# A SECOND-ORDER OVERLAPPING SCHWARZ METHOD FOR A 2D SINGULARLY PERTURBED SEMILINEAR REACTION-DIFFUSION PROBLEM

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ABSTRACT. An overlapping Schwarz domain decomposition is applied to a semilinear reaction-diffusion equation posed in a smooth two-dimensional domain. The problem may exhibit multiple solutions; its diffusion parameter $\varepsilon^2$ is arbitrarily small, which induces boundary layers. The Schwarz method invokes a boundary-layer subdomain and an interior subdomain, the narrow subdomain overlap being of width $O(\varepsilon|\ln h|)$, where $h$ is the maximum side length of mesh elements, and the global number of mesh nodes does not exceed $O(h^{-2})$. We employ finite differences on layer-adapted meshes of Bakhvalov and Shishkin types in the boundary-layer subdomain, and lumped-mass linear finite elements on a quasiuniform Delaunay triangulation in the interior subdomain. For this iterative method, we present maximum norm error estimates for $\varepsilon \in (0, 1]$. It is shown, in particular, that when $\varepsilon \leq C|\ln h|^{-1}$, one iteration is sufficient to get second-order convergence (with, in the case of the Shishkin mesh, a logarithmic factor) in the maximum norm uniformly in $\varepsilon$. Numerical results are presented to support our theoretical conclusions.

## 1. INTRODUCTION

Consider the singularly perturbed semilinear reaction-diffusion boundary-value problem

$$(1.1a) \qquad Fu := -\varepsilon^2 \Delta u + f(x, u) = 0, \qquad x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2,$$

$$(1.1b) \qquad u(x) = g_0(x), \qquad x \in \partial\Omega,$$

where $\varepsilon$ is a small positive parameter, $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ is the Laplace operator, $f$ and $g_0$ are sufficiently smooth functions, and $\Omega$ is a bounded two-dimensional domain whose boundary $\partial\Omega$ is sufficiently smooth.

We shall examine solutions of (1.1) that exhibit sharp boundary layers, which are narrow regions where solutions change rapidly (see Figure 1). To obtain reliable numerical approximations of layer solutions in an efficient way, one has to use locally refined meshes that are fine and anisotropic in layer regions and standard outside. When multidimensional meshes of different nature are introduced in different non-overlapping subdomains (e.g., in layer regions and outside), it may be rather
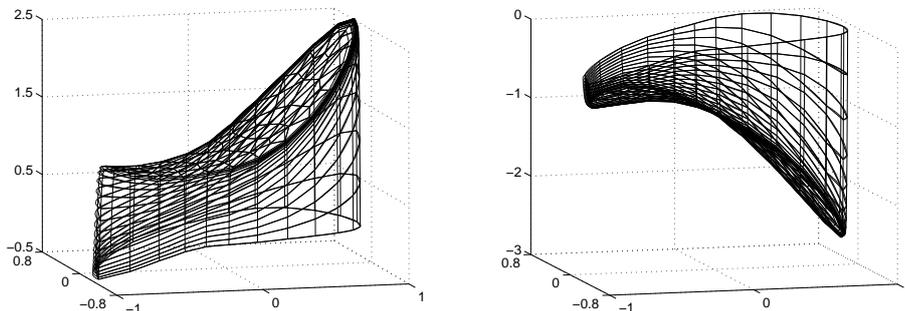
FIGURE 1. *Multiple boundary-layer solutions of model problem (5.1); in the interior subdomain $u(x) \approx z(x)$ (left) or $u(x) \approx -z(x)$ (right), where $\pm z(x)$ are stable solutions of the reduced problem (1.4).*

inconvenient to match them, while non-overlapping non-matching meshes require a special treatment (see, e.g., [6] for non-matching meshes used to solve a problem of type (1.1)). Furthermore, different discretizations of differential equations may be used in layer regions and outside, in which case they should be matched along the interface boundaries (see, e.g., [8]).

Handling non-overlapping non-matching meshes and matching different discretizations along the interface boundaries can be entirely avoided by invoking iterative overlapping domain decomposition methods of Schwarz-Chimera type; see, e.g., [20, §1.5]. Note that non-overlapping domain decomposition methods, at best, have conventional geometric rates of convergence when applied to singularly perturbed problems of type (1.1). In contrast, overlapping methods, with the subdomain overlap being as narrow as $O(\varepsilon|\ln h|)$, where $h$ is the triangulation diameter, may enjoy much faster convergence. To be more precise, we prove in this paper that one iteration is sufficient to achieve second-order accurate computed solutions when $\varepsilon \leq C|\ln h|^{-1}$, where the global number of mesh nodes does not exceed $O(h^{-2})$; see Theorems 3.9 and 4.4 for details.

We now present a continuous version of the discrete Schwarz method that we investigate. Define, for some $0 \leq a < b$, subdomains of $\Omega$:

$$\Omega_a := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > a\}, \qquad \Omega_{[a,b]} := \{x \in \Omega : a < \operatorname{dist}(x, \partial\Omega) < b\},$$

so we have $\Omega_0 = \Omega$, $\Omega_{[a,b]} = \Omega_a \backslash \Omega_b$, and $\partial\Omega_{[a,b]} = \partial\Omega_a \cup \partial\Omega_b$. Consider the **overlapping subdomains** $\Omega_\sigma$ and $\Omega_{[0,2\sigma]} = \Omega\backslash\bar\Omega_{2\sigma}$, where $\sigma > 0$ is sufficiently small so that these subdomains are well-defined and smooth; see Figure 2 (left). Let $u_\sigma$ and $u_{[0,2\sigma]}$ be solutions of the following boundary value problems

$$(1.2a) \qquad Fu_{[0,2\sigma]} = 0 \text{ for } x \in \Omega_{[0,2\sigma]}, \qquad \begin{array}{ll} u_{[0,2\sigma]}(x) = g_0(x) & \text{for } x \in \partial\Omega, \\ u_{[0,2\sigma]}(x) = g_{2\sigma}(x) & \text{for } x \in \partial\Omega_{2\sigma}, \end{array}$$

$$(1.2b) \qquad Fu_\sigma = 0 \text{ for } x \in \Omega_\sigma, \qquad u_\sigma(x) = u_{[0,2\sigma]}(x) \text{ for } x \in \partial\Omega_\sigma.$$

Here $g_0$ is from the boundary condition of our original problem (1.1), while $g_{2\sigma}$ is updated for each iteration by

$$(1.3a) \qquad g_{2\sigma}(x) = g_{2\sigma}^{[k]}(x) := \left\{ \begin{array}{ll} g_{2\sigma}^{[1]}(x) & \text{for } k = 1, \\ u^{[k-1]}(x) & \text{for } k = 2, 3, \ldots, \end{array} \right. \quad x \in \partial\Omega_{2\sigma},$$
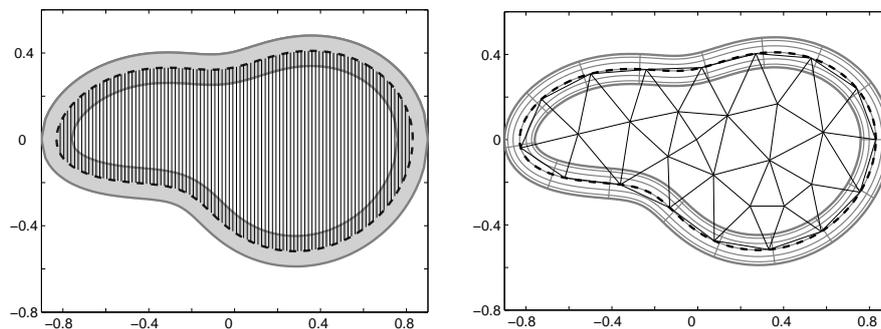
FIGURE 2. *Overlapping boundary-layer subdomain $\Omega_{[0,2\sigma]}$ and interior subdomain $\Omega_\sigma$ (left); layer-adapted tensor-product mesh in $\Omega_{[0,2\sigma]}$ and quasiuniform Delaunay triangulation in $\Omega_\sigma$ (right).*

with some suitable initial guess $g_{2\sigma}^{[1]}$. Successively solving problems (1.2a) and (1.2b) with $g_{2\sigma} = g_{2\sigma}^{[k]}$, for $k = 1, 2 \ldots$, we get the $k$**th-iteration approximations**:

$$(1.3b) \qquad u^{[k]}(x) := \begin{cases} u_{[0,2\sigma]}(x) & \text{for } x \in \bar{\Omega}_{[0,\sigma]} = \bar{\Omega} \backslash \Omega_\sigma, \\ u_\sigma(x) & \text{for } x \in \bar{\Omega}_\sigma. \end{cases}$$

We discretize the domain $\Omega_{[0,2\sigma]}$ as in Figure 2 (right), using layer-adapted tensor-product meshes of Bakhvalov and Shishkin types whose number of mesh nodes does not exceed $Ch^{-2}$. We then solve problem (1.2a) in this domain using standard finite differences in curvilinear coordinates. For problem (1.2b) in the domain $\Omega_\sigma$, we use lumped-mass linear finite elements on a quasiuniform Delaunay triangulation of diameter $h$.

When considering semilinear problems of type (1.1), it is frequently assumed in the numerical analysis literature (see, e.g., [3, 23]) that $f_u(x, u) > \gamma^2 > 0$ for all $(x, u) \in \bar{\Omega} \times \mathbb{R}$ and some positive constant $\gamma$. Under this assumption, our problem (1.1) and the associated reduced problem (1.1), i.e.

$$(1.4) \qquad f(x, z(x)) = 0 \qquad \text{for } x \in \Omega,$$

defined by setting $\varepsilon = 0$ in (1.1), have unique solutions $u$ and $z$. This global assumption is however rather restrictive. E.g., mathematical models of biological and chemical processes frequently involve problems related to (1.1) with $f(x, u)$ that is non-monotone with respect to $u$. Therefore in the most important case of $\varepsilon \leq Ch$ (see §3), we examine problem (1.1) under the following weaker **assumptions** also used in [5, 18]:

- it has a *stable reduced solution*, i.e., there exists a sufficiently smooth solution $z$ of (1.4) such that

$$(1.5a) \qquad f_u(x, z) > \gamma^2 > 0 \qquad \text{for all } x \in \bar{\Omega};$$

- the boundary condition $g_0$ on $\partial\Omega$, also denoted $\partial\Omega_0$, satisfies the assumption, with $d = 0$, that

$$(1.5b) \qquad \int_{z(x)}^{v} f(x, s)\, ds > 0 \qquad \text{for all } v \in \big(z(x), g_d(x)\big]', \qquad x \in \partial\Omega_d.$$

Here the notation $(a, b]'$ is defined to be $(a, b]$ when $a < b$ and $[b, a)$ when $a > b$, while $(a, b]' = \emptyset$ when $a = b$.

Note that if $g_0(x) \approx u_0(x)$, then (1.5b) follows from (1.5a) combined with (1.4), while if $g_0(x) = u_0(x)$ at some point $x \in \partial\Omega$, then (1.5b) does not impose any restriction on $g_0$ at this point. (Problem (1.2a) should also satisfy (1.5b) with $d = 2\sigma$; otherwise the nonlinear problem (1.2a) may have no solutions. When $\sigma$ is small, one can simply take $g_{2\sigma}^{[1]} \approx g_0$.)

Conditions (1.5) intrinsically arise from the asymptotic analysis of problem (1.1) and guarantee that there exists a boundary-layer solution $u$ such that $u \approx u_0$ in the interior part of $\Omega$, while the boundary layer is of width $O(\varepsilon |\ln \varepsilon|)$; see, e.g., [5, 18, 7]. Note that assumption (1.5a) is local, i.e. the reduced problem (1.4) is permitted to have more than one stable solution. Furthermore, if multiple stable solutions of the reduced problem satisfy (1.5), then problem (1.1) has multiple boundary-layer solutions.

The discrete Schwarz method that we consider is a domain decomposition version of the numerical method of [8], where problem (1.1), (1.5) was posed in a smooth two-dimensional domain, and it was shown that one gets second-order convergence in the discrete maximum norm under the condition $\varepsilon \leq Ch$. Note that in the present paper we give convergence estimates for all $\varepsilon \in (0, 1]$. In one dimension, similar domain decomposition methods using layer-adapted meshes have been analyzed for linear [16, 25] and semilinear [10] equations of type (1.1); in particular, faster convergence of the algorithm for small values of $\varepsilon$ was addressed in [10, 25]. The numerical analysis literature addressing problems of type (1.1) posed in various two-dimensional domains is discussed in [8]. In particular, the semilinear equation (1.1) under the condition $f_u > \gamma^2 > 0$ was considered in [3, 23], while linear equations of this type were considered in [1, 4, 14, 17].

The paper is organized as follows. In §2 we introduce independent meshes and discretizations in the subdomains $\Omega_{[0,2\sigma]}$ and $\Omega_\sigma$, and then present a discrete version of the continuous Schwarz method (1.2), (1.3). The errors in the discrete Schwarz method are estimated in two regimes: for $\varepsilon \leq Ch$ in §3 and $\varepsilon \geq Ch$ in §4. So throughout §3 we let $\varepsilon \leq Ch$. Asymptotic properties of solutions in particular subdomains are discussed, and then appropriate sub- and super-solutions are constructed in §3.1. Errors in the continuous and discrete Swartz methods are estimated, respectