_i$ for $i = 1,2,\ldots,N$. Set $h = \max h_i$.

In the error estimates that follow, the generic constants C depend on the choice of collocation parameters $\{c_j\}$ and on r, but are independent of h.

Let $m \in \mathbb{N}$. Our computed solution z_h will lie in the space

$$S_{m-1}^{-1} := \left\{ v : v\big|_{(x_i,x_{i+1})} \in \pi_{m-1},\ i = 0,1,\ldots,N-1 \right\}$$

comprising piecewise polynomials of degree at most $m-1$ that may be discontinuous at interior mesh points x_i. The set of collocation points is

$$X_h := \left\{x_i + c_j h_i : 0 \le c_1 < c_2 < \cdots < c_m \le 1,\ i = 0, 1, \ldots, N-1\right\}$$

where the collocation parameters $\{c_j\}$ are yet to be chosen. If $c_1 = 0$ and $c_m = 1$, then z_h will lie in the space $S_{m-1}^{-1} \cap C[0,1] =: S_{m-1}^{0}$, and (to make the number of equations equal to the number of unknowns) we require z_h to satisfy the initial condition $z_h(0) = 0$ because the integrals in (2.4) vanish when $x = 0$.

The *collocation solution* $z_h \in S_{m-1}^{-1}$ of (2.4) is defined by

$$\begin{aligned} z_h(x) - \frac{1}{\Gamma(1-\delta)} \int_{t=0}^{x} (x-t)^{-\delta} \left[b(t) z_h(t) + c(t) \int_0^t z_h(s)\, ds \right] dt \\ = \frac{1}{\Gamma(1-\delta)} \int_{t=0}^{x} (x-t)^{-\delta} g(t)\, dt \quad \text{for all } x \in X_h \cup \{1\}. \end{aligned} \tag{3.1}$$

It is shown in [2, Theorem 6.2.1] that for sufficiently small h, the collocation solution z_h is well defined.

One solves (3.1) iteratively, mesh interval by mesh interval: when z_h has been computed on $[0, x_i)$, one can then compute z_h on $[x_i, x_{i+1}]$ using $x = x_i + c_j h_i$ in (3.1) for $j = 1, 2, \ldots, m$; this is a system of m equations which is cheap to solve (it's a system of $m-1$ equations if $c_1 = 0$ and $c_m = 1$).

We have the following convergence result for our collocation method. (A general analysis of the attainable order of convergence of collocation solutions for weakly singular Volterra integral equations is given in [3].)

Lemma 3.1 *Assume that* $g \in C^m[0,1]$. *Let* h *be sufficiently small. Let the mesh grading exponent* r *satisfy*

$$r = \frac{\sigma}{1-\delta} \quad \textit{with} \quad \sigma \ge 1-\delta.$$

Then the collocation solution z_h *of* (2.4) *satisfies the error bound*

$$\|z - z_h\|_\infty \le C h^{\min\{\sigma, m\}}. \tag{3.2}$$

Proof By Lemma 2.1, for $0 \le x \le 1$ one has

$$z(x) = z(0) + O(x^{1-\delta}). \tag{3.3}$$

One can replace [2, Theorem 6.1.6] by (3.3) in the proof of [2, Theorem 6.2.9], whose argument then yields the error bound (3.2).

Lemma 3.1 assumes that in the collocation method all integrals are evaluated exactly. In practice quadrature rules must be applied to these integrals, and we now discuss the effect of these.

Assume that product quadrature formulas with the collocation points as nodes are used to evaluate the integrals in (3.1), viz., on each mesh interval $[x_{i-1}, x_i]$ the function multiplying $(x-t)^{-\delta}$ is replaced by a polynomial of degree $m-1$ that interpolates to this function at the collocation points $x_{i-1} + c_j h_i$, $j = 1, 2, \ldots, m$, then the resulting integrals are evaluated exactly. See [2, Section 6.2.2] for details. Write $\hat{z}_h$ for the numerical solution in S_{m-1}^{-1} that is then obtained. Theorem 6.2.2 of [2] shows that $\hat{z}_h$ is well defined for all sufficiently small h.

The error $z_h - \hat{z}_h$ due to quadrature is analyzed as in [2, Section 6.2.7]; the only difference between this section and our situation is the non-smooth right-hand side of (3.1), but the quadrature error for this term is bounded in [2, Theorems 6.2.7 and 6.2.8]. Consequently a bound on $z_h - \hat{z}_h$ is supplied by [2, Theorem 6.2.14], which we will now state in a slightly simplified form. First, we define a quantity that depends on the choice of collocation parameters $\{c_j\}$. Set

$$J = \int_0^1 \prod_{j=1}^m (s - c_j)\,ds.$$

Note that $J = 0$ is precisely the condition under which the m-point quadrature rule on each interval $[x_i, x_{i+1}]$ becomes $O(h_i^{m+1})$ accurate and not just $O(h_i^m)$.

Lemma 3.2 *Assume that $b, c, g \in C^m[0,1]$ if $r = 1$, and $b, c, g \in C^{m+1}[0,1]$ if $r > 1$. Let z_h be the collocation solution of (3.1) and $\hat{z}_h$ its solution when product quadrature with the collocation points as nodes is used. Then*

$$\|z_h - \hat{z}_h\|_\infty \le \begin{cases} Ch^m & \text{if } J \neq 0, \\ Ch^{m+1-\delta} & \text{if } J = 0. \end{cases}$$

Proof See [2, Theorem 6.2.14].

When $r > 1$, this result is weaker than the result stated in [2] but it suffices for our purposes.

