A POSTERIORI ERROR ESTIMATION FOR PARABOLIC PROBLEMS USING ELLIPTIC RECONSTRUCTIONS. I: BACKWARD-EULER AND CRANK-NICOLSON METHODS*
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Abstract. A semilinear second-order parabolic equation is considered in a regular and a singularly-perturbed regime. For this equation, we give computable a posteriori error estimates in the maximum norm. Fully discrete Bakward-Euler and Crank-Nicolson methods are addressed, for which we employ elliptic reconstructions that are, respectively, piecewise-constant and piecewise-linear in time. We also use certain bounds for the Green's function of the parabolic operator.

Key words. a posteriori error estimate, maximum norm, singular perturbation, elliptic reconstruction, Backward-Euler, Crank-Nicolson, parabolic equations, reaction-diffusion.

AMS subject classifications. 65M15 , 65M60.

1. Introduction. Consider a semilinear parabolic equation in the form
\[
\mathcal{M} u := \partial_t u + \mathcal{L} u + f(x, t, u) = 0 \quad \text{for } (x, t) \in Q := \Omega \times (0, T],
\]
with a second-order linear elliptic operator \( \mathcal{L} = \mathcal{L}(t) \) in a spatial domain \( \Omega \subset \mathbb{R}^n \) with Lipschitz boundary, subject to the initial and Dirichlet boundary conditions
\[
u(x, 0) = \varphi(x) \quad \text{for } x \in \bar{\Omega}, \quad u(x, t) = 0 \quad \text{for } (x, t) \in \partial \Omega \times [0, T].
\]
We assume that \( f \) is differentiable in the third argument and, for some positive constants \( \gamma \) and \( \bar{\gamma} \), satisfies
\[
0 \leq \gamma \leq \partial_z f(x, t, z) \leq \bar{\gamma} \quad \text{for } (x, t, z) \in \bar{\Omega} \times [0, T] \times \mathbb{R}.
\]

The purpose of this paper is to obtain computable a posteriori error estimates for fully discrete methods applied to problem (1.1). We consider the first-order Backward-Euler and the second-order Crank-Nicolson discretizations in time. Furthermore, in a forthcoming paper [9], a similar approach will be used to analyze the third-order Discontinuous Galerkin method dG(1).

These results are applied to the model equation with \( \mathcal{L} := -\varepsilon^2 \Delta = -\varepsilon^2 \sum_{i=1}^n \partial_{x_i}^2 \):
\[
\mathcal{M} u := \partial_t u - \varepsilon^2 \Delta u + f(x, t, u) = 0
\]
posed in a bounded polyhedral spatial domain \( \Omega \subset \mathbb{R}^n \), with \( n = 1, 2, 3 \). This equation will be considered in the two regimes:

(i) \( \varepsilon = 1, \ \gamma \geq 0 \); \hspace{1cm} (ii) \( \varepsilon \ll 1, \ \gamma > 0 \).

Note that regime (ii) yields a singularly perturbed reaction-diffusion equation, whose solutions may exhibit sharp layer phenomena. So it is important in this regime that a posteriori error estimates are robust in the sense that any dependence on the small perturbation parameter \( \varepsilon \) should be shown explicitly [13, 16].

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We will give error estimates in the maximum norm, which is sufficiently strong to capture sharp layers and singularities that may occur, in particular, if problem (1.1) is of singularly-perturbed type. Our estimates will be of interpolation type in the sense that they will include certain terms that may be interpreted as approximating $\tau_n^p |\partial_t^p u|$, where $p$ and $\tau_n$ are the discretization order and local step size in time, respectively.

We employ the elliptic reconstruction technique, which was introduced in the recent papers [14, 11, 3] as a counterpart of the Ritz-projection in the a posteriori error estimation for parabolic problems. We also use certain bounds for the Green's function of the continuous parabolic operator in a manner similar to [3], only for a more general semilinear parabolic operator of (1.3) (compared to $\partial_t - \Delta$ in [3]).

One distinctive feature of our analysis in this paper (compared, e.g., to [1, 3]) is that we use computed solutions and elliptic reconstructions that are piecewise-constant in time when dealing with the Backward-Euler method, and piecewise-linear in time when dealing with the Crank-Nicolson method. Consequently, we allow the residuals of computed solutions, as well as other functions, to be understood as distributions; this inclusion plays a crucial role in our analysis.

The paper is organized as follows. In Section 2, we introduce the Green's function and obtain a certain stability lemma, which is then used in Sections 3–4 to obtain a posteriori error estimates for semidiscrete Backward-Euler and Crank-Nicolson methods (with no spatial discretization). Next, in Section 5 we cite some elliptic a posteriori error estimates, which are used in Sections 6–7 to derive a posteriori error estimates for fully discrete Backward-Euler and Crank-Nicolson methods. The final Section 8 gives a proof of certain Green's function bounds deferred from Section 2.

Notation. Throughout the paper, $C$, as well as $c$, denotes a generic positive constant that may take different values in different formulas, but is independent of the diffusion coefficient $\varepsilon$ and any mesh sizes. We use $|x|$ for the Euclidian norm of $x \in \mathbb{R}^n$. The usual spaces $C(\bar{\Omega})$ and $H^1_0(\Omega)$ are used, as well as the spaces $L_p$, $1 \leq p \leq \infty$, with the norm $\| \cdot \|_{p, \Omega}$, while $\langle \phi, \psi \rangle = \int_{\bar{\Omega}} \phi(x) \psi(x) \, dx$ denotes the inner product in $L_2(\Omega)$.

Distributions and left-continuity convention. Certain functions will be understood as distributions [7], which will in most cases be indicated. By contrast, if a certain function is Lebesgue-integrable in $\Omega \times (0, T)$, we shall refer to it as a regular function. Whenever we deal with a regular function, it will be understood right-continuous at $t = 0$ and left-continuous for all $t \in (0, T]$. In particular, this convention will be applied to all piecewise-continuous temporal derivatives.

2. The Green's function of the parabolic operator. In this section we consider the Green's function $\mathcal{G}$ associated with the operator $\mathcal{M}$ of (1.1). Our interest in the Green's function is in that it will be used to express the error of a numerical approximation in terms of its residual.

For definitions and properties of fundamental solutions and Green's functions of parabolic operators with variable coefficients, we refer the reader to [6, Chap. 1 and §7 of Chap. 3]. In particular, for fixed $(x, t) \in Q$, the Green's function $\mathcal{G}(x; t; \xi, s) := \Gamma(\xi, s)$ solves the adjoint terminal-value problem

$$
[-\partial_t - \mathcal{L}^* + a(\xi, s)] \Gamma(\xi, s) = 0 \quad \text{for } (\xi, s) \in \Omega \times [0, t),
$$

$$
\Gamma(\xi, t) = \delta(\xi - x) \quad \text{for } \xi \in \Omega,
$$

$$
\Gamma(\xi, s) = 0 \quad \text{for } (\xi, s) \in \partial \Omega \times [0, t].
$$

Here $\delta(\cdot)$ is the Dirac $\delta$-distribution in $\mathbb{R}^n$ [7], and $\mathcal{L}^*$ is the adjoint operator to the linear operator $\mathcal{L}$. We set $a(x, t) := \partial_z f(x, t, z)$ if $f$ is linear in the third argument, i.e.
$f(x, t, z) = a(x, t)z + b(x, t)$. If $f$ is nonlinear, we associate $a(x, t) := \int_0^1 \partial_x f(x, t, w + z)\, dz$ with a pair of bounded functions $v$ and $w$ that vanish on $\partial \Omega$. As the standard linearization now yields $\mathcal{M}v - \mathcal{M}w = [\partial_t + L + a(x, t)](v - w)$, so with the help of this Green’s function, the difference $v - w$ is represented as

$$[v - w](x, t) = \int_\Omega G(x, t; \xi, 0) [v - w](\xi, 0) \, d\xi$$

$$+ \int_0^t \int_\Omega G(x, t; \xi, s) [\mathcal{M}v - \mathcal{M}w](\xi, s) \, d\xi \, ds.$$  \hfill (2.2)

The analysis in this paper will be carried out under the following condition.

**Condition 2.1.** There are constants $\kappa_0, \kappa_1 > 0$ and $\kappa_2 \geq 0$ such that the Green’s function $G$ of (2.1), (1.2) satisfies

$$\|G(x, t; \cdot, s)\|_{1, \Omega} \leq \kappa_0 e^{-\gamma(t - s)} \int_0^{t - \tau} \|\partial_x G(x, t; \cdot, s)\|_{1, \Omega} \, ds \leq \kappa_1 \ell(t, t) + \kappa_2,$$

where $x \in \Omega$, $\tau \in (0, t]$, $t \in (0, T]$, and $\ell(t, t) := \int_t^t s^{-\gamma^2} e^{-\frac{\gamma^2}{2} s} \, ds \leq \ln(t/\tau)$.

Note that our model problem satisfies this condition as follows.

**Remark 2.3.** The constants $\kappa_0$ and $\kappa_1$ given by Lemma 2.2 are reasonably sharp. E.g., for the constant-coefficient version $\partial_t u - \varepsilon^2 \partial_x^2 u + \gamma^2 u = b(x, t)$ of (1.3) in the spatial domain $\Omega := \mathbb{R}$, a calculation yields $\|G(x, t; \cdot, s)\|_{1, \Omega} = 1$ and $\|\partial_x G(x, t; \cdot, s)\|_{1, \Omega} \leq (\sqrt{\frac{2}{\pi \varepsilon}}(t - s)^{-1} + \gamma^2) e^{-\gamma^2(t - s)}$ so Condition 2.1 is satisfied with $\kappa_0 = 1$ (as in Lemma 2.2), $\kappa_1 = \sqrt{\frac{2}{\pi \varepsilon}} \approx 0.48$, $\kappa_2 = 1$, while Lemma 2.2 gives $\kappa_1 = \frac{3}{2\gamma^2} \approx 1.06$.

The above Condition 2.1 will be employed by means of the following lemma, which plays a crucial role in our analysis here and in the forthcoming paper [9]. The lemma is formulated in the context of an arbitrary nonuniform mesh in the time direction

$$0 = t_0 < t_1 < t_2 < \ldots < t_M = T, \quad \text{with } \tau_j = t_j - t_{j-1} \text{ for } j = 1, \ldots, M. \hfill (2.3)$$

**Lemma 2.4.** Suppose the parabolic operator $\mathcal{M}$ of (1.1) satisfies (1.2) and Condition 2.1, and $v, w$ are bounded in $\bar{\Omega} \times [0, T]$. Furthermore, let $v(\cdot, t), w(\cdot, t) \in H_0^1(\Omega) \cap C(\bar{\Omega})$ for $t \in [0, T]$, and

$$\mathcal{M}v - \mathcal{M}w = \partial_t \mu + \vartheta \quad \text{in } Q,$$

where the function $\mu$ is continuous and bounded on $[t_0, t_1]$ and each $(t_j, t_{j+1})$, while $\partial_t \mu$ is continuous and bounded on $(t_{m-1}, t_m)$ for some $1 < m < M$, and $\|\vartheta(\cdot, s)\|_{\infty, \Omega}$ is integrable on $(0, t_m)$ (possibly, in the sense of distributions). Then

$$\|v - \vartheta(\cdot, t_m)\|_{\infty, \Omega} \leq \kappa_0 e^{-\gamma^2 t_m} \|v - \vartheta(\cdot, 0)\|_{\infty, \Omega} + (\kappa_1 \epsilon_1 + \kappa_2) \sup_{s \in [0, t_{m-1}]} \|\mu(\cdot, s)\|_{\infty, \Omega}$$

$$+ \kappa_0 \sup_{s \in [0, t_{m-1}]} \|\partial_t \mu(\cdot, s)\|_{\infty, \Omega} + \kappa_0 \sup_{s \in [0, t_m]} \|\partial_x \mu(\cdot, s)\|_{\infty, \Omega}$$

$$+ \kappa_0 \sup_{s \in [0, t_m]} \|\partial_t \mu(\cdot, s)\|_{\infty, \Omega} ds,$$
we get the desired result.

Remark 2.5. The term $\partial_t \mu$ in the right-hand side of (2.4) is understood in the sense of distributions.

Remark 2.6. One can easily check that if $\gamma = 0$, then $\ell_m = \ln(t_m/\tau_m)$. Otherwise, if $\gamma > 0$, one has $\ell_m(\gamma) = E_1(\frac{1}{2} \gamma^2 \tau_m) - E_1(\frac{1}{2} \gamma^2 t_m)$, where $E_1(t) = \int_t^{\infty} s^{-1} e^{-s} ds$; so $\ell_m(\gamma) \leq |\ln(\frac{1}{2} \gamma^2 \tau_m)|$ provided that $\frac{1}{2} \gamma^2 \tau_m \leq 0.67$ (this is easily checked by finding the only root $\approx 0.67$ of the equation $E_1(s) = |\ln s|$ on $(0,1)$). Note also that $\ell_1 = 0$ for any $\gamma \geq 0$.

Proof. Combining representation (2.2) with the notation $\Gamma(\xi,s) := G(x,t_m;\xi,s)$ for the Green’s function of (2.1), one gets

$$[v - w](x,t_m) = \langle [v - w](\cdot,0), \Gamma(\cdot,0) \rangle + \int_0^{t_m} \langle [\mathcal{A} v - \mathcal{A} w](\cdot,s), \Gamma(\cdot,s) \rangle ds.$$ 

Here, in view of (2.4), the integral on the right-hand side involves $\mu$ and $\vartheta$, so can be represented as a sum $J_\mu + J_\vartheta$ of the corresponding integrals, which we consider separately. We use the notation $\int^{b+} := \lim_{b \to -0+} \int^{b+\beta}$ and so split $J_\mu$ as

$$J_\mu = J^{(1)}_\mu + J^{(2)}_\mu := \int_0^{t_m-1} \langle \partial_s \mu, \Gamma(\cdot,s) \rangle ds + \int_{t_m-1}^{t_m} \langle \partial_s \mu, \Gamma(\cdot,s) \rangle ds.$$ 

Here, for $J^{(1)}_\mu$, an integration by parts yields

$$J^{(1)}_\mu = \langle \mu(\cdot,t_m-1), \Gamma(\cdot,t_m-1) \rangle - \langle \mu(\cdot,0), \Gamma(\cdot,0) \rangle - \int_0^{t_m-1} \langle \mu(\cdot,s), \partial_t \Gamma(\cdot,s) \rangle ds.$$ 

Consequently, we arrive at

$$[v - w](x,t_m) = \langle [v - w - \mu](\cdot,0), \Gamma(\cdot,0) \rangle - \int_0^{t_m-1} \langle \mu(\cdot,s), \partial_s \Gamma(\cdot,s) \rangle ds$$

$$+ \langle \mu(\cdot,t_m-1), \Gamma(\cdot,t_m-1) \rangle + \int_{t_m-1}^{t_m} \langle \partial_s \mu, \Gamma(\cdot,s) \rangle ds$$

$$+ \int_0^{t_m} \langle \vartheta(\cdot,s), \Gamma(\cdot,s) \rangle ds,$$

where the last term represents $J_\vartheta$. Finally, Condition 2.1 implies that

$$\|\Gamma(\cdot,s)\|_{1,\Omega} \leq \kappa_0 e^{-\gamma(t_m-s)} \leq \kappa_0, \quad \int_0^{t_m-1} \|\partial_s \Gamma(\cdot,s)\|_{1,\Omega} ds \leq \kappa_1 \ell_m + \kappa_2,$$

so we get the desired result. $\Box$


Consider an arbitrary nonuniform mesh (2.3) in the time direction and discretize the abstract parabolic problem (1.1) in time using the first-order Backward Euler method (also referred to as the implicit Euler method or the Discontinuous Galerkin method $dG(0)$) as follows. We associate an approximate solution $U^j \in H^1_0(\Omega) \cap C(\Omega)$ with the time level $t_j$ and require it to satisfy

$$\delta_t U^j + \mathcal{L}^j U^j + f^j = 0 \quad \text{in } \Omega, \quad j = 1, \ldots, M; \quad U^0 = \varphi,$$  

(3.1a)
where

\[
\delta t U^j := \frac{U^j - U^{j-1}}{\tau_j}, \quad \mathcal{L}^j := \mathcal{L}(t_j) \quad \text{and} \quad f^j := f(\cdot, t_j, U^j). \quad (3.1b)
\]

For this discretization, we give the following a posteriori error estimate.

**Theorem 3.1.** Let \( u \) solve the problem (1.1) with the parabolic operator \( \mathcal{M} \) satisfying (1.2) and Condition 2.1, and \( U^j \) solve the corresponding semidiscrete problem (3.1). Then, for \( m = 1, \ldots, M \), one has

\[
\left\| U^m - u(\cdot, t_m) \right\|_{\infty, \Omega} \leq (\kappa_1 t_m + \kappa_2) \max_{j=1,\ldots,m-1} \left\| U^j - U^{j-1} \right\|_{\infty, \Omega} + 2\kappa_0 \left\| U^m - U^{m-1} \right\|_{\infty, \Omega} + \kappa_0 \sum_{j=1}^m \int_{t_{j-1}}^{t_j} e^{-\tau_j(t - s)} \left\| [\partial_\mathcal{L} + \partial f](\cdot, s) \right\|_{\infty, \Omega} ds,
\]

where \( \partial_\mathcal{L} \) and \( \partial f \) are regular functions defined, for \( t \in (t_{j-1}, t_j) \), \( j = 1, \ldots, M \), by

\[
\partial_\mathcal{L}(\cdot, t) := [\mathcal{L}(t) - \mathcal{L}^j] U^j, \quad \partial f(\cdot, t) := f(\cdot, t, U^j) - f(\cdot, t_j, U^j). \quad (3.3)
\]

**Proof.** Let \( I_t U \) be the standard piecewise-linear interpolant of \( U^j \) in time:

\[
I_t U(\cdot, t) := \frac{t - t_j}{\tau_j} U^{j-1} + \frac{t - t_{j-1}}{\tau_j} U^j \quad \text{for} \quad t \in [t_{j-1}, t_j], \quad j = 1, \ldots, M. \quad (3.4)
\]

Furthermore, we define a **piecewise-constant** interpolant \( \tilde{U} \) of \( U^j \) by

\[
\tilde{U}(\cdot, t) := U^j \quad \text{for} \quad t \in (t_{j-1}, t_j), \quad j = 1, \ldots, M; \quad \tilde{U}(\cdot, 0) := U^1, \quad (3.5)
\]

(so \( \tilde{U} \) is continuous on \([t_0, t_1]\)). Note that the temporal derivative \( \partial_t \tilde{U} \) is understood as a distribution, while \( \partial_t(I_t U) \) is a regular function, equal to \( \delta t U^j \) for \( t \in (t_{j-1}, t_j) \) (in agreement with our left-continuity convention). Consequently, (3.1a) implies that

\[
\partial_t(I_t U) + \mathcal{L}(t) \tilde{U} + f(x, t, \tilde{U}) = \partial_\mathcal{L} + \partial f \quad \text{for} \quad (x, t) \in \Omega \times (0, T). \quad (3.6)
\]

Here we also used the observation that, by (3.5), the regular functions \( \partial_\mathcal{L} \) and \( \partial f \) of (3.3) can be rewritten for \( t \in (t_{j-1}, t_j) \) as \( \partial_\mathcal{L} = \mathcal{L}(t) \tilde{U} - \mathcal{L}^j U^j \) and \( \partial f = f(\cdot, t, \tilde{U}) - f(\cdot, t_j, U^j) \).

Next, combining (3.6) with (1.1a) yields

\[
\mathcal{M} \tilde{U} - \mathcal{M} u = \partial_t \tilde{U} + \mathcal{L}(t) \tilde{U} + f(x, t, \tilde{U}) = \partial_t \tilde{U} - I_t U + [\partial_\mathcal{L} + \partial f] \quad \text{in} \quad Q.
\]

Now the desired bound for \( U^m - u(\cdot, t_m) = [\tilde{U} - u](\cdot, t_m) \) is obtained by an application of Lemma 2.4 with \( \mu := \tilde{U} - I_t U \) and \( \vartheta := \partial_\mathcal{L} + \partial f \), using the following two observations. First, we note that \([\tilde{U} - u - \mu](\cdot, 0) = U^1 - \vartheta - (U^1 - \vartheta) = 0 \). Second, for \( t \in (t_{j-1}, t_j) \), one has

\[
\mu = \frac{t_j - t}{\tau_j} (U^j - U^{j-1}) \implies |\mu| \leq |U^j - U^{j-1}|, \quad \tau_j |\partial_t \mu| = |U^j - U^{j-1}|.
\]

This completes the proof. \( \Box \)

**Corollary 3.2.** Under assumption (1.2), the a posteriori error estimate (3.2) applies to the model problem (1.3) with \( \vartheta \) from (3.3), \( \vartheta = 0 \), and the constants \( \kappa_0, \kappa_1, \kappa_2 \) from Lemma 2.2.
Consider an arbitrary nonuniform mesh (2.3) in the time direction and discretize the abstract parabolic problem (1.1) in time using the second-order Crank-Nicolson method as follows. We associate an approximate solution \( U^j \in H_0^1(\Omega) \cap C(\Omega) \) with the time level \( t_j \) and require it to satisfy
\[
\delta_\tau U^j + \frac{1}{2}(\mathcal{L}^{-1}U^{j-1} + \mathcal{L}'U^j) + \frac{1}{2}(f^{j-1} + f^j) = 0 \quad \text{in } \Omega, \quad j = 1, \ldots, M, \tag{4.1a}
\]
where we again let
\[
U^0 = \varphi, \quad \delta_\tau U^j := \frac{U^j - U^{j-1}}{\tau_j}, \quad \mathcal{L}^j := \mathcal{L}(t_j) \quad \text{and} \quad f^j := f(\cdot, t_j, U^j). \tag{4.1b}
\]
To give an a posteriori error estimate for this discretization, we will use the standard piecewise linear interpolation \( I_t \), which, for any continuous function \( w = w(t) \), is defined by
\[
I_tw(t) := \frac{1}{\tau_j} w(t_{j-1}) + \frac{t-t_{j-1}}{\tau_j} w(t_j) \quad \text{for } t \in [t_{j-1}, t_j], \quad j = 1, \ldots, M, \tag{4.2}
\]
(this definition is almost identical with (3.4)).

**Theorem 4.1.** Let \( u \) solve the problem (1.1) with the parabolic operator \( \mathcal{M} \) satisfying (1.2) and Condition 2.1, and \( U^j \) solve the corresponding semidiscrete problem (4.1). Then for \( m = 1, \ldots, M \), one has
\[
\|U^m - u(\cdot, t_m)\|_{\infty, \Omega} \leq \frac{1}{8} (\kappa_1 \ell_m + \kappa_2) \max_{j=1, \ldots, m-1} \tau_j^2 \|\delta_\tau (\mathcal{L}'U^j + f^j)\|_{\infty, \Omega} + \frac{1}{8} \kappa_0 \tau_m^2 \|\delta_\tau (\mathcal{L}^m U^m + f^m)\|_{\infty, \Omega} + \kappa_0 \sum_{j=1}^m \int_{t_{j-1}}^{t_j} e^{-\gamma(t_m-s)} \|\partial_{\mathcal{L}} + \partial_f(\cdot, s)\|_{\infty, \Omega} ds. \tag{4.3}
\]
Here \( \partial_{\mathcal{L}} \) and \( \partial_f \) are regular functions defined, for \( t \in (t_{j-1}, t_j) \), \( j = 1, \ldots, M \), by
\[
\partial_{\mathcal{L}} := \mathcal{L}(t)U - I_t[\mathcal{L}(t)U], \quad \partial_f := f(\cdot, t, U) - I_t[f(\cdot, t, U)], \tag{4.4}
\]
where we use \( U(\cdot, t) := I_tU(\cdot, t) \) of (3.4) and \( I_t \) of (4.2).

**Remark 4.2.** As the exact solution \( u \) of (1.1) satisfies \( \partial_t^2 u = -\partial_t (\mathcal{L}u + f(x, t, u)) \), so the terms \( \tau_j^2 \|\delta_\tau (\mathcal{L}'U^j + f^j)\|_{\infty, \Omega} \) in (4.3) can be considered discrete analogues of \( \tau_j^2 \|\partial_t^2 u\|_{\infty, \Omega} \). We also note that these terms can be bounded using \( \tau_j^2 \|\partial_t w^j\|_{\infty, \Omega} \leq \tau_j (\|w^{j-1}\|_{\infty, \Omega} + \|w^j\|_{\infty, \Omega}) \) with \( w^j := \mathcal{L}'U^j + f^j \).

**Proof.** Consider \( t \in [t_{j-1}, t_j] \). In view of (4.2), for any function \( w = w(t) \) with the notation \( w^j := w(t_j) \), one has \( I_tw(t) = \frac{1}{2}(w^{j-1} + w^j) + (t - t_{j-1/2}) \delta_t w^j \). So using (4.4) and then this property, for \( t \in [t_{j-1}, t_j] \), one easily gets
\[
f(\cdot, t, U) - \frac{1}{2}(f^{j-1} + f^j) = \{I_t[f(\cdot, t, U)] - \frac{1}{2}(f^{j-1} + f^j)\} + \partial_f
\]
\[
= (t - t_{j-1/2}) \delta_t f^j + \partial_f,
\]
and a similar relation
\[
\mathcal{L}(t)U - \frac{1}{2}(\mathcal{L}'U^{j-1} + \mathcal{L}'U^j) = \{I_t[\mathcal{L}(t)U] - \frac{1}{2}(\mathcal{L}'U^{j-1} + \mathcal{L}'U^j)\} + \partial_{\mathcal{L}}
\]
\[
= (t - t_{j-1/2}) \delta_t [\mathcal{L}'U^j] + \partial_{\mathcal{L}}.
\]
Note also that $U(\cdot, t) = I_t U(\cdot, t)$ implies that $\delta_t U^j = \partial_t U$ for $t \in (t_{j-1}, t_j]$.

Combining these three observations with (4.1a), one deduces that

$$\partial_t U + \mathcal{L}(t) U + f(\cdot, t, U) = (t - t_{j-1}/2) \delta_t \mathcal{L} U^j + f^j + [\partial_L + \partial_f],$$  \tag{4.5}$$

for $t \in (t_{j-1}, t_j]$ (here the left-hand side is a regular function). Next, combining (4.5) with (1.1a) yields

$$\mathcal{M} u - \mathcal{M} \mu = \partial_t U + \mathcal{L}(t) U + f(x, t, U) = \partial_t \mu + [\partial_L + \partial_f] \text{ in } Q,$$  \tag{4.6}$$

where $\mu = \mu(x, t)$ is a continuous function defined by

$$\mu(\cdot, t) := -\frac{1}{\tau} (t_j - t)(t_j - t_{j-1}) \cdot \delta_t \mathcal{L} U^j + f^j \quad \text{for } t \in [t_{j-1}, t_j].$$  \tag{4.7}$$

This is easily checked by using the relation $\frac{d}{dt}[-\frac{1}{\tau}(t_j - t)(t_j - t_{j-1})] = t_j - t_{j-1}/2$ to evaluate $\partial_t \mu$.

Now the desired bound (4.3) for $U^m - u(\cdot, t_m)$ is obtained by an application of Lemma 2.4 to the equation (4.6) with $\mu$ defined by (4.7), and $\vartheta := \partial_L + \partial_f$, using the following two observations. First, note that $|U - u - \mu(\cdot, 0)| = U^1 - \varphi - 0 = 0$. Second, for $t \in (t_{j-1}, t_j)$, one has

$$|\mu| \leq \frac{1}{\tau} \left| \sigma_j \mathcal{L} U^j + f^j \right| \quad \text{and} \quad \tau_j |\partial_t \mu| \leq \frac{\kappa}{2} \left| \sigma_j (\mathcal{L} U^j + f^j) \right|.$$  

This completes the proof. \[\Box\]

**Corollary 4.3.** Under assumption (1.2), the a posteriori error estimate (4.3) applies to the model problem (1.3) with $\vartheta_f$ from (4.4), $\vartheta_L = 0$, and the constants $\kappa_0, \kappa_1, \kappa_2$ from Lemma 2.2.

**Remark 4.4.** If the term $\frac{1}{2}(f^{j-1} + f^j)$ in the Crank-Nicolson discretization (4.1) is replaced by $f^j_h := \tau^{-1}_j \int_{t_{j-1}}^{t_j} f(\cdot, t, U)\, dt$, then the proof of Theorem 4.1 remains applicable with the only modification that the right-hand side in (4.6) involves, for $t \in (t_{j-1}, t_j)$, an additional term $-\vartheta_f := \frac{1}{2}(f^{j-1} + f^j) - f^j_h$. Consequently, the statement of Theorem 4.1 remains valid with $[\vartheta_L + \vartheta_f]$ in the final line of (4.3) replaced by $[\partial_L + \partial_f - \vartheta_f]$, where, as one can easily deduce, $\vartheta_f = \tau^{-1}_j \int_{t_{j-1}}^{t_j} \vartheta_f\, dt$ is the average value of $\vartheta_f$ on $[t_{j-1}, t_j]$.

**Remark 4.5.** The a posteriori error estimates given by Theorem 4.1 and Remark 4.4 resemble (but are not identical with) error estimates of [1]. Our analysis of the semidiscrete Crank-Nicolson method seems more straightforward as we work with the standard piecewise linear interpolant of the computed solution, while the analysis in [1] involves a construction of a certain piecewise-quadratic polynomial of the computed solution in time. Furthermore, in Section 7, we derive a posteriori error estimates for fully discrete Crank-Nicolson methods, which were not considered in [1].

**5. Elliptic a posteriori error estimators.** In this section, we consider a steady-state version of the abstract parabolic problem (1.1):

$$\mathcal{L} v + g(x, v) = 0 \quad \text{for } x \in \Omega, \quad v = 0 \quad \text{for } x \in \partial \Omega,$$  \tag{5.1}$$

and its discretizations in the form

Find $v_h \in \hat{V}_h : \quad \mathcal{L}_h v_h + P_h [g(\cdot, v_h)] = 0, \quad \text{where } \hat{V}_h := V_h \cap H^1_0(\Omega).$  \tag{5.2a}$$
Here \( V_h \subset C(\hat{\Omega}) \) is some finite-element space, and for the related Lagrange interpolation operator \( I_h \) onto \( V_h \), we use some linear operators \( \mathcal{L}_h \) and \( \mathcal{P}_h \) such that

\[
\mathcal{L}_h : H^1_0(\Omega) \to \hat{V}_h = I_h[g(\cdot,0)],
\mathcal{P}_h v \in \hat{V}_h + I_h v \quad \forall v \in C(\hat{\Omega}), \quad \mathcal{P}_h v_h = v_h \quad \forall v_h \in V_h.
\]  

(5.2h)

Note that as any \( v_h \in \hat{V}_h \) vanishes on \( \partial \Omega \), so \( \hat{V}_h = I_h[g(\cdot,0)] = I_h[g(\cdot,v_h)] \), so the definition (5.2) is consistent.

**Assumptions.** We assume, for any admissible \( g \), that

(i) there exist unique solutions \( v \) and \( v_h \) of problems (5.1) and (5.2), respectively;

(ii) an a posteriori error estimate is available for these solutions in the form

\[
\|v - v_h\|_{\infty,\Omega} \leq \eta(V_h,v_h,g(\cdot,v_h)).
\]  

(5.3)

Note that the availability of elliptic a posteriori error estimates, such as (5.3), enables one to employ elliptic reconstructions in the a posteriori error estimation of the related parabolic problems. Moreover, \( \mathcal{L}_h \) and \( \mathcal{P}_h \) are not necessarily needed to be evaluated explicitly to compute the a posteriori estimator either for the elliptic problem or the parabolic problem.

**5.1. Elliptic model problem.** Many standard finite element discretizations of elliptic equations (including those with quadrature) allow a representation of type (5.2). For example, consider a steady-state elliptic version of our model problem (1.3) posed in a bounded polyhedral domain \( \Omega \subset \mathbb{R}^n \):

\[
-\varepsilon^2 \Delta v + g(x,v) = 0 \quad \text{for} \ x \in \Omega, \quad v = 0 \quad \text{for} \ x \in \partial \Omega, \quad \partial_z g(x,z) \geq \gamma^2 > 0.
\]  

(5.4)

With a finite-element space \( V_h \subset C(\Omega) \) and \( \hat{V}_h := V_h \cap H^1_0(\Omega) \), a standard Galerkin finite element method for this problem can be described by

Find \( v_h \in \hat{V}_h : \quad \varepsilon^2 \langle \nabla v_h, \nabla \chi \rangle + \langle g(\cdot,v_h), \chi \rangle_h = 0 \quad \forall \chi \in \hat{V}_h.
\]  

(5.5)

where \( \langle \cdot, \cdot \rangle_h \) is either exactly the inner product \( \langle \cdot, \cdot \rangle \) in \( L^2(\Omega) \), or some quadrature formula for \( \langle \cdot, \cdot \rangle \).

**Remark 5.1.** The discretization (5.5) is of type (5.2). Suppose, for example, that \( \langle \psi, \chi \rangle_h = \langle \psi, \chi \rangle \) for all \( \psi, \chi \in V_h \). Then \( \langle \mathcal{L}_h \varphi, \chi \rangle = \varepsilon^2 \langle \nabla \varphi, \nabla \chi \rangle \) and \( \langle \mathcal{P}_h \psi, \chi \rangle = \langle \psi, \chi \rangle_h \), subject to (5.2b), for all \( \varphi \in H^1_0(\Omega) \), \( \psi \in C(\hat{\Omega}) \) and \( \chi \in \hat{V}_h \). In particular,

(i) if \( \langle \cdot, \cdot \rangle_h := \langle \cdot, \cdot \rangle \) (i.e. no quadrature is used), then \( \mathcal{P}_h \) is the \( L^2 \) projection;

(ii) if a quadrature of type \( \langle \psi, \chi \rangle_h := \langle I_h \psi, \chi \rangle \) is used, where \( I_h \) is the Lagrange interpolation operator associated with \( V_h \), then \( \mathcal{P}_h := I_h \).

**Remark 5.2.** Suppose that one employs a quadrature of lumped-mass type defined by \( \langle \psi, \chi \rangle_h = \langle I_h(\psi) \chi, 1 \rangle = \langle \psi \chi, 1 \rangle \) for all basis functions \( \chi_i \) of \( V_h \), where \( \psi \in C(\Omega) \) and \( \sum \psi_i \chi_i = I_h \psi \). Then again \( \mathcal{P}_h := I_h \), but \( \mathcal{L}_h \varphi := \sum a_i \chi_i \) with \( a_i := \varepsilon^2 \frac{\langle \nabla \varphi, \nabla \chi_i \rangle}{\langle \chi_i, \chi_i \rangle} \) for interior mesh nodes, and \( a_i := -[g(\cdot,0)]_i \) for boundary mesh nodes. Consequently, \( \mathcal{L}_h v_h \) is easily computable for any \( v_h \in \hat{V}_h \) by applying the normalized stiffness matrix to the column vector of nodal values \( \{v_{h,i}\} \).

We now cite elliptic estimators of type (5.3) for particular cases of (5.4) and (5.5).
5.2. Elliptic model problem: regular regime. We first consider the steady-state version (5.4) of our model problem (1.3) in the regular regime of \( \epsilon = 1 \).

Let \( v \) solve the problem (5.4) with \( \epsilon = 1, \gamma \geq 0 \), posed in a bounded polyhedral domain \( \Omega \subset \mathbb{R}^n \), \( n = 2, 3 \), and \( v_h \) solve the discrete problem (5.5) with \( V_h \) and \( \langle \cdot, \cdot \rangle_h \) defined as follows. Given a conforming and shape-regular triangulation \( T_h \) of \( \Omega \) made of elements \( T \), we let \( V_h \) be the space of continuous piecewise polynomial finite element functions of degree \( l \geq 1 \), and \( V_h \) is the space of continuous piecewise polynomial finite element functions of degree \( l \geq 1 \) and \( V_h \) be the space of continuous piecewise polynomial finite element functions of degree \( l \geq 1 \), and \( V_h \) of continuous finite element functions on an arbitrary nonuniform mesh \( \{ x_i \}_{i=0}^{N} \) with \( 0 = x_0 < x_1 < \cdots < x_N = 1 \) and \( h_i := x_i - x_{i-1} \). Note that here we make absolutely no mesh regularity assumptions (as solutions of our problem typically exhibit sharp layers so a suitable mesh is expected to be highly-nonuniform; see, e.g., [13]).

Consider two choices of \( \langle \cdot, \cdot \rangle_h \), which are defined using the standard piecewise-linear Lagrange interpolation operator \( I_h \) onto the space of piecewise linear polynomials of degree \( \leq q' \).

5.3. Elliptic model problem: singularly-perturbed regime in one dimension. We now consider the steady-state version (5.4) of our model problem (1.3) in the singularly-perturbed regime of \( \epsilon \ll 1 \).

Let \( v \) solve the problem (5.4) with \( \epsilon \in (0,1] \) and \( \gamma > 0 \), posed in the domain \( \Omega := (0,1) \), and \( v_h \) solve the discrete problem (5.5) using the space \( V_h \) of continuous piecewise-linear finite element functions on an arbitrary nonuniform mesh \( \{ x_i \}_{i=0}^{N} \) with \( 0 = x_0 < x_1 < \cdots < x_N = 1 \) and \( h_i := x_i - x_{i-1} \). Note that here we make absolutely no mesh regularity assumptions (as solutions of our problem typically exhibit sharp layers so a suitable mesh is expected to be highly-nonuniform; see, e.g., [13]).

Consider two choices of \( \langle \cdot, \cdot \rangle_h \), which are defined using the standard piecewise-
linear Lagrange polynomial \( I_h \) onto \( V_h \):

\[
\langle \varphi, \psi \rangle_h := \langle I_h \varphi, \psi \rangle \quad \text{(quadrature)} \quad \text{(5.7a)}
\]
\[
\langle \varphi, \psi \rangle_h := \langle I_h[\varphi], \psi \rangle \quad \text{(lumped-mass quadrature)} \quad \text{(5.7b)}
\]

**Remark 5.3.** To illustrate Remarks 5.1 and 5.2, note that the described two discretizations using either (5.7a) or (5.7b) are of type (5.2). In particular, for (5.7a), we get \( L_h := -\epsilon^2 \partial_x^2 \) and \( P_h := I_h \). Here the operator \( \partial_x^2 \) is defined by \( \langle -\partial_x^2 \varphi, \chi \rangle = \langle \varphi', \chi', \varphi \rangle \chi \in V_h \).

Consequently, the discrete problem using (5.7a) may be represented as

\[
-\epsilon^2 \partial_x^2 v_h + I_h g(\cdot, v_h) = 0. \quad \text{(5.8a)}
\]

By contrast, (5.7b) can be rewritten as a difference scheme: 
\[
-\epsilon^2 \partial_x^2 v_h + g(x_i, v_h) = 0, \quad \text{for } i = 1, \ldots, N - 1,
\]
where \( \partial_x^2 v_h := \frac{2}{h_i} \left[ \frac{1}{h_i} (v_{h,i+1} - v_{h,i}) - \frac{1}{h_i} (v_{h,i} - v_{h,i-1}) \right] \]
is the standard finite-difference operator. Letting \( \delta^2_v v_{h,i} := \varepsilon^{-2} g(x_i, v_{h,i}) \) for \( i = 0, N \) and applying the linear interpolation \( I_h \) to \( \{ \delta^2_v v_{h,i}\}_{i=0}^N \), we can represent the discrete problem using (5.7b) as

\[
-\varepsilon^2 I_h[\delta^2_v v_h] + I_h[g(\cdot, v_h)] = 0, 
\]

where the values \( \delta^2_v v_{h,i} \) are easily explicitly computable.

We cite a posteriori error bounds [8, 12, 13] of type (5.3) with \( \eta := \eta_\varepsilon(V_h, g(\cdot, v_h)) \) for (5.7a) and \( \eta := \eta_{h;1.m.}(V_h, g(\cdot, v_h)) \) for (5.7b), respectively, defined by

\[
\eta_\varepsilon(V_h, g_\varepsilon) := \max_{i=1,\ldots,N} \left\{ \frac{h_i^2}{\varepsilon^2} \| I_h g_\varepsilon \|_{\infty,(x_{i-1},x_i)} \right\} + \gamma^{-2} \| g_\varepsilon - I_h g_\varepsilon \|_{\infty,(0,1)}, 
\]

\[
\eta_{h;1.m.}(V_h, g_\varepsilon) := \eta_\varepsilon + \max_{i=1,\ldots,N} \left\{ \frac{h_i^2}{\varepsilon^2} \| \partial_x(I_h g_\varepsilon) \|_{\infty,(x_{i-1},x_i)} \right\},
\]

where \( g_\varepsilon := g(\cdot, v_h) \).

**Remark 5.4.** The error estimators (5.9a) and (5.9b) are robust although they involve negative powers of the small parameter \( \varepsilon \). Indeed, an inspection of representations (5.7a) and (5.7b) for the two considered numerical methods shows that \( \varepsilon^{-2} h_i^2 |I_h g_\varepsilon = \varepsilon^{-2} h_i^2 |I_h g(\cdot, v_h) | \) becomes \( h_i^2 \| \partial^2_x v_h \| \) or \( h_i^2 \| \delta^2_v v_h \| \), so it approximates \( h_i^2 \| \partial^2_x v \| \), where \( v \) is the exact solution of our equation \( -\varepsilon^2 \partial^2_x v + g(\cdot, v) = 0 \). Similarly, the term \( \varepsilon^{-2} h_i^2 |\partial_x(I_h g_\varepsilon) | \) approximates \( \varepsilon \| \partial^2_x v \| \), which has similar magnitude to \( h_i^2 \| \partial^2_x v \| \) in the layer regions.

By contrast, if \( \gamma_\varepsilon := \gamma(\cdot, \cdot) \) (i.e. no quadrature is used), then one can obtain a simpler-looking error estimate of type (5.3) with \( \eta := \max_{i=1,\ldots,N} \left\{ \frac{h_i^2}{\varepsilon^2} \| g_\varepsilon \|_{\infty,(x_{i-1},x_i)} \right\} \). However, this estimate is not robust. To see this, split \( g_\varepsilon = P_h g_\varepsilon + (g_\varepsilon - P_h g_\varepsilon) \) using the standard \( L_2 \) projection \( P_h \). Then, instead of (5.8a), we have the representation

\[
-\varepsilon^2 \partial^2_x v_h + P_h g_\varepsilon = 0
\]

for our numerical method. The component \( \varepsilon^{-2} h_i^2 |P_h g_\varepsilon | \) approximates \( h_i^2 \| \partial^2_x v \| \) so it yields a robust part of the estimator. But the other component \( \varepsilon^{-2} h_i^2 |g_\varepsilon - P_h g_\varepsilon | \) may be as large as \( \mathcal{O}(\varepsilon^2 h_i^2) \), which may become quite large if \( \varepsilon \) is small compared to the local mesh size. For this numerical method one can, in fact, obtain a robust error estimator, which is almost identical with (5.9a), only \( I_h \) in \( \eta_\varepsilon \) should be replaced by \( P_h \) (but this latter estimator is less practical, as it requires the \( L_2 \) projection \( P_h g_\varepsilon \), which is explicitly computable).

**6. Fully discrete Backward Euler method.** To fully discretize the abstract parabolic problem (1.1), we now apply a spatial discretization of type (5.2) to the semidiscrete problem (3.1) as follows. We associate a finite-element space \( V_h \subset C(\Omega) \) and a computed solution \( u_h^j \in V_h^j := V_h^j \cap H^1_0(\Omega) \) with the time level \( t_j \) and require, for \( j = 1, \ldots, M \), that

\[
\mathcal{L}_h^j u_h^j + \mathcal{P}_h^j[f(\cdot, t_j, u_h^j)] + \delta_t^j u_h^j = 0, 
\]

with some \( u_0 \approx \varphi \) in \( V_h^0 \). Here \( \mathcal{L}_h^j \) and \( \mathcal{P}_h^j \) are also associated with the time level \( t_j \) and, in agreement with (5.2b), for the Lagrange interpolation operator \( I_h^j \) onto \( V_h^j \), we let

\[
\mathcal{L}_h^j : H^1_0(\Omega) \to V_h^j - I_h^j[f(\cdot, t_j, 0)], 
\]

\[
\mathcal{P}_h^j v \in V_h^j + I_h^j v \quad \forall v \in C(\Omega), 
\]

\[
\mathcal{P}_h^j u_h = v_h \quad \forall v_h \in V_h^j. 
\]

Note that as both \( u_h^j \) and \( \delta_t^j u_h^j \) vanish on \( \partial \Omega \), so \( V_h^j - I_h^j[f(\cdot, t_j, 0)] \) coincides with \( V_h^j - I_h^j[f(\cdot, t_j, u_h^j) + \delta_t^j u_h^j] \), so the definition (6.1) is consistent.
The term $\delta^*_j u^j_h$ in (6.1a) approximates $\partial_t u$ and is defined by
\[
\delta^*_j u^j_h := \frac{u^j_h - \hat{u}^{j-1}_h}{\tau_j}, \text{ where } \hat{u}^0_h := u^0_h. \tag{6.1c}
\]
The operator $\delta^*_j$ is identical with $\delta_j$ of (3.1b) for $j = 1$, while for $j > 1$ it involves the intermediate computed solution $\hat{u}^{j-1}_h \in H^1_0(\Omega)$ that we associate with the time level $t^{j-1}_j$ (this is indicated by the hat notation).

We do not presently specify $\hat{u}^{j-1}_h$, but note two particular cases of interest:

Case A: $\hat{u}^{j-1}_h := u^{j-1}_h$ \quad $\Rightarrow$ $\delta^*_j u^j_h \in \text{span}(\hat{V}^{j-1}_h, \hat{V}_h^j)$, \tag{6.2a}

Case B: $\hat{u}^{j-1}_h := I_j^h u^{j-1}_h$, $I_j^h : \hat{V}^{j-1}_h \to \hat{V}_h^j$ \quad $\Rightarrow$ $\delta^*_j u^j_h \in \hat{V}_h^j$. \tag{6.2b}

Here $I_j^h$ is some interpolation operator such that $I_j^h u^{j-1}_h := u^{j-1}_h$ if $V^{j-1}_h \subset V^j_h$.

Note that if $V^{j-1}_h \subset V^j_h$ (which includes the case of $V^j_h = V_h$ being fixed for all $j = 0, \ldots, M$), then Cases A and B are identical. Note also that to define $I_j^h$ in Case B, one may employ, e.g., the standard Lagrange interpolation or the $L_2$ projection.

6.1. A posteriori error estimate using piecewise-constant elliptic reconstructions. To estimate the error of the fully discrete Backward-Euler method (6.1), we shall employ the elliptic reconstruction, which was introduced in the recent papers [14, 11, 3] as a counterpart of the Ritz-projection in the a posteriori error estimation for parabolic problems.

We associate an elliptic reconstruction $R^j \in H^1_0(\Omega) \cap C(\bar{\Omega})$ with the time level $t_j$ and require, for $x \in \Omega$, $j = 1, \ldots, M$, that
\[
\mathcal{L}^j R^j + g^j(x, R^j) = 0, \quad \text{where } g^j(x, v) := f(x, t_j, v) + \delta^*_j u^j_h, \tag{6.3}
\]
so this relation may be considered somewhat between (3.1a) and (6.1a). On the other hand, (6.3) is a version of the elliptic problem (5.1) with $\mathcal{L} := \mathcal{L}^j$ and $g := g^j$, while the numerical method (5.2) applied to this problem is identical with (6.1) and yields the computed solution $u^j_h$. Furthermore, applying the elliptic a posteriori error estimate (5.3) to the exact solution $R^j$ and the corresponding computed solution $u^j_h$, one gets
\[
\eta^j := \| R^j - u^j_h \|_{\infty, \Omega} \leq \eta(V^j_h, u^j_h, g^j(\cdot, u^j_h)) \quad \text{for } j = 1, \ldots, M. \tag{6.4}
\]

We now give an a posteriori error estimate for the fully discrete method (6.1).

**Theorem 6.1.** Let $u$ solve the problem (1.1), (1.2) with the parabolic operator $\mathcal{M}$ satisfying Condition 2.1, $u^j_h$ solve the discrete problem (6.1), and $R^j$ be the elliptic reconstruction defined by (6.3) and satisfying (6.4). Then for $m = 1, \ldots, M$, one has
\[
\| u^m_h - u(\cdot, t_m) \|_{\infty, \Omega} \leq \kappa_0 e^{-\gamma^2 t_m} \| u^0_h - \varphi \|_{\infty, \Omega}
+ (\kappa_1 \ell_m + \kappa_2) \max_{j=1,\ldots,m-1} \left\{ \| u^j_h - u^{j-1}_h \|_{\infty, \Omega} + \eta^j \right\}
+ 2\kappa_0 \| u^m_h - u^{m-1}_h \|_{\infty, \Omega} + (\kappa_0 + 1) \eta^m
+ \kappa_0 \sum_{j=1}^m \int_{t_{j-1}}^{t_j} e^{-\gamma^2 (t-s)} \| [\vartheta_{L,R} + \vartheta_{f,R}(\cdot, s)] \|_{\infty, \Omega} \, ds
+ \kappa_0 \sum_{j=2}^m e^{-\gamma^2 (t_{j-1})} \| \hat{u}^{j-1}_h - u^{j-1}_h \|_{\infty, \Omega}, \tag{6.5}
\]
where \( \vartheta_{L,R} \) and \( \vartheta_{f,R} \) are regular functions defined, for \( t \in (t_{j-1}, t_j], \) \( j = 1, \ldots, M, \) by

\[
\vartheta_{L,R}(\cdot, t) := [L(t) - L'] R^j, \quad \vartheta_{f,R}(\cdot, t) := f(\cdot, t, R^j) - f(\cdot, t_j, R^j). \tag{6.6}
\]

**Remark 6.2.** The final term in (6.5) vanishes in Case A of (6.2); in particular, when one has \( V_j^{m-1} \subset V_j^l \) for all \( j = 1, \ldots, M. \)

**Proof.** In view of (6.4), to get the desired bound (6.5) for \( u_h^m - u(\cdot, t_m) \), it suffices to obtain a bound of type (6.5) for \( R^m - u(\cdot, t_m) \) only with \( (\kappa_0 + 1) \) replaced by \( \kappa_0 \), and then apply the triangle inequality. So we focus on estimating \( R^m - u(\cdot, t_m) \).

We partially imitate the proof of Theorem 3.1. Let \( I_t u_h \) be a standard piecewise-linear interpolant of \( u_h^j \) in time:

\[
I_t u_h(\cdot, t) := \frac{t - t_{j-1}}{\tau_j} u_h^{j-1} + \frac{t - t_j}{\tau_j} u_h^j \quad \text{for} \quad t \in [t_{j-1}, t_j], \quad j = 1, \ldots, M. \tag{6.7}
\]

Furthermore, we define a **piecewise-constant** interpolant \( \tilde{R} \) of \( R^j \) in time by

\[
\tilde{R}(\cdot, t) := R^j \quad \text{for} \quad t \in (t_{j-1}, t_j], \quad j = 1, \ldots, M; \quad \tilde{R}(\cdot, 0) := R^1, \tag{6.8}
\]

(so \( \tilde{R} \) is continuous on \([t_0, t_1]\); compare with \( \tilde{U} \) of (3.5)). Note that the temporal derivative \( \partial_t \tilde{R} \) is understood in the sense of distributions, while \( \partial_t (I_t u_h) \) is a regular function. Now, in view of (6.3), one gets

\[
\partial_t (I_t u_h) + L(t) \tilde{R} + f(\cdot, t, \tilde{R}) = \vartheta_{L,R} + \vartheta_{f,R} + \vartheta_* \quad \text{in} \quad Q, \tag{6.9}
\]

where \( \vartheta_* \) is a regular function defined by

\[
\vartheta_*(\cdot, t) := \partial_t (I_t u_h) - \delta_t^* u_h^j \quad \text{for} \quad t \in (t_{j-1}, t_j]. \tag{6.10}
\]

To get (6.9), we also used the observation that, by (6.8), the regular functions \( \vartheta_{L,R} \) and \( \vartheta_{f,R} \) of (6.6) can be rewritten for \( t \in (t_{j-1}, t_j] \) as \( \vartheta_{L,R} = L(t) \tilde{R} - L' \tilde{R}^j \) and \( \vartheta_{f,R} = f(\cdot, t, \tilde{R}) - f(\cdot, t_j, R^j). \)

Next, combining (6.9) with (1.1a) yields

\[
\mathcal{M} \tilde{R} - Mu = \partial_t \tilde{R} + L(t) \tilde{R} + f(\cdot, t, \tilde{R}) = \partial_t [\tilde{R} - I_t u_h] + [\vartheta_{L,R} + \vartheta_{f,R} + \vartheta_*].
\]

Now the desired bound of type (6.5) for \( R^m - u(\cdot, t_m) = [\tilde{R} - u](\cdot, t_m) \), only with \( (\kappa_0 + 1) \) replaced by \( \kappa_0 \), is obtained by an application of Lemma 2.4 with \( \mu := \tilde{R} - I_t u_h \) and \( \vartheta := \vartheta_{L,R} + \vartheta_{f,R} + \vartheta_* \), using the following three observations. First, note that

\[
[\tilde{R} - u - \mu](\cdot, 0) = R^1 - \vartheta - (R^1 - u_h^0) = u_h^0 - \vartheta. \tag{6.11}
\]

Next, for \( t \in (t_{j-1}, t_j] \), we have \( \mu = R^j - u_h^j + \frac{t - t_{j-1}}{\tau_j} (u_h^j - u_h^{j-1}) \). Thus,

\[
|\mu| \leq |R^j - u_h^j| + |u_h^j - u_h^{j-1}| \quad \text{and} \quad |\partial_j \mu| = |u_h^j - u_h^{j-1}|, \tag{6.12}
\]

where \( \|R^j - u_h^j\|_{\infty, \Omega} = \eta^j \). Finally, (6.10) combined with (6.1c), (6.7) implies that \( \vartheta_*(\cdot, t) = \frac{1}{\tau_j}(u_h^{j-1} - u_h^{j-2}) \) for \( t \in (t_{j-1}, t_j] \). Therefore,

\[
\int_{t_{j-1}}^{t_j} e^{-\gamma(t_{m-s})} \|\vartheta_*(\cdot, s)\|_{\infty, \Omega} \, ds \leq e^{-\gamma(t_{m-t_j})} \|u_h^{j-1} - u_h^{j-2}\|_{\infty, \Omega}, \tag{6.13}
\]
where \( \hat{u}_h^0 - u_h^0 = 0 \). The three observations (6.11), (6.12), (6.13) yield the required bound for \( \|R^n - u(\cdot, t_m)\|_{\infty, \Omega} \).

**Theorem 6.1.** The statement of Theorem 6.1 remains valid with the terms 
\[
\|u_h^j - u_h^{j-1}\|_{\infty, \Omega} \text{ and } \|u_h^m - u_h^{m-1}\|_{\infty, \Omega} \text{ in (6.5) respectively replaced by } \|u_h^j - \hat{u}_h^{j-1}\|_{\infty, \Omega} \text{ and } \|u_h^m - \hat{u}_h^{m-1}\|_{\infty, \Omega}.
\]

**Proof.** We imitate the proof of Theorem 6.1, but with \( I_T u_h \) of (6.7) replaced by the piecewise-continuous interpolant 
\[
I_T^* u_h(\cdot, t) := \frac{\tau}{\tau_j} \hat{u}_h^{j-1} + \frac{\tau_j - \tau}{\tau_j} u_h^j \quad \text{for } t \in (t_{j-1}, t_j), \quad j = 1, \ldots, M,
\]
with \( I_T^* u_h(\cdot, 0) := \hat{u}_h^0 = u_h^0 \). Furthermore, \( \vartheta_* \) is defined not by (6.10), but by 
\[
\vartheta_*(\cdot, t) := \partial_t(I_T^* u_h) - \delta_* u_h^j = [\hat{u}_h^{j-1} - u_h^{j-1}] \delta(t - t_{j-1}^+) \quad \text{for } t \in (t_{j-1}, t_j),
\]
where \( \delta(\cdot) \) is the one-dimensional Dirac \( \delta \)-distribution. (Note that \( \hat{u}_h^0 = u_h^0 \) and the right-continuity convention at \( t = 0 \) imply that \( \vartheta_* = 0 \) on \([0, t_1]) \). So instead of (6.13) we get 
\[
\int_0^{t_m} e^{-\gamma(t_m - s)} \|\vartheta_*(\cdot, s)\|_{\infty, \Omega} ds \leq \sum_{j=2}^m e^{-\gamma(t_m - t_{j-1})} \|\hat{u}_h^{j-1} - u_h^{j-1}\|_{\infty, \Omega}.
\]

The required bound for \( R^n - u(\cdot, t_m) = [\hat{R} - u(\cdot, t_m) \) is again obtained by an application of Lemma 2.4 only with \( \mu := \hat{R} - I_T^* u_h \), for which we have a version of (6.12) with \( u_h^0 \) replaced by \( \hat{u}_h^{-1} \).

**Remark 6.3.** The terms \( \vartheta_{f,R} \) and \( \vartheta_{f,R} \) in (6.5) involve the elliptic reconstruction \( \hat{R} \). In view of the bound (6.4), their discrepancy from \( \vartheta_{f,u_h} \) and \( \vartheta_{f,u_h} \), respectively, can be easily estimated. E.g., for \( \vartheta_{f,R} \) with \( t \in (t_{j-1}, t_j) \), we have 
\[
\|\vartheta_{f,R} - \vartheta_{f,u_h}(\cdot, t)\|_{\infty, \Omega} \leq \eta_j \sup_{(t_{j-1}, t_j) \times \mathbb{R}} \|\partial_z f(\cdot, t, z) \|
\leq \eta_j \sup_{(t_{j-1}, t_j) \times \mathbb{R}} \|\partial_z f(\cdot, t, z) \|
\]

where \( \eta_j \) is estimated using (6.4). In fact, if \( |\partial_t \partial_z f| \leq C \), then the discrepancy 
\[
\|\vartheta_{f,R} - \vartheta_{f,u_h}(\cdot, t)\|_{\infty, \Omega}
\]

between \( \vartheta_{f,R} \) and \( \vartheta_{f,u_h} \) becomes \( \mathcal{O}(\eta_j) \), i.e. negligible compared with the terms \( n_j \) that explicitly appear in (6.5).

### 6.2. Applications to the model problem (1.3). Consider a fully discrete Backward-Euler method for (1.3), obtained by applying the spatial discretization (5.5) to a version of the semidiscrete Backward-Euler method (3.1):

Find \( u_h^j \in \hat{V}_h^j \) such that 
\[
e^2 \langle \nabla u_h^j, \nabla \chi \rangle + \langle f(\cdot, t_j, u_h^j) + \delta^j_* u_h^j, \chi \rangle_h = 0 \quad \forall \chi \in \hat{V}_h^j,
\]

where \( \langle \cdot, \cdot \rangle_h \) is either exactly the inner product \( \langle \cdot, \cdot \rangle \) in \( L^2(\Omega) \), or some quadrature formula for \( \langle \cdot, \cdot \rangle \), and \( \delta_*^j u_h^j \) is defined by (6.1e), (6.2).

Note that the full discretization (6.17) is of type (6.1). For some particular cases of \( \langle \cdot, \cdot \rangle_h \), the operators \( L_h^j \) and \( P_h^j \) are defined as in Remarks 5.1 and 5.2 only using \( V_h^j \) instead of \( V_h \).
6.2.1. Model problem (1.3): regular regime. Let \( u \) solve the problem (1.3) with \( \varepsilon = 1, \gamma \geq 0 \), posed in a bounded polyhedral spatial domain \( \Omega \subset \mathbb{R}^n \), \( n = 2, 3 \), and \( u_0 \) solve the discrete problem (6.17) with \( V_h^j \) and \( \langle \cdot, \cdot \rangle_h \) defined, for each time level \( t_j \), as in \( \S 5.2 \). To be more specific, we let \( T_h \) be a conforming and shape-regular triangulation of \( \Omega \) made of elements \( T \), \( V_h \) be the space of continuous piecewise polynomial finite element functions of degree \( l \geq 1 \), and \( V_h^j := V_h^j \cap H^1(\Omega) \). We then employ a quadrature formula \( \langle \varphi, \psi \rangle_h := \sum_{T \in T_h} QT \langle \varphi \psi \rangle_h \), as described in \( \S 5.2 \).

**Corollary 6.4.** Let the above numerical method be applied to problem (1.3) with \( \varepsilon = 1, \gamma \geq 0 \). Then the a posteriori error estimates of Theorems 6.1 and 6.1* are valid with \( \vartheta_{E,R} = 0 \), \( \vartheta_{f,R} \) computed as described in Remark 6.3, and

\[
\eta_j^q := \eta_0(V_h^j, u_0^j, f(\cdot, t_j, u_h^j) + \delta^* u_h^j) \quad \text{for} \quad j = 1, \ldots, M,
\]

where \( \eta_0 \) is defined in (5.6).

**Remark 6.5.** The Backward-Euler method for a linear version of (1.3) with \( \varepsilon = 1 \) was considered in [4, 3]. The a posteriori error estimate (6.5) of Corollary 6.4 resembles (but is not identical with) the one of [4, (1.13)] in that it involves terms such as \( |u_h^j - u_{h-1}^j| \), that may be interpreted as approximating \( \tau_j \partial_u \). (Note also that [4, (1.13)] is given without proof, and does not appear to be proved elsewhere.)

By contrast, the a posteriori error estimates of [3] include terms (denoted there by \( \tau_j |g^j - g^{j-1}| \) that may be interpreted as approximating the quantity \( \tau_j |\partial^2 u + \ldots| \), which seems less suitable for a first-order method in time.

6.2.2. Model problem (1.3): singularly perturbed regime in one dimension. Now, consider \( \varepsilon \ll 1 \). Let \( u \) solve (1.3) with \( \varepsilon \in (0, 1], \gamma > 0 \), posed in the domain \( \Omega : = (0, 1) \). Let \( u_0 \) solve the discrete problem (6.17) with \( V_h^j \) and \( \langle \cdot, \cdot \rangle_h \) defined, for each time level \( t_j \), as in \( \S 5.3 \). Thus \( V_h^j \) is the space of continuous piecewise-linear finite element functions on an arbitrary nonuniform mesh \( \{x_i^j\}_{i=1}^N \) with \( 0 = x_0^j < x_1^j < \cdots < x_N^j = 1 \) under absolutely no mesh regularity assumptions. We consider the two choices (5.7a) and (5.7b) of \( \langle \cdot, \cdot \rangle_h \), using the piecewise-linear Lagrange interpolant \( I_h := I_h^O \) onto \( V_h^j \).

**Corollary 6.6.** Let the above numerical method be applied to problem (1.3) with \( \varepsilon \in (0, 1], \gamma > 0, \Omega := (0, 1) \). Then the a posteriori error estimates of Theorems 6.1 and 6.1* are valid with \( \vartheta_{E,R} = 0 \), \( \vartheta_{f,R} \) computed as described in Remark 6.3, and

\[
\eta_j^q := \begin{cases} 
\eta_c(V_h^j, f(\cdot, t_j, u_h^j) + \delta^* u_h^j) & \text{for } (5.7a), \\
\eta_{c,1.m}(V_h^j, f(\cdot, t_j, u_h^j) + \delta^* u_h^j) & \text{for } (5.7b), 
\end{cases}
\]

for \( j = 1, \ldots, M \), where \( \eta_c \) and \( \eta_{c,1.m.} \) are defined in (5.9), with \( I_h \) replaced by \( I_h^O \).

**Remark 6.7.** The a posteriori error estimators of Corollary 6.6 are robust as the argument of Remark 5.4 applies to \( \eta^q \) with \( \nu \) replaced by \( u(\cdot, t_j) \), so \( \varepsilon^{-2} h^2 |I_h g_*| \) approximates \( h^2 |\partial^2 u(\cdot, t_j)| \), while the term \( \varepsilon^{-1} h^2 |\partial_u (I_h g_*)| \) approximates \( \varepsilon |\partial^2 u(\cdot, t_j)| \), which has similar magnitude to \( h^2 |\partial^2 u(\cdot, t_j)| \) in the layer regions.

**Remark 6.8.** Consider the term \( |g_* - I_h^O g_*|_{\infty, \Omega} \) in the error estimators of Corollary 6.6 for Cases A and B of (6.2). In Case B, one has \( \delta^* u_h^j - I_h^O [\delta^* u_h^j] = 0 \), hence \( |g_* - I_h^O g_*|_{\infty, \Omega} \) simplifies to \( ||f(\cdot, t_j, u_h^j) - I_h^O f(\cdot, t_j, u_h^j)||_{\infty, \Omega} \). In Case A, the final term in (6.5) vanishes (see Remark 6.2). However, \( g_* - I_h^O g_* \) again involves \( f(\cdot, t_j, u_h^j) - I_h^O f(\cdot, t_j, u_h^j) \) and, furthermore, \( \delta^* u_h^j - I_h^O [\delta^* u_h^j] = -\frac{1}{\tau_j} (u_h^{j-1} - I_h^O [u_h^{j-1}]) \).
Combining (7.2) with (7.1a) immediately yields
\[ \delta_t^* u_h^j + \frac{1}{2} (\hat{L}_h^{j-1} \hat{u}_h^{j-1} + \mathcal{L}_h^j u_h^j) + \frac{1}{2} \mathcal{P}_h^j [f(\cdot, t_{j-1}, \hat{u}_h^{j-1}) + f(\cdot, t_j, u_h^j)] = 0 \] (7.1a)
with some \( u_h^0 \approx \varphi \) in \( V_h^0 \). Here \( \mathcal{L}_h^j \) and \( \mathcal{P}_h^j \) are associated with the time level \( t_j \), while \( \hat{L}_h^{j-1} \) is associated with the time level \( t_{j-1} \). In agreement with (5.2b), for the Lagrange interpolation operator \( I_h^j \) onto \( V_h^j \), we let
\[ \mathcal{L}_h^j : H_0^1(\Omega) \to \hat{V}_h^j = I_h^j f(\cdot, t_j, 0), \quad \hat{L}_h^{j-1} : H_0^1(\Omega) \to \hat{V}_h^j = I_h^j f(\cdot, t_{j-1}, 0), \]
\[ \mathcal{P}_h^j v \in \hat{V}_h^j + I_h^j v \quad \forall v \in C(\Omega), \quad \mathcal{P}_h^j u_h = u_h \quad \forall u_h \in \hat{V}_h^j. \] (7.1b)
Note that any \( u_h \in \hat{V}_h^j \) vanishes on \( \partial \Omega \), so \( \hat{V}_h^j = L_h^j f(\cdot, t_k, 0) = \hat{V}_h^j = L_h^j f(\cdot, t_k, v_h) \) for \( k = j-1, j \), while we also impose that \( \delta_t^* u_h^j \in \hat{V}_h^j \). So the definition (7.1) is consistent.

The term \( \delta_t^* u_h^j \) in (7.1a) approximates \( \partial_t u \) and is identical with (6.1c):
\[ \delta_t^* u_h^j := \frac{u_h^j - \hat{u}_h^{j-1}}{t_j}, \quad \text{where} \quad \hat{u}_h^0 := u_h^0. \] (7.1c)
The operator \( \delta_t^* \) is identical with \( \delta_t \) of (4.1b) for \( j = 1 \), while for \( j > 1 \) it involves \( \hat{u}_h^{j-1} \) defined by
\[ \hat{u}_h^{j-1} := I_h^j u_h^{j-1}, \quad I_h^j : \hat{V}_h^{j-1} \to \hat{V}_h^j \quad \Rightarrow \quad \delta_t^* u_h^j \in \hat{V}_h^j. \] (7.1d)
Here \( I_h^j \) is some linear interpolation operator such that \( I_h^j u_h^{j-1} = u_h^{j-1} \) if \( V_h^{j-1} \subset V_h^j \).

Thus, in this section we restrict our analysis to Case B of (6.2b). Note that if \( V_h^{j-1} \subset V_h^j \) (which includes the case of \( V_h^j = V_h \) being fixed for all \( j \)), then \( u_h^{j-1} = \hat{u}_h^{j-1} \). Also, to define \( I_h^j \), one may employ, e.g., the standard Lagrange interpolation or the \( L_2 \) projection.

### 7.1. A posteriori error estimate using piecewise-linear elliptic reconstructions
We associate elliptic reconstructions \( R^j, \hat{R}^{j-1} \in H_0^1(\Omega) \cap C(\Omega) \), respectively, with the time levels \( t_j \) and \( t_{j-1} \), and require, for \( x \in \Omega, j = 1, \ldots, M \), that
\[ \mathcal{L}^j R^j + g^j(x, R^j) = 0, \quad \mathcal{L}^{j-1} \hat{R}^{j-1} + \hat{g}^{j-1}(x, \hat{R}^{j-1}) = 0, \] (7.2a)
where
\[ g^j(\cdot, v) := f(\cdot, t_j, v) - \left\{ \mathcal{L}_h^j u_h^j + \mathcal{P}_h^j [f(\cdot, t_j, u_h^j)] \right\}, \] (7.2b)
\[ \hat{g}^{j-1}(\cdot, v) := f(\cdot, t_{j-1}, v) - \left\{ \hat{L}_h^{j-1} \hat{u}_h^{j-1} + \mathcal{P}_h^j [f(\cdot, t_{j-1}, \hat{u}_h^{j-1})] \right\}. \] (7.2c)
Combining (7.2) with (7.1a) immediately yields
\[ \delta_t^* u_h^j + \frac{1}{2} (\mathcal{L}^{j-1} \hat{R}^{j-1} + \mathcal{L}^j R^j) + \frac{1}{2} [f(\cdot, t_{j-1}, \hat{R}^{j-1}) + f(\cdot, t_j, R^j)] = 0, \] (7.3)
which may be considered somewhat between (4.1a) and (7.1a).

Note that (7.2a) describes two versions of the elliptic problem (5.1) with $L := L^j$, $g := g^j$, and with $L := L^{j-1}$, $g := \hat{g}^{j-1}$, and exact solutions $R^j$ and $\hat{R}^{j-1}$, respectively. Furthermore, the numerical method (5.2), using the finite element space $V_h^j$, applied to these two problems yields

\[
L_h^j R_h^j + P^j_h [g^j(x, R_h^j)] = 0, \quad \hat{L}_h^{j-1} \hat{R}_h^{j-1} + P^j_h [\hat{g}^{j-1}(x, \hat{R}_h^{j-1})] = 0. \tag{7.4}
\]

We have assumed that solutions of these two discrete problems are unique. Thus, $R_h^j = u_h^j$ and $\hat{R}_h^{j-1} = \hat{u}_h^{j-1}$. This is easily checked by combining (7.4) with the definitions of $g^j$ and $\hat{g}^{j-1}$ in (7.2), in which the terms $\{ \cdots \} \in V_h^j$. Consequently, applying the elliptic a posteriori error estimate (5.3) to the exact solutions $R^j$ and $\hat{R}^{j-1}$ and the corresponding computed solutions $u_h^j$ and $\hat{u}_h^{j-1}$, one gets, for $j = 1, \ldots, M$,

\[
\eta^j := \| R^j - u_h^j \|_{\infty, \Omega} \leq \eta(V_h^j, u_h^j, g^j(\cdot, u_h^j)), \tag{7.5a}
\]

\[
\hat{\eta}^{j-1} := \| \hat{R}^{j-1} - \hat{u}_h^{j-1} \|_{\infty, \Omega} \leq \eta(V_h^j, \hat{u}_h^{j-1}, \hat{g}^{j-1}(\cdot, \hat{u}_h^{j-1})). \tag{7.5b}
\]

To formulate our a posteriori error estimate for $u_h - u$, we generalize the piecewise linear interpolation $I_t$ of (4.2) to any left-continuous function $w = w(t)$ by setting

\[
I_t^* w(t) := \frac{t - t_{j-1}}{\tau_j} w(t_{j-1}) + \frac{t - t_j}{\tau_j} w(t_j) \quad \text{for } t \in (t_{j-1}, t_j], \quad j = 1, \ldots, M. \tag{7.6}
\]

In a similar manner, we apply $I_t^*$ to the elliptic reconstruction (7.2) and define

\[
R(\cdot, t) := I_t^* R(t, \cdot) := \frac{t - t_{j-1}}{\tau_j} \hat{R}^{j-1} + \frac{t - t_j}{\tau_j} R^j \quad \text{for } t \in (t_{j-1}, t_j], \quad j = 1, \ldots, M. \tag{7.7}
\]

(In agreement with the adapted left-/right-continuity convention, both $R$ and $I_t^* w$ are right-continuous at $t = 0$.)

**Theorem 7.1.** Let $u$ solve (1.1), (1.2) with a parabolic operator $M$ satisfying Condition 2.1, $u_h^j$ solve the discrete problem (7.1), and $R^j, \hat{R}^{j-1}$ be the elliptic reconstructions defined by (7.2) and satisfying (7.5). Then, for $m = 1, \ldots, M$, one has

\[
\| u_h^m - u(\cdot, t_m) \|_{\infty, \Omega} \leq \kappa_0 e^{-\gamma^2 t_m} \| u_h^0 - \varphi \|_{\infty, \Omega}
\]

\[
+ (\kappa_1 T_m + \kappa_2) \max_{j=1, \ldots, m-1} \left\{ \frac{1}{\tau_j} \left[ \| \delta_j^* \Psi^j \|_{\infty, \Omega} + \eta^j + \hat{\eta}^{j-1} \right] \right\}
\]

\[
+ \frac{5}{8} \kappa_0 T_m \left[ \left\| \delta_j^* \Psi^m \right\|_{\infty, \Omega} + (2\kappa_0 + 1) \eta^m + 2\kappa_0 \hat{\eta}^{m-1} \right] + \kappa_0 \int_{t_{j-1}}^{t_j} e^{-\gamma^2(s-t_{j-1})} \left[ \| \vartheta_{L,R} + \vartheta_{f,R}(\cdot, s) \|_{\infty, \Omega} \right] ds + \kappa_0 \int_{t_{j-1}}^{t_j} e^{-\gamma^2(s-t_{j-1})} \left[ \| u_h^j - u_h^{j-1} \|_{\infty, \Omega} \right]. \tag{7.8}
\]

Here, for $j = 1, \ldots, M$, we use $\delta_j^* \Psi^j = \frac{1}{\tau_j} (\Psi^j - \hat{\Psi}^{j-1})$ with

\[
\Psi^j := L^j_h u_h^j + P^j_h [f(\cdot, t_j, u_h^j)], \quad \hat{\Psi}^{j-1} := \hat{L}^{j-1}_h \hat{u}_h^{j-1} + P^j_h [f(\cdot, t_{j-1}, \hat{u}_h^{j-1})]. \tag{7.9}
\]

while $\vartheta_{L}$ and $\vartheta_{f}$ are regular functions defined, for $t \in (t_{j-1}, t_j)$, $j = 1, \ldots, M$, by

\[
\vartheta_{L,R} := L(t) R - I_t^*[L(t) R], \quad \vartheta_{f,R} := f(\cdot, t, R) - I_t^*[f(\cdot, t, R)] . \tag{7.10}
\]
with $I^*_t$ and $R$ from (7.6) and (7.7).

Proof. In view of (7.5a), to get the desired bound (7.8) for $u_h^m - u(\cdot, t_m)$, it suffices to obtain a bound of type (7.8) for $R^m - u(\cdot, t_m)$, with $(2\kappa_0 + 1)$ replaced by $2\kappa_0$, and then apply the triangle inequality. So we consider $R^m - u(\cdot, t_m)$ only.

We partially imitate the proof of Theorem 4.1. Let $t \in [t_{j-1}, t_j]$. In view of (7.6), for any left-continuous function $w = w(t)$ with the notation $w^j := w(t_j)$ and $\hat{w}^{j-1} := w(t_{j-1})$, one has $I^*_t w(t) = \frac{1}{2}(\hat{w}^{j-1} + w^j) + (t - t_{j-1/2}) \delta_t^j w^j$. So combining (7.10) and this property, for $t \in (t_{j-1}, t_j)$, one easily gets

$$f(\cdot, t, R) - \frac{1}{2} \left(f(\cdot, t_{j-1}, \hat{R}^{j-1}) + f(\cdot, t_j, R^j)\right) = (t - t_{j-1/2}) \delta_t^j [f(\cdot, t, R)]^j + \vartheta_{f,R},$$

and a similar relation

$$L(t) R - \frac{1}{2} \left(L^{j-1} \hat{R}^{j-1} + L^j R^j \right) = (t - t_{j-1/2}) \delta_t^j [L(t) R]^j + \vartheta_{L,R}.$$ 

Combining these two observations with (7.3), one deduces that

$$\delta_t^j u_h^j + L(t) R + f(\cdot, t, R) = (t - t_{j-1/2}) \delta_t^j [L(t) R + f(\cdot, t, R)]^j + [\vartheta_{L,R} + \vartheta_{f,R}]$$

$$= (t - t_{j-1/2}) \delta_t^j \Psi \cdot 1 + [\vartheta_{L,R} + \vartheta_{f,R}],$$

where $\delta_t^j u_h^j = \frac{1}{\gamma_1} (\Psi^j - \hat{\Psi}^{j-1})$ with

$$\Psi^j := L^j R_j^j + f(\cdot, t_j, R^j) \quad \text{and} \quad \hat{\Psi}^{j-1} := L^{j-1} \hat{R}^{j-1} + f(\cdot, t_{j-1}, \hat{R}^{j-1}).$$

In view of (7.2), these definitions of $\Psi^j$ and $\hat{\Psi}^{j-1}$ are equivalent to (7.9). Finally, for $t \in (t_{j-1}, t_j)$, where $\mu_1 = \mu_1(x, t)$ is a continuous function defined by

$$\mu_1(\cdot, t) := -\frac{1}{2} (t_j - t) (t - t_{j-1}) \cdot \delta_t^j \Psi \quad \text{for} \ t \in [t_{j-1}, t_j].$$

This is easily checked by using the relation $\frac{d}{d\tau} [-\frac{1}{2} (t_j - t) (t - t_{j-1})] = t - t_{j-1/2}$ to evaluate $\partial_t \mu_1$.

Next, we shall invoke $I^*_t u_h$ defined by (6.14), for which we have (6.15) and (6.16). Combining (7.11) with (1.1a) and then (6.15) yields

$$M R - M u = \partial_t R + L(t) R + f(\cdot, t, R)$$

$$= \partial_t (R - I^*_t u_h) + \partial t \mu(\cdot, t) + [\vartheta_{L,R} + \vartheta_{f,R} + \vartheta_\mu] \quad \text{in} \ Q.$$  

(7.13)

So the desired bound of type (7.8) for $R^m - u(\cdot, t_m)$, only with $(2\kappa_0 + 1)$ replaced by $2\kappa_0$, is obtained by an application of Lemma 2.4 to (7.13) with $\mu := \mu_0 + \mu_1 := (R - I^*_t u_h) + \mu_1$ and $\vartheta := \vartheta_{L,R} + \vartheta_{f,R} + \vartheta_\mu$, using (6.16) and the following two observations. First, $[R - u(\cdot, t)](\cdot, 0) = \hat{R} - \vartheta = [(\hat{R} - u_h^0) + 0] = u_h^0 - \vartheta$. Second, for $t \in [t_{j-1}, t_j)$, we have

$$|\mu_0| \leq |R^j - u_h^j| + |\hat{R}^{j-1} - \hat{u}_h^{j-1}|, \quad \tau_j |\partial_t \mu_0| \leq |R^j - u_h^j| + |\hat{R}^{j-1} - \hat{u}_h^{j-1}|,$$

$$|\mu_1| \leq \frac{1}{2} \tau_j^2 |\delta_t^j \Psi|, \quad \tau_j |\partial_t \mu_1| \leq \frac{1}{2} \tau_j^2 |\delta_t^j \Psi|,$$

where we used $\mu_0 = R - I^*_t u_h = I^*_t (R - u_h)$ and (7.12). Finally, recall (7.5) to obtain the required bound for $\| R^m - u(\cdot, t_m) \|_{\infty, \Omega}$.
Remark 7.2. It is essential for the computability of the error estimator (7.8) that some explicitly computable bounds are available for \( \eta^l, \hat{\eta}^{-1} \) and \( \| \delta^l \Psi \|_{\infty, \Omega} \). For this, it is sufficient (but not necessary) that one can explicitly compute \( \Psi^i \) and \( \hat{\Psi}^{-1} \).

Indeed, (7.5) and (7.9) show that \( \eta^l \) and \( \hat{\eta}^{-1} \) are computable via the functions \( g^i \) and \( \hat{g}^{-1} \) of (7.2b), (7.2c), while the latter can be represented as

\[
g^j(\cdot, v) = f(\cdot, t_j, v) - \Psi^i, \quad \hat{g}^{-1}(\cdot, v) = f(\cdot, t_{j-1}, v) - \hat{\Psi}^{-1}.
\]

(7.14)

The remaining terms \( \vartheta_{\varepsilon, R} \) and \( \vartheta_{f, R} \) in (6.5) can be computed using \( \vartheta_{\varepsilon, I} \) and \( \vartheta_{f, I} \) as described in Remark 6.3. For example, if \( \| \partial \varphi \|^2 f \| \leq C \) for some constant \( C \), then \( \vartheta_{f, R} \) can be effectively replaced by \( \vartheta_{f, I} \) as a calculation shows, for \( t \in (t_{j-1}, t_j] \), that \( \| \vartheta_{f, R - \vartheta_{f, I} u_h} (\cdot, t) \|_{\infty, \Omega} \leq C \tau_h^{-2} (\eta^l + \hat{\eta}^{-1}) \).


Consider a fully discrete Crank-Nicolson method for (1.3), obtained by applying the spatial discretization (5.5) to the semidiscrete problem (4.1): Find \( u_h^1 \in V_h^1 \) such that

\[
\varepsilon^2 \left( \frac{1}{2} \nabla (\varphi_h^{j-1} + u_h^j), \nabla \chi \right) + \left( \frac{1}{2}[f(\cdot, t_{j-1}, \varphi_h^{j-1}) + f(\cdot, t_j, u_h^j)] + \delta_h^j u_h^j, \chi \right)_h = 0,
\]

(7.15)

\( \forall \chi \in V_h^1 \), where \( \langle \cdot, \gamma \rangle_h \) is either exactly the inner product \( \langle \cdot, \cdot \rangle \) in \( L_2(\Omega) \), or some quadrature formula for \( \langle \cdot, \cdot \rangle \), and \( \delta^j u_h^j \) is defined by (7.1c), (7.1d).

Note that the full discretization (7.15) is of type (7.1) with \( \mathcal{L}_h^{-2} := \mathcal{L}_h^n \). For some particular cases of \( \langle \cdot, \gamma \rangle_h \), the operators \( \mathcal{L}_h^n \) and \( \mathcal{P}_h^n \) are defined as in Remarks 5.1 and 5.2 only using \( V_h^j \) instead of \( V_h \).

7.2.1. Model problem (1.3): regular regime. Let \( u \) solve problem (1.3) with \( \varepsilon = 1, \gamma \geq 0, \) posed in a bounded polyhedral spatial domain \( \Omega \subset \mathbb{R}^n \), \( n = 2, 3 \), and \( u_h \) solve the discrete problem (7.15) with \( V_h^j \) and \( \langle \cdot, \cdot \rangle_h \) defined, for each time level \( t_j \), as in §5.2. To be more specific, we let \( T_h \) be a conforming and shape-regular triangulation of \( \Omega \) made of elements \( T \), \( V_h \) be the space of continuous piecewise polynomial finite element functions of degree \( l \geq 1 \), and \( V_h^j := V_h^j \cap H_0^1(\Omega) \). We then employ a quadrature formula \( \langle \varphi, \psi \rangle_h := \sum_{T \in T_h} Q_T(\varphi \psi) \), as described in §5.2.

Corollary 7.3. Let the above numerical method be applied to problem (1.3) with \( \varepsilon = 1, \gamma \geq 0 \). Then the a posteriori error estimate of Theorem 7.1 is valid with \( \vartheta_{\varepsilon, R} = 0 \) and \( \vartheta_{f, R} \) computed as described in Remark 7.2. The quantities \( \eta^l \) and \( \hat{\eta}^{-1} \) satisfy (7.5) combined with (7.14), where \( \eta := \eta_0 \) is defined in (5.6).

7.2.2. Model problem (1.3): singularly perturbed regime in one dimension.

Now consider the regime of \( \varepsilon \ll 1 \). Let \( u \) solve the problem (1.3) with \( \varepsilon \in (0, 1], \gamma > 0, \) posed in the domain \( \Omega := (0, 1) \), and \( u_h \) solve the discrete problem (7.15) with \( V_h^j \) and \( \langle \cdot, \cdot \rangle_h \) defined, for each time level \( t_j \), as in §5.3. Thus \( V_h^j \) is the space of continuous \text{piecewise-linear} \ finite element functions on an arbitrary nonuniform mesh \( \{x_i^j\}_{i=1}^{N_j} \) with \( 0 = x_0^j < x_1^j < \cdots < x_{N_j}^j = 1 \) under absolutely no mesh regularity assumptions. We consider the two choices (5.7a) and (5.7b) of \( \langle \cdot, \cdot \rangle_h \), using the piecewise-linear Lagrange polynomial \( I_h := I_h^j \) onto \( V_h^j \).

Corollary 7.4. Let the above numerical method be applied to problem (1.3) with \( \varepsilon \in (0, 1], \gamma > 0, \Omega := (0, 1) \). Then the a posteriori error estimate of Theorem 7.1 is valid with \( \vartheta_{\varepsilon, R} = 0 \) and \( \vartheta_{f, R} \) computed as described in Remark 7.2. The quantities \( \eta^l \) and \( \hat{\eta}^{-1} \) satisfy (7.5) combined with (7.14), where \( \eta := \eta_e \) for (5.7a), and \( \eta := \eta_{e, 1, \text{m}} \) for (5.7b), while \( \eta_e \) and \( \eta_{e, 1, \text{m}} \) are defined in (5.9a) and (5.9b), respectively, in which \( I_h \) is now replaced by \( I_h^j \).
7.2.3. Computability. In view of Remark 7.2, we now further discuss the computability of the error estimator (7.8) when applied to the model problem (1.3).

(i) Suppose that in (7.15), one employs a lumped-mass quadrature \( \langle \psi, \chi \rangle_h \). Then \( P_h^j := I_h^j \) is the Lagrange interpolation operator onto \( V_h^j \), and \( L_h^j v_h = L_h^{j-1} v_h \) is easily computable for any \( v_h \in V_h^j \) by applying the normalized stiffness matrix to the column vector of nodal values \( \{ v_{h,i} \} \); see Remark 5.2. Consequently, \( \eta^j, \hat{\eta}^{j-1} \) and \( \| \hat{\Psi}^j \|_\infty, \Omega \) are explicitly computable, as described in Remark 7.2.

(ii) Let \( V_h^j \subset V_h^{j-1} \) for all \( j \). In general, (7.1a) combined with (7.9) yields \( \delta^*_h u_h^j + \frac{1}{2} (\Psi_h^{-1} + \Psi_h^j) = 0 \). In our case, one has \( \Psi_h^{-1} = \Psi_h^j \) so \( \Psi_h^j = -\Psi_h^{-1} - 2 \delta^*_h u_h^j \) is explicitly computable for \( j = 1, \ldots, M \). Note that both \( u_h^j \) and \( \hat{\Psi}^j \) are explicitly computable, as described in Remark 7.2.

(iii) In the general case, the computation of \( L_h^j v_h \) and hence of \( \hat{\Psi}^j \) may be more expensive. Roughly speaking, \( L_h^j v_h \) can be obtained by an application of \( M_j^* K_j \) to the column vector of nodal values \( \{ v_{h,i} \} \), where \( M_j \) is the mass matrix and \( K_j \) is the stiffness matrix associated with the time level \( t_j \). Note that the computation of \( u_h^j \) at each time level by the Crank-Nicolson method already involves an application of \( M_j^{-1} \). Furthermore, in some cases, an inversion of \( M_j \) can be entirely avoided by using bounds of the type \( \| \delta^*_h \Psi^j \|_\infty, \Omega \leq \| M_j^{-1} \|_\infty \cdot \| \delta^*_h (M_j \Psi^j) \|_\infty, \Omega \), where \( \| M_j^{-1} \|_\infty \) denotes the associated matrix norm (which may be bounded a priori).

8. Proof of Lemma 2.2. First, note that the Green’s function \( \mathcal{G} \) associated with our problem (1.3) in the spatial domain \( \Omega \) and the Green’s function \( \hat{\mathcal{G}} \) for the related problem \( Ma := \partial_t u - \Delta u + f(x/\varepsilon, t, \bar{u}) = 0 \) in the spatial domain \( \hat{\Omega} := \Omega \varepsilon \) satisfy \( \| \partial_t^i \hat{\mathcal{G}}(x,t; \cdot, \cdot) \|_{1, \hat{\Omega}} = \| \partial_t^i \mathcal{G}(x,t; \cdot, \cdot) \|_{1, \Omega} \) for \( k = 0, 1 \). Consequently, it suffices to prove Condition 2.1 for the case of \( \varepsilon = 1 \) with \( \kappa_0, \kappa_1 \) and \( \kappa_2 \) independent of \( |\Omega| \), so throughout the proof we set \( L^* = -\Delta \) in (2.1a).

(i) We start by proving the first bound in Condition 2.1. The Green’s function \( \mathcal{G} \) associated with \( M := \partial_t - \Delta + \gamma^2 \) in the domain \( \hat{\Omega} := \mathbb{R}^n \) can be easily obtained from the fundamental solution of the heat equation (the latter can be found, e.g., in [17, §III.3], [5, §2.3.1]). So one gets
\begin{align}
\mathcal{G}(x,t; \xi, s) = g(x - \xi, t - s), \quad \text{where} \quad g(x,t) := \frac{e^{-\gamma^2 t}}{(4\pi t)^{n/2}} \exp \left( -\frac{|x|^2}{4t} \right). \tag{8.1}
\end{align}

Next, note that, by (1.2), the coefficient \( a \) in (2.1a) satisfies \( a \geq \gamma^2 \) so an application of the maximum principle to problem (2.1) yields \( 0 \leq \mathcal{G} \leq \hat{\mathcal{G}} \). Finally, note that
\begin{align}
\mathcal{G}(x,t; \xi, s) d\xi = e^{-\gamma^2(t-s)} \psi(\zeta) d\zeta, \quad \text{where} \quad \psi(\zeta) := \frac{e^{-|\zeta|^2}}{\pi^{n/2}}, \quad \zeta := \frac{\xi - x}{2\sqrt{t - s}}, \tag{8.2}
\end{align}

As \( \int_{\mathbb{R}^n} \psi(\zeta) d\zeta = 1 \), we immediately get \( \| \mathcal{G}(x,t; \cdot, s) \|_{1, \Omega} \leq 1 \), which yields the first bound in Condition 2.1 with \( \kappa_0 = 1 \).

(ii) Next, we prove the second bound in Condition 2.1 in the linear case of \( f(x, t, z) = a(x) z + b(x, t) \) with \( \kappa_2 = 0 \). In this case, the differential operator in (2.1) does not involve \( s \), so one can invoke [2, Corollary 5] (in using this result, we imitate the proof of [3, Lemma 2.1]). In view of the above bound \( 0 \leq \mathcal{G} \leq \hat{\mathcal{G}} \), an application of [2, Corollary 5] with \( \beta = 2, \gamma = 1, c_1 = \frac{1}{4}, c_2 = \frac{3}{4} c_1 \) and \( \alpha(t) = \frac{e^{-\gamma^2 t}}{(4\pi t)^{n/2}} \) yields \( |\partial_s \mathcal{G}(x,t; \xi, s)| \leq 18 c_1 c_2 (t - s)^{-1} \alpha \left( \frac{1}{2} |t - s| \right) e^{-\gamma^2 (c_1^2 + c_2)^2 |\zeta|^2} \), which immediately implies the second bound in Condition 2.1 with \( \kappa_2 = 0 \).
with \( \kappa \). For any fixed \((x, t) \in \Omega \times (0, T)\), consider the Green’s function \( G(x, t; \xi, s) := \hat{\Gamma}(\xi, s) \) associated with the operator \( \hat{\partial}_t - \Delta + \gamma^2 \) in the domain \( \Omega \) so \( \hat{\Gamma}(\xi, s) \) satisfies a version of (2.1) with \( \kappa \) replaced by \( \gamma^2 \). Comparing this problem with the problem (2.1) for \( \Gamma \) and noting that \( \mathcal{L} = \mathcal{L}^* = \Delta \), we find that for any fixed \((x, t)\), the function \( v(\xi, s) := \hat{\Gamma}(\xi, s) - \Gamma(\xi, s) \) solves the terminal-value problem

\[
[-\partial_s - \Delta + \gamma^2] v(\xi, s) = F(\xi, s) \quad \text{for} \quad (\xi, s) \in \Omega \times [0, t), 
\]

\[
v(\xi, t) = 0 \quad \text{for} \quad \xi \in \Omega, 
\]

\[
v(\xi, s) = 0 \quad \text{for} \quad (\xi, s) \in \partial\Omega \times [0, t],
\]

where \( F(\xi, s) := |a(\xi, s) - \gamma^2| \Gamma(\xi, s) \) so, using \( \Gamma \leq \hat{\Gamma} \) and (8.1),

\[
0 \leq F(\xi, s) \leq (\gamma^2 - \gamma^2) g(x - \xi, t - s).
\]

Note that in part (ii) we have shown that \( \hat{\Gamma} \) satisfies the second bound in Condition 2.1 with \( \kappa_2 = 0 \). So it remains to show that \( v \) satisfies the second bound in Condition 2.1 with \( \kappa_1 = 0 \) and \( \kappa_2 = (\gamma^2 - \gamma^2) \kappa_2 \). This latter bound is immediately obtained by an application of Lemma 8.1 below to the terminal-value problem (8.3).

The next lemma is applied to the terminal-value problem (8.3), but it is convenient to formulate it in the context of an initial-value problem.

**Lemma 8.1.** Let \( v \) satisfy \([\hat{\partial}_t - \Delta + \gamma^2] v = F \) in \( Q \) and vanish for \( t = 0 \) and \( x \in \partial \Omega \), where \( 0 \leq F(x, t) \leq g(x - x_0, t) \) with \( g \) from (8.1) and some \( x_0 \in \Omega \). Then

\[
\int_0^T \| \hat{\partial}_t v(\cdot, t) \|_{1, \Omega} dt \leq \hat{\kappa}_2, \quad \text{where} \quad \hat{\kappa}_2 = \hat{\kappa}_2(\gamma) \quad \text{if} \quad \gamma > 0, \quad \text{while} \quad \hat{\kappa}_2 = \hat{\kappa}_2(T) \quad \text{if} \quad \gamma = 0.
\]

**Proof.** Without loss of generality, assume that \( x_0 = 0 \in \Omega \) so \( F(x, t) \leq g(x, t) \). Recall that \( \mathcal{M} g = 0 \) with \( \mathcal{M} = \hat{\partial}_t - \Delta + \gamma^2 \); this implies that \( \mathcal{M}[g] = g \), so an application of the maximum principle yields

\[
0 \leq v(x, t) \leq t g(x, t).
\]

(i) First we establish the desired estimate with \( \hat{\kappa}_2 \) that depends on \( |\Omega| \). Let \( w(x, t) := \varrho(t) v \) with the weight \( \varrho := \frac{1}{2} t^\gamma e^\frac{1}{2} t^\gamma \) so \( \varrho' = \left( \frac{1}{2} t^{\gamma - 1} + \frac{1}{2} t^\gamma \right) \varrho \). Note that

\[
\| \hat{\partial}_t v \|_{1, \Omega \times [0, T]} \leq \| \varrho^{-1} \|_{2, \Omega \times [0, T]} \| \varrho \hat{\partial}_t v \|_{2, \Omega \times [0, T]} \\
\leq \hat{\kappa}_3 |\Omega|^\frac{1}{2} \left( \| \hat{\partial}_t w \|_{2, \Omega \times [0, T]} + \| \varrho' v \|_{2, \Omega \times [0, T]} \right),
\]

where we used \( \varrho \hat{\partial}_t v = \hat{\partial}_t w - \varrho' v \) and

\[
\| \varrho^{-1} \|_{2, \Omega \times [0, T]} = |\Omega| \int_0^T t^{-\frac{\gamma}{2}} e^{-\gamma t} dt =: |\Omega| \hat{\kappa}_3^2
\]

(so \( \hat{\kappa}_3^2 \leq 3T^{1/3} \) for \( \gamma \geq 0 \), and \( \int_0^\infty t^{-\frac{\gamma}{2}} v e^{-t} dt \approx 2.7 \) implies \( \hat{\kappa}_3^2 \leq 2.7 \gamma^{-2/3} \) for \( \gamma > 0 \)). To estimate \( \hat{\partial}_t w \) in (8.5), we note that \( \mathcal{M} w = \varrho F + \varrho' v \leq \varrho g + \varrho' v \) and so apply an a priori estimate \( \| \mathcal{M} w \|_{2, \Omega \times [0, T]} \leq 3T^{1/3} \) of Chapter III:

\[
\| \hat{\partial}_t w \|_{2, \Omega \times [0, T]} \leq \| \mathcal{M} w \|_{2, \Omega \times [0, T]} \leq 3T^{1/3}
\]
(in fact, the cited estimate is given for a slightly different differential operator, but the argument also applies to $\mathcal{M}$). In view of $\dot{g} \; v \leq \left(\frac{1}{4} + \frac{1}{2}\gamma^2 \; t\right) \; g \; g$ (which follows from (8.4)), one gets

$$
\|\partial_t v\|_{L^2(\Omega \times [0,T])} \leq 2 \hat{k}_3 \|\Omega\|^{\frac{1}{2}} \|\dot{\theta} g\|_{L^2(\Omega \times [0,T])}, \quad \text{where} \quad \dot{\theta} := \left(\frac{1}{4} + \frac{1}{2}\gamma^2 \; t\right) \theta.
$$

(8.7)

Finally, a calculation using $\zeta := \frac{\sqrt{T}}{\sqrt{2\pi}}$ and $\psi(\zeta)$ from (8.2) yields

$$
\|\theta g\|^2_{L^2(\Omega \times [0,T])} \leq \int_0^T \dot{\theta}^2 e^{-\gamma^2 t} \frac{1}{2\pi T} \int_{\mathbb{R}^n} \psi(\zeta) \; d\zeta \; dt = \int_0^T \dot{\theta}^2 e^{-\gamma^2 t} \frac{1}{2\pi T} \; dt + \kappa_3^2.
$$

(this integral is convergent as $n \geq 3$). Combining this with (8.7), we arrive at the desired bound with $\hat{k}_2 := 2 \hat{k}_3 \kappa_4 \|\Omega\|^{\frac{1}{2}}$.

(ii) Now we shall show the desired result with $\hat{k}_2$ independent of $|\Omega|$ (which requires a more subtle estimation). Divide $\mathbb{R}^n$ into the non-overlapping subdomains $\Omega_0 := \{|x| < 2\}$ and $\Omega_j := \{2^j < |x| < 2^{j+1}\}$ for $j = 1, \ldots$; furthermore let $\Omega_0' := \Omega$ and $\Omega_j' := \{2^{-j-1} < |x| < 2^{j+2}\} \subset \Omega_j$. Note that

$$
|\Omega_j|^{\frac{1}{2}} \leq c_n 2^{\frac{1}{2}j}. \quad (8.8)
$$

Now we partially imitate the proof in part (i). First, note that one has the bound (8.5) with $\Omega$ replaced by $\Omega_j$ for $j = 0, 1, \ldots$. So for $j = 0$, using the results of part (i), one immediately gets

$$
\|\partial_t v\|_{L^2(\Omega \cap \Omega_0 \times [0,T])} \leq 2 \hat{k}_3 \kappa_4 \|\Omega_0\|^{\frac{1}{2}} \quad (8.9)
$$

(compare with $\hat{k}_2$ from part (i)).

For $j \geq 1$, we combine the local version of (8.5) with a local version of the global estimate (8.6) from

$$
\|\partial_t w\|_{L^2(\Omega \cap \Omega_j \times [0,T])} \leq C \left\{ \|\mathcal{M} w\|_{L^2(\Omega \cap \Omega_j' \times [0,T])} + \|w\|_{L^2(\Omega \cap \Omega_j' \times [0,T])} \right\},
$$

with the constant $C$ independent of $\Omega$ and $T$ (this estimate is obtained similarly to [10, (6.6), (6.11) of Chapter III]). Here $\mathcal{M} w$ is estimated as in part (i), while $w = \varrho \; v \leq t \varrho \; g$ by (8.4). This yields a local version of (8.7):

$$
\|\partial_t v\|_{L^2(\Omega \cap \Omega_j \times [0,T])} \leq 2 \hat{k}_3 \kappa_4 \|\Omega_j\|^{\frac{1}{2}} \hat{C} \|\dot{\theta} + t \varrho\|_{L^2(\Omega_j' \times [0,T])} \quad \text{for} \quad j \geq 1. \quad (8.10)
$$

Next, we use $\zeta := \frac{\sqrt{T}}{\sqrt{2\pi}}$ and $\psi(\zeta)$ from (8.2), and also the observation that as $j \geq 1$ so

$$
(\exp(-\frac{|x|^2}{T}))^2 \leq e^{-\frac{1}{4} t} e^{-|\zeta|^2} \leq c_n \left(\frac{1}{4}\right)^n e^{-|\zeta|^2}.
$$

So for $j \geq 1$ a calculation shows that

$$
\|\dot{\theta} + t \varrho\|_{L^2(\Omega_j' \times [0,T])} \leq c_n 4^{-jn} \int_0^T (\dot{\theta} + t \varrho)^2 e^{-\gamma^2 t} \frac{1}{4\pi T} \int_{\mathbb{R}^n} \psi(\zeta) \; d\zeta \; dt = c_n' 4^{-jn}.
$$

Combining this with (8.10) and then with (8.9) and (8.8), we arrive at

$$
\|\partial_t v\|_{L^2(\Omega \cap \Omega_j \times [0,T])} \leq 2 \hat{k}_3 c_n \left\{ \begin{array}{ll}
\hat{k}_4 & \text{for} \; j = 0, \\
\sqrt{c_n' 2^{\frac{1}{2}j}} & \text{for} \; j \geq 1.
\end{array} \right.
$$

This immediately yields the desired bound with $\hat{k}_2 := 2 \hat{k}_3 c_n [\hat{k}_4 + \sqrt{c_n' (2^{j+2} - 1)^{-1}}]$ independent of $|\Omega|$. \qed

REFERENCES