Corollary 3.1 *Assume that $b, c, g \in C^m[0,1]$. Let the mesh grading exponent r satisfy $r = \sigma/(1-\delta)$ with $\sigma \ge 1-\delta$. Then the collocation solution $\hat{z}_h$ computed using product quadrature satisfies*

$$\|z - \hat{z}_h\|_\infty \le Ch^{\min\{\sigma, m\}}.$$

Proof Use a triangle inequality with Lemmas 3.1 and 3.2.

3.1 Superconvergence at collocation points

Lemma 3.1 deals with the accuracy of the collocation solution in the norm $\|\cdot\|$. However, the collocation solution z_h is often more accurate at the collocation points than when the error is measured in the norm $\|\cdot\|_\infty$. This observation is important as z_h approximates $z(x) = u'(x) - \mu$, so our main interest lies in the accurate approximation of $\int_0^x z(t)\,dt$ (rather than z), for which the collocation points will be used as quadrature points.

Lemma 3.3 *Assume that $J = 0$ and $b, c, g \in C^{m+1}[0,1]$. Then*

$$|(z - z_h)(x_i + c_j h_i)| \le \begin{cases} Ch^{2(1-\delta)} & \text{if } r = 1, \\ Ch^{m+1-\delta} & \text{if } r \ge m/(1-\delta), \end{cases}$$

for $i = 0, 1, \dots, N$ and $j = 1, 2, \dots, m$, with the convention that $h_N = 0$.

Proof Let z_h^{it} be the iterated collocation solution as defined in [2]. It is clear that $(z_h^{it} - z_h)(x_i + c_j h_i) = 0$. One can verify that Theorem 6.2.13 of [2] applies to $z - z_h^{it}$; the result follows.

Corollary 3.2 *Assume that $J = 0$, that $b, c, g \in C^{m+1}[0,1]$, and that the mesh grading exponent satisfies $r \geq m/(1-\delta)$. Then*

$$|(z - \hat{z}_h)(x_i + c_j h_i)| \leq C h^{m+1-\delta}$$

for $i = 0, 1, \ldots, N$ and $j = 1, 2, \ldots, m$, with the convention that $h_N = 0$.

Proof Combine the $J = 0$ case of Lemma 3.2 with Lemma 3.3.

4 Numerical solution u_h of (1.5)

In this section we prove that one can construct an accurate approximation of the solution u of (1.5) from the approximate solutions of (2.4) with $g = g_1$ and $g = g_2$ that were discussed in Section 3. To do this, we need first to examine the relationship of u to the exact solutions of these latter problems.

Let the solutions of (2.4) with

$$(i) \quad g(t) = g_1(t) = b(t) + (t + \alpha_0) c(t) \qquad\qquad (ii) \quad g(t) = g_2(t) = \gamma_0 c(t) - f(t)$$

be v and w respectively.

Lemma 4.1 *One has*

$$1 + \alpha_0 + \alpha_1 + \alpha_1 v(1) + \int_0^1 v(x) \geq \alpha_0 > 1. \tag{4.1}$$

Proof Set $v_1(x) = 1 + v(x)$ and $V_1(x) = \int_0^x v_1(t)\,dt$ for $0 \leq x \leq 1$. Then

$$1 + \alpha_0 + \alpha_1 + \alpha_1 v(1) + \int_0^1 v(x)\,dx = \alpha_0 + \alpha_1 v_1(1) + \int_0^1 v_1(x)\,dx. \tag{4.2}$$

By its definition, $v \in C[0,1] \cap C^m(0,1]$ is the solution of

$$v(x) - J^{1-\delta}(bv + cV)(x) = (J^{1-\delta} g_1)(x) \quad \text{for } 0 \leq x \leq 1,$$

where $V(t) := \int_0^t v(s)\,ds$. Hence

$$v_1(x) - J^{1-\delta}(bv_1 + cV_1)(x) = 1 + \alpha_0 (J^{1-\delta} c)(x) \quad \text{for } 0 \leq x \leq 1, \tag{4.3}$$

where $V_1(t) := \int_0^t v_1(s)\,ds$.

We claim that (4.3) implies that $v_1(x) \geq 0$ for $0 \leq x \leq 1$. Suppose this is false. Set $x^* = \inf_{x \in [0,1]}\{x : v_1(x) < 0\}$. As $v_1 \in C[0,1]$ and $v_1(0) = 1$ (because of the definition of v), it follows that $x^* \in (0,1)$, $v_1(x^*) = 0$, and $v_1 \geq 0$ on $[0, x^*)$. Applying the

Riemann-Liouville derivative $D^{1-\delta}$ to (4.3), by [5, Theorem 2.14 and Example 2.4] we get

$$D^{1-\delta} v_1(x) - (bv_1 + cV_1)(x) = \frac{x^{\delta-1}}{\Gamma(\delta)} + \alpha_0 c(x) \quad \text{for } 0 \le x \le 1.$$

In particular, taking $x = x^*$, one has

$$D^{1-\delta} v_1(x^*) - (cV_1)(x^*) = \frac{(x^*)^{\delta-1}}{\Gamma(\delta)} + \alpha_0 c(x^*). \tag{4.4}$$

But by (1.2),

$$\begin{aligned} D^{1-\delta} v_1(x^*) &= \frac{d}{dx}\left(\frac{1}{\Gamma(\delta)} \int_0^x (x-t)^{\delta-1} v_1(t)\,dt \right)\Bigg|_{x=x^*} \\ &= \frac{\delta-1}{\Gamma(\delta)} \int_0^{x^*} (x^*-t)^{\delta-2} v_1(t)\,dt, \end{aligned} \tag{4.5}$$

as can be seen by integrating by parts before and after differentiating with respect to x; here one needs to use $|v_1(t)| \le C(x^*-t)$ for $t \in [0,x^*]$, which follows from $v_1(x^*) = 0$ and $v_1 \in C[0,1] \cap C^m(0,1]$. Now, combining (4.4) and (4.5) yields

$$-\frac{1-\delta}{\Gamma(\delta)} \int_0^{x^*} (x^*-t)^{\delta-2} v_1(t)\,dt = (cV_1)(x^*) + \frac{(x^*)^{\delta-1}}{\Gamma(\delta)} + \alpha_0 c(x^*).$$

But $v_1(0) = 1$ and $v_1 \ge 0$ on $[0,x^*)$, so $V_1(x^*) > 0$ and the left-hand side of the equation is strictly negative—but all terms on the right-hand side are non-negative since $c \ge 0$. From this contradiction we conclude that $v_1 \ge 0$ on $[0,1]$.

Now (4.2) yields immediately $1 + \alpha_0 + \alpha_1 + \alpha_1 v(1) + \int_0^1 v(x)\,dx \ge \alpha_0$, and the proof is complete since $\alpha_0 \ge 1/(\delta-1) > 1$.

Theorem 4.1 *For the solution u of* (1.5) *one has*

$$(i)\ u'(x) = \mu[v(x)+1] + w(x), \tag{4.6}$$

$$(ii)\ u(x) = \gamma_0 + \mu\alpha_0 + \int_0^x u'(t)\,dt, \tag{4.7}$$

where

$$\mu = \frac{\gamma_1 - \gamma_0 - \alpha_1 w(1) - \int_0^1 w}{1 + \alpha_0 + \alpha_1 + \alpha_1 v(1) + \int_0^1 v}. \tag{4.8}$$

Proof The identity (2.2) and the definitions of v and w imply that $y = \mu v + w$. But by definition $y(x) = u'(x) - \mu$, so

$$u(x) = u(0) + \int_0^x u'(t)\,dt = u(0) + \mu x + \int_0^x y(t)\,dt = u(0) + \mu x + \int_0^x (\mu v + w)(t)\,dt$$

for $0 \le x \le 1$. Thus to satisfy the boundary conditions (1.5b), we need

$$\gamma_0 = u(0) - \alpha_0 u'(0) = u(0) - \alpha_0 \mu,$$
$$\gamma_1 = u(1) + \alpha_1 u'(1) = u(0) + \mu + \int_0^1 (\mu v + w) + \alpha_1 [\mu + \mu v(1) + w(1)].$$

Subtracting these equations yields

$$\gamma_1 - \gamma_0 = \mu \left[1 + \alpha_0 + \alpha_1 + \alpha_1 v(1) + \int_0^1 v \right] + \alpha_1 w(1) + \int_0^1 w, \tag{4.9}$$

which by Lemma 4.1 we can solve for μ, obtaining (4.8).

Now that μ is known, (4.6) follows from $u'(x) = \mu + y(x) = \mu + \mu v(x) + w(x)$, while (4.7) is immediate from $u(x) = u(0) + \int_0^x u'(t)\,dt$ and the boundary condition (1.5b) at $x = 0$.

Remark 4.1 Theorem 4.1 implies that a solution of (2.4) is also a solution of (1.5). Our derivation of (2.4) from (1.5) in Section 2 proved the converse implication. Thus the problems (1.5) and (2.4) are equivalent.

Let the computed collocation solutions of the problems defining v and w (using the same set of collocation points X_h for each problem) be $\hat{v}_h \in S_{m-1}^{-1}$ and $\hat{w}_h \in S_{m-1}^{-1}$ respectively, where quadrature (as described in Section 3) is used to evaluate the integrals arising in the method. Then we define our computed solution of (2.2) to be $y_h := \mu_h \hat{v}_h + \hat{w}_h$, where the constant μ_h will be chosen in a moment. Repeating the calculations in the proof of Theorem 4.1 with μ, v, w, y replaced by $\mu_h, \hat{v}_h, \hat{w}_h, y_h$ respectively, we see that we can obtain a computed solution $u_h \in S_m^0$ of (1.5) that satisfies the boundary conditions (1.5b) provided that

$$1 + \alpha_0 + \alpha_1 + \alpha_1 \hat{v}_h(1) + \int_0^1 \hat{v}_h \neq 0. \tag{4.10}$$

But Lemma 4.1 and Corollary 3.1 imply that for h sufficiently small, inequality (4.10) is satisfied. Consequently the analogues of (4.6)–(4.8) hold true:

$$(i)\ u_h'(x) = \mu_h[\hat{v}_h(x) + 1] + \hat{w}_h(x), \tag{4.11}$$
$$(ii)\ u_h(x) = \gamma_0 + \mu_h \alpha_0 + \int_0^x u_h'(t)\,dt, \tag{4.12}$$

where

$$(iii)\ \mu_h = \frac{\gamma_1 - \gamma_0 - \alpha_1 \hat{w}_h(1) - \int_0^1 \hat{w}_h}{1 + \alpha_0 + \alpha_1 + \alpha_1 \hat{v}_h(1) + \int_0^1 \hat{v}_h}. \tag{4.13}$$

We come now to our main result.

Theorem 4.2 *Assume that $b,c,f \in C^m[0,1]$ for some $m \in \mathbb{N}$. Let h be sufficiently small. Let the mesh grading exponent r satisfy*

$$r = \frac{\sigma}{1-\delta} \quad \textit{with} \quad \sigma \geq 1-\delta.$$

Then the collocation solution u_h of (1.5)*, when product quadrature with the collocation points as nodes is used, satisfies the error bound*

$$\|u-u_h\|_\infty + \|u'-u_h'\|_\infty \leq \begin{cases} Ch^\sigma & \textit{if } 1-\delta \leq \sigma \leq m, \\ Ch^m & \textit{if } \sigma \geq m. \end{cases} \tag{4.14}$$

If in addition $J = 0$ and $b,c,f \in C^{m+1}[0,1]$, then for $\sigma \geq m$ one obtains

$$\|u-u_h\|_\infty + |(u'-u_h')(x_i + c_j h_j)| \leq Ch^{m+1-\delta} \tag{4.15}$$

for $i = 0,1,\ldots,N$ and $j = 1,2,\ldots,m$, with the convention that $h_N = 0$.

Proof For convenience write $H(\sigma,m)$ for the right-hand side of (4.14), and recall that C is a generic constant that is independent of h. Invoking Corollary 3.1 twice, we have $\|v-\hat{v}_h\|_\infty \leq H(\sigma,m)$ and $\|w-\hat{w}_h\|_\infty \leq H(\sigma,m)$. Then $|\mu-\mu_h| \leq H(\sigma,m)$ by (4.8), (4.13) and Lemma 4.1. Consequently (4.6) and (4.11) yield $\|u'-u_h'\|_\infty \leq H(\sigma,m)$. Now $\|u-u_h\|_\infty \leq H(\sigma,m)$ follows from (4.7) and (4.12), and the proof of (4.14) is complete.

Next, suppose that $J = 0$ and $b,c,f \in C^{m+1}[0,1]$. Corollary 3.2 immediately gives $|w(1)-\hat{w}_h(1)| \leq Ch^{m+1-\delta}$ and also, by virtue of [2, Theorem 6.2.8], implies that $|\int_0^1 (w-\hat{w}_h)| \leq Ch^{m+1-\delta}$. Similar inequalities are valid for $v-\hat{v}_h$. Combining these results with Lemma 4.1, (4.8) and (4.13), we get $|\mu-\mu_h| \leq Ch^{m+1-\delta}$. The bound $|(u'-u_h')(x_i+c_jh_j)| \leq Ch^{m+1-\delta}$ now follows from (4.6), (4.11) and Corollary 3.2.

Finally, the bound $\|u-u_h\|_\infty \leq Ch^{m+1-\delta}$ is a consequence of the inequality just proved and [2, Theorem 6.2.8].

Remark 4.2 The analysis of this section provides an alternative proof of the a priori bounds on the derivatives of the solution u of (1.5) that are derived in [12, Corollary 3.5]. For we have shown in Theorem 4.1 that

$$u(x) = \gamma_0 + \mu\alpha_0 + \int_0^x u'(t)\,dt = \gamma_0 + \mu\alpha_0 + \int_0^x \{\mu[v(t)+1] + w(t)\}\,dt. \tag{4.16}$$

Suppose that $b,c,f \in C^q[0,1]$ for some $q \geq 1$. (This hypothesis is stronger than the assumption in [12, Corollary 3.5] that $b,c,f \in C^{q,\delta}[0,1]$.) By Lemma 2.1 we have $v,w \in C^{q,\delta}[0,1]$, so $|v(x)|+|w(x)| \leq C$ and $|v^{(k)}(x)|+|w^{(k)}(x)| \leq Cx^{1-\delta-k}$ for $k = 1,\ldots,q$. By differentiating (4.16) and invoking these bounds on v and w, we get $|u(x)|+|u'(x)| \leq C$ and $|u^{(k)}(x)| \leq Cx^{2-\delta-k}$ for $k = 1,\ldots,q$, as desired.

5 Numerical results

In this section we present the results of numerical experiments for two test problems, only one of which satisfies the regularity hypotheses of Theorem 4.2. Both problems are solved numerically for various values of $\delta \in (0,1)$ on a graded mesh $x_i = (i/N)^r$ for $i = 0,1,\ldots,N$ with $r = m/(1-\delta)$, where the collocation method uses piecewise polynomials of degree $m-1$. The values $m = 1,2,3$ are examined in our experiments.

The caption to each table of results states the value of m, the choice of the collocation parameters $c_1, c_2, \ldots, c_m$, and whether $J = 0$.

Example 5.1 Consider (1.5) with $b(x) = \cos x - x^2,\ c(x) \equiv 0,\ \alpha_0 = 1/(1-\delta)$ and $\alpha_1 = 3/5$. We choose f, γ_0 and γ_1 to agree with the exact solution

$$u(x) = 2x^{2-\delta} - x^{3-2\delta} + 1 + 2x - 3x^3 + \tfrac{1}{2}x^4, \tag{5.1}$$

whose regularity is typical of solutions of (1.5); see Theorem 2.1 and [12, Example 3.7].

Tables 5.1–5.6 list, for a range of values of δ and N, the discrete maximum nodal errors $\max_i |(u-u_h)(x_i)|$ and the rates of convergence of these maximum errors when N changes while δ is fixed. Tables 5.7 and 5.8 give corresponding results for the derivative approximation errors $\max_{i,k} |(u' - u_h')(x_i + c_k h_i)|$ when $m = 1$. We devote only two tables to these errors because our main interest is in the errors $u - u^h$ and we do not wish to make the presentation excessively long; for other values of m we observe the rates of convergence predicted by Theorem 4.2.

Table 5.1 Example 5.1, $\max_i |(u-u_h)(x_i)|$ for $m = 1,\ c_k = \{0\};\ J \neq 0$

	N=128	N=256	N=512	N=1024	N=2048	N=4096	N=8192
$\delta = 0.1$	4.474e-2 1.00	2.238e-2 1.00	1.119e-2 1.00	5.595e-3 1.00	2.797e-3 1.00	1.399e-3 1.00	6.994e-4
$\delta = 0.2$	4.768e-2 1.00	2.382e-2 1.00	1.190e-2 1.00	5.950e-3 1.00	2.974e-3 1.00	1.487e-3 1.00	7.434e-4
$\delta = 0.3$	5.309e-2 1.00	2.647e-2 1.00	1.321e-2 1.00	6.597e-3 1.00	3.296e-3 1.00	1.647e-3 1.00	8.232e-4
$\delta = 0.4$	6.220e-2 1.01	3.091e-2 1.01	1.538e-2 1.00	7.667e-3 1.00	3.825e-3 1.00	1.910e-3 1.00	9.539e-4
$\delta = 0.5$	7.765e-2 1.02	3.837e-2 1.01	1.901e-2 1.01	9.443e-3 1.01	4.70e-3 1.01	2.341e-3 1.00	1.167e-3
$\delta = 0.6$	1.055e-1 1.03	5.175e-2 1.02	2.547e-2 1.02	1.258e-2 1.01	6.228e-3 1.01	3.090e-3 1.01	1.536e-3
$\delta = 0.7$	1.612e-1 1.04	7.857e-2 1.03	3.840e-2 1.03	1.883e-2 1.02	9.258e-3 1.02	4.565e-3 1.02	2.256e-3
$\delta = 0.8$	2.947e-1 1.04	1.437e-1 1.04	7.001e-2 1.04	3.416e-2 1.03	1.670e-2 1.03	8.185e-3 1.03	4.019e-3
$\delta = 0.9$	7.648e-1 1.01	3.791e-1 1.02	1.867e-1 1.03	9.166e-2 1.03	4.498e-2 1.03	2.208e-2 1.03	1.084e-2

The first-order rate of convergence that is apparent in each row of Table 5.1 is in agreement with (4.14). For $m = 1$, we observed very similar results when we took

$c_k = \{1/3\}$ and $c_k = \{1\}$. But for $c_k = \{1/2\}$ one now has $J = 0$ and a higher order of convergence is predicted by (4.15); this is manifested in Table 5.2.

Table 5.2 Example 5.1, $\max_i |(u-u_h)(x_i)|$ for $m=1,\ c_k=\{1/2\};\ J=0$

	N=128	N=256	N=512	N=1024	N=2048	N=4096	N=8192
$\delta = 0.1$	7.911e-5 1.97	2.020e-5 1.97	5.163e-6 1.97	1.321e-6 1.97	3.381e-7 1.96	8.663e-8 1.96	2.222e-8
$\delta = 0.2$	1.431e-4 1.74	4.273e-5 1.75	1.268e-5 1.76	3.745e-6 1.77	1.101e-6 1.77	3.228e-7 1.77	9.434e-8
$\delta = 0.3$	4.130e-4 1.66	1.303e-4 1.67	4.091e-5 1.68	1.279e-5 1.68	3.987e-6 1.69	1.240e-6 1.69	3.848e-7
$\delta = 0.4$	1.098e-3 1.58	3.682e-4 1.58	1.230e-4 1.59	4.095e-5 1.59	1.360e-5 1.59	4.511e-6 1.59	1.494e-6
$\delta = 0.5$	2.836e-3 1.48	1.015e-3 1.49	3.621e-4 1.49	1.288e-4 1.49	4.572e-5 1.50	1.621e-5 1.50	5.743e-6
$\delta = 0.6$	7.298e-3 1.38	2.796e-3 1.39	1.067e-3 1.39	4.059e-4 1.40	1.542e-4 1.40	5.855e-5 1.40	2.221e-5
$\delta = 0.7$	1.912e-2 1.28	7.853e-3 1.29	3.210e-3 1.29	1.309e-3 1.30	5.326e-4 1.30	2.166e-4 1.30	8.803e-5
$\delta = 0.8$	5.320e-2 1.18	2.350e-2 1.19	1.031e-2 1.19	4.506e-3 1.20	1.966e-3 1.20	8.567e-4 1.20	3.731e-4
$\delta = 0.9$	1.804e-1 1.06	8.645e-2 1.08	4.089e-2 1.09	1.921e-2 1.09	8.993e-3 1.10	4.203e-3 1.10	1.963e-3

Next, in Table 5.3 we show results for $m = 2$ and $c_k = \{0, 1\}$. These are in agreement with (4.14). If instead one chooses $c_k = \{0, 2/3\}$, then $J = 0$ and one obtains the higher rates of convergence displayed in Table 5.4 and predicted by (4.15).

Table 5.3 Example 5.1, $\max_i |(u-u_h)(x_i)|$ for $m=2,\ c_k=\{0,1\};\ J\neq 0$

	N=64	N=128	N=256	N=512	N=1024	N=2048
$\delta = 0.1$	3.787e-4 2.00	9.476e-5 2.00	2.370e-5 2.00	5.925e-6 2.00	1.481e-6 2.00	3.704e-7
$\delta = 0.2$	4.051e-4 2.00	1.015e-4 2.00	2.541e-5 2.00	6.357e-6 2.00	1.590e-6 2.00	3.976e-7
$\delta = 0.3$	4.277e-4 1.99	1.076e-4 2.00	2.698e-5 2.00	6.758e-6 2.00	1.691e-6 2.00	4.232e-7
$\delta = 0.4$	4.361e-4 1.98	1.104e-4 1.99	2.783e-5 1.99	6.991e-6 2.00	1.753e-6 2.00	4.392e-7
$\delta = 0.5$	6.530e-4 2.01	1.619e-4 2.01	4.026e-5 2.00	1.003e-5 2.00	2.502e-6 2.00	6.244e-7
$\delta = 0.6$	1.337e-3 2.00	3.346e-4 2.00	8.376e-5 2.00	2.097e-5 2.00	5.249e-6 2.00	1.313e-6
$\delta = 0.7$	2.992e-3 1.97	7.619e-4 1.98	1.933e-4 1.98	4.891e-5 1.99	1.235e-5 1.99	3.111e-6
$\delta = 0.8$	7.721e-3 1.94	2.015e-3 1.95	5.214e-4 1.96	1.342e-4 1.96	3.437e-5 1.97	8.774e-6
$\delta = 0.9$	2.963e-2 1.90	7.953e-3 1.92	2.102e-3 1.93	5.507e-4 1.94	1.435e-4 1.95	3.724e-5

Table 5.4 Example 5.1, $\max_i |(u-u_h)(x_i)|$ for $m=2,\ c_k=\{0,2/3\};\ J=0$

	N=64	N=128	N=256	N=512	N=1024	N=2048
$\delta=0.1$	3.178e-6 2.82	4.493e-7 2.83	6.312e-8 2.84	8.822e-9 2.85	1.228e-9 2.85	1.702e-10
$\delta=0.2$	9.732e-6 2.73	1.472e-6 2.74	2.201e-7 2.75	3.269e-8 2.76	4.828e-9 2.77	7.099e-10
$\delta=0.3$	2.395e-5 2.63	3.871e-6 2.65	6.164e-7 2.66	9.731e-8 2.67	1.527e-8 2.68	2.387e-9
$\delta=0.4$	5.125e-5 2.52	8.913e-6 2.55	1.518e-6 2.57	2.555e-7 2.58	4.273e-8 2.59	7.118e-9
$\delta=0.5$	9.242e-5 2.39	1.761e-5 2.44	3.236e-6 2.47	5.840e-7 2.48	1.044e-7 2.49	1.859e-8
$\delta=0.6$	1.036e-4 2.13	2.366e-5 2.28	4.884e-6 2.34	9.634e-7 2.37	1.860e-7 2.39	3.554e-8
$\delta=0.7$	2.336e-4 2.68	3.634e-5 2.52	6.322e-6 2.40	1.201e-6 2.33	2.384e-7 2.30	4.826e-8
$\delta=0.8$	3.023e-3 2.22	6.486e-4 2.24	1.376e-4 2.23	2.925e-5 2.23	6.247e-6 2.22	1.341e-6
$\delta=0.9$	2.904e-2 1.93	7.603e-3 2.01	1.893e-3 2.06	4.540e-4 2.09	1.067e-4 2.10	2.482e-5

Finally, we present our results for $\max_i |(u-u_h)(x_i)|$ with $m=3$ in Table 5.5, where $c_k=\{0,1/3,1\}$ and $J\neq 0$, and Table 5.6, where $c_k=\{0,1/2,1\}$ and $J=0$. Once again the rates of convergence agree with Theorem 4.2.

Table 5.5 Example 5.1, $\max_i |(u-u_h)(x_i)|$ for $m=3,\ c_k=\{0,1/3,1\};\ J\neq 0$

	N=64	N=128	N=256	N=512	N=1024	N=2048
$\delta=0.1$	4.885e-7 2.99	6.138e-8 3.00	7.690e-9 3.00	9.620e-10 3.00	1.203e-10 3.00	1.504e-11
$\delta=0.2$	1.346e-6 3.00	1.688e-7 3.00	2.111e-8 3.00	2.637e-9 3.00	3.294e-10 3.00	4.116e-11
$\delta=0.3$	3.005e-6 3.00	3.764e-7 3.00	4.696e-8 3.00	5.855e-9 3.00	7.301e-10 3.00	9.110e-11
$\delta=0.4$	6.449e-6 2.99	8.091e-7 3.00	1.008e-7 3.01	1.254e-8 3.01	1.561e-9 3.01	1.944e-10
$\delta=0.5$	1.409e-5 2.98	1.785e-6 3.00	2.230e-7 3.01	2.774e-8 3.01	3.447e-9 3.01	4.287e-10
$\delta=0.6$	3.251e-5 2.95	4.209e-6 2.98	5.319e-7 3.00	6.652e-8 3.01	8.284e-9 3.01	1.031e-9
$\delta=0.7$	8.312e-5 2.90	1.117e-5 2.95	1.445e-6 2.98	1.835e-7 2.99	2.307e-8 3.00	2.888e-9
$\delta=0.8$	2.604e-4 2.81	3.718e-5 2.90	4.998e-6 2.94	6.521e-7 2.96	8.369e-8 2.98	1.064e-8
$\delta=0.9$	1.508e-3 2.69	2.342e-4 2.83	3.303e-5 2.89	4.463e-6 2.92	5.887e-7 2.94	7.659e-8

In Tables 5.7 and 5.8 we present our results for the derivative approximation errors $\max_{i,k} |(u'-u'_h)(x_i+c_kh_i)|$ when $m=1$. We observe that the rates of convergence computed agree with the rates predicted by Theorem 4.2.

Table 5.6 Example 5.1, $\max_i |(u-u_h)(x_i)|$ for $m=3,\ c_k=\{0,1/2,1\};\ J=0$

	N=64	N=128	N=256	N=512	N=1024	N=2048
$\delta=0.1$	8.453e-9 3.81	6.041e-10 3.80	4.342e-11 3.81	3.092e-12 3.82	2.196e-13 3.86	1.510e-14
$\delta=0.2$	5.696e-8 3.68	4.447e-9 3.70	3.410e-10 3.72	2.582e-11 3.74	1.937e-12 3.74	1.448e-13
$\delta=0.3$	2.410e-7 3.58	2.019e-8 3.61	1.653e-9 3.63	1.332e-10 3.65	1.062e-11 3.65	8.4387e-13
$\delta=0.4$	8.402e-7 3.47	7.585e-8 3.51	6.650e-9 3.54	5.725e-10 3.56	4.870e-11 3.57	4.107-12
$\delta=0.5$	2.653e-6 3.35	2.599e-7 3.41	2.454e-8 3.44	2.265e-9 3.46	2.062e-10 3.47	1.860e-11
$\delta=0.6$	7.976e-6 3.22	8.556e-7 3.29	8.752e-8 3.33	8.708e-9 3.35	8.519e-10 3.37	8.245e-11
$\delta=0.7$	2.382e-5 3.07	2.834e-6 3.16	3.168e-7 3.21	3.417e-8 3.24	3.609e-9 3.26	3.763e-10
$\delta=0.8$	7.566e-5 2.89	1.023e-5 3.01	1.266e-6 3.08	1.492e-7 3.13	1.710e-8 3.15	1.928e-9
$\delta=0.9$	3.014e-4 2.57	5.084e-5 2.80	7.307e-6 2.92	9.632e-7 2.99	1.211e-7 3.03	1.481e-8

Table 5.7 Example 5.1, $\max_{i,k} |(u'-u_h')(x_i+c_kh_i)|$ for $m=1,\ c_k=\{0\};\ J\neq 0$

	N=128	N=256	N=512	N=1024	N=2048	N=4096	N=8192
$\delta=0.1$	3.265e-2 1.00	1.635e-2 1.00	8.178e-3 1.00	4.090e-3 1.00	2.045e-3 1.00	1.023e-3 1.00	5.113e-4
$\delta=0.2$	3.148e-2 1.00	1.575e-2 1.00	7.873e-3 1.00	3.936e-3 1.00	1.968e-3 1.00	9.840e-4 1.00	4.920e-4
$\delta=0.3$	3.185e-2 1.00	1.591e-2 1.00	7.948e-3 1.00	3.971e-3 1.00	1.985e-3 1.00	9.920e-4 1.00	4.959e-4
$\delta=0.4$	3.451e-2 1.01	1.711e-2 1.00	8.529e-3 1.00	4.256e-3 1.00	2.125e-3 1.00	1.061e-3 1.00	5.302e-4
$\delta=0.5$	4.533e-2 1.03	2.217e-2 1.02	1.091e-2 1.02	5.390e-3 1.01	2.672e-3 1.01	1.328e-3 1.01	6.612e-4
$\delta=0.6$	6.561e-2 1.05	3.165e-2 1.04	1.539e-2 1.03	7.534e-3 1.02	3.706e-3 1.02	1.830e-3 1.01	9.063-4
$\delta=0.7$	1.071e-1 1.07	5.097e-2 1.06	2.449e-2 1.05	1.186e-2 1.04	5.773e-3 1.03	2.825e-3 1.03	1.387e-3
$\delta=0.8$	2.074e-1 1.08	9.806e-2 1.07	4.684e-2 1.06	2.253e-2 1.05	1.089e-2 1.04	5.2844e-3 1.04	2.573e-3
$\delta=0.9$	5.710e-1 1.08	2.700e-1 1.06	1.295e-1 1.05	6.263e-2 1.04	3.042e-2 1.04	1.4809e-2 1.04	7.224e-3

All these results show that our main convergence result (Theorem 4.2) is sharp. We see also that our collocation method is much more accurate (and less expensive) than the finite difference methods for (1.5) that are discussed in [6, 12].

In our second test problem we modify Example 5.1 by replacing the smooth function b by a function that lies only in $C[0,1]$.

Table 5.8 Example 5.1, $\max_{i,k} |(u' - u_h')(x_i + c_k h_i)|$ for $m = 1,\ c_k = \{1/2\};\ J = 0$

	N=128	N=256	N=512	N=1024	N=2048	N=4096	N=8192
$\delta = 0.1$	1.930e-4 1.94	5.035e-5 1.94	1.315e-5 1.94	3.439e-6 1.93	9.000e-7 1.93	2.358e-7 1.93	6.184e-8
$\delta = 0.2$	3.522e-4 1.85	9.785e-5 1.84	2.727e-5 1.84	7.622e-6 1.84	2.136e-6 1.83	6.002e-7 1.83	1.690e-7
$\delta = 0.3$	7.040e-4 1.74	2.103e-4 1.74	6.313e-5 1.73	1.903e-5 1.73	5.755e-6 1.72	1.746e-6 1.72	5.310e-7
$\delta = 0.4$	1.470e-3 1.63	4.734e-4 1.63	1.533e-4 1.62	4.983e-5 1.62	1.625e-5 1.61	5.315e-6 1.61	1.742e-6
$\delta = 0.5$	3.107e-3 1.53	1.079e-3 1.52	3.761e-4 1.51	1.316e-4 1.51	4.620e-5 1.51	1.625e-5 1.51	5.723e-6
$\delta = 0.6$	6.559e-3 1.42	2.452e-3 1.41	9.200e-4 1.41	3.462e-4 1.41	1.306e-4 1.40	4.931e-5 1.40	1.864e-5
$\delta = 0.7$	1.388e-2 1.31	5.593e-3 1.31	2.258e-3 1.31	9.128e-4 1.30	3.696e-4 1.30	1.498e-4 1.30	6.077e-5
$\delta = 0.8$	3.057e-2 1.20	1.332e-2 1.20	5.791e-3 1.20	2.518e-3 1.20	1.095e-3 1.20	4.762e-4 1.20	2.072e-4
$\delta = 0.9$	8.118e-2 1.07	3.861e-2 1.09	1.818e-2 1.09	8.521e-3 1.10	3.984e-3 1.10	1.860e-3 1.10	8.683e-4

Example 5.2 Consider (1.5) with

$$b(x) = \begin{cases} x + 0.2 & \text{for } 0 \le x \le 0.4, \\ 1.4 - 2x & \text{for } 0.4 < x \le 1, \end{cases}$$

$c(x) \equiv 0$ and the same solution $u(x)$ as in (5.1). Then f is chosen to satisfy (1.5a). The values of $\alpha_0, \alpha_1, \gamma_0$ and γ_1 are the same as in Example 5.1.

The numerical results for Example 5.2 resemble those for Example 5.1 so we present only two tables of them (Tables 5.9 and 5.10). The lack of smoothness in b does not cause any deterioration in the accuracy of the method because u remains as smooth as in Example 5.1 and this is the key requirement for the accuracy predicted by Theorem 4.2.

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Table 5.9 Example 5.2, $\max_i |(u-u_h)(x_i)|$ for $m=1,\ c_k=\{0\};\ J\neq 0$

	N=128	N=256	N=512	N=1024	N=2048	N=4096	N=8192
$\delta=0.1$	4.745e-2 1.00	2.372e-2 1.00	1.186e-2 1.00	5.928e-3 1.00	2.964e-3 1.00	1.482e-3 1.00	7.409e-4
$\delta=0.2$	5.043e-2 1.00	2.518e-2 1.00	1.258e-2 1.00	6.288e-3 1.00	3.143e-3 1.00	1.571e-3 1.00	7.856e-4
$\delta=0.3$	5.588e-2 1.00	2.786e-2 1.00	1.390e-2 1.00	6.941e-3 1.00	3.468e-3 1.00	1.733e-3 1.00	8.661e-4
$\delta=0.4$	6.496e-2 1.01	3.228e-2 1.01	1.607e-2 1.00	8.008e-3 1.00	3.996e-3 1.00	1.995e-3 1.00	9.965e-4
$\delta=0.5$	8.015e-2 1.02	3.961e-2 1.01	1.963e-2 1.01	9.754e-3 1.01	4.854e-3 1.00	2.419e-3 1.00	1.206e-3
$\delta=0.6$	1.070e-1 1.03	5.252e-2 1.02	2.587e-2 1.02	1.278e-2 1.01	6.332e-3 1.01	3.143e-3 1.01	1.563e-3
$\delta=0.7$	1.596e-1 1.04	7.784e-2 1.03	3.808e-2 1.03	1.869e-2 1.02	9.197e-3 1.02	4.539e-3 1.02	2.245e-3
$\delta=0.8$	2.825e-1 1.04	1.376e-1 1.04	6.709e-2 1.03	3.277e-2 1.03	1.604e-2 1.03	7.868e-3 1.02	3.867e-3
$\delta=0.9$	7.015e-1 1.02	3.453e-1 1.03	1.696e-1 1.03	8.320e-2 1.03	4.084e-2 1.03	2.006e-2 1.02	9.858e-3

Table 5.10 Example 5.2, $\max_i |(u-u_h)(x_i)|$ for $m=3,\ c_k=\{0,1/2,1\};\ J=0$

	N=64	N=128	N=256	N=512	N=1024	N=2048
$\delta=0.1$	9.370e-9 3.79	6.771e-10 3.81	4.838e-11 3.82	3.428e-12 3.84	2.394e-13 4.01	1.488e-14
$\delta=0.2$	6.281e-8 3.69	4.875e-9 3.71	3.722e-10 3.73	2.808e-11 3.74	2.101e-12 3.72	1.594e-13
$\delta=0.3$	2.631e-7 3.59	2.192e-8 3.62	1.788e-9 3.64	1.437e-10 3.65	1.143e-11 3.66	9.035e-13
$\delta=0.4$	9.088e-7 3.48	8.166e-8 3.52	7.137e-9 3.54	6.130e-10 3.56	5.206-11 3.57	4.386e-12
$\delta=0.5$	2.836e-6 3.36	2.767e-7 3.41	2.606e-8 3.44	2.403e-9 3.46	2.185e-10 3.47	1.970e-11
$\delta=0.6$	8.383e-6 3.23	8.964e-7 3.29	9.160e-8 3.33	9.112e-9 3.35	8.917e-10 3.37	8.634e-11
$\delta=0.7$	2.446e-5 3.08	2.902e-6 3.16	3.244e-7 3.21	3.503e-8 3.24	3.706e-9 3.26	3.871e-10
$\delta=0.8$	7.541e-5 2.89	1.016e-5 3.01	1.257e-6 3.08	1.484e-7 3.12	1.705e-8 3.15	1.928e-9
$\delta=0.9$	2.903e-4 2.58	4.864e-5 2.80	6.976e-6 2.92	9.199e-7 2.99	1.158e-7 3.03	1.420e-8

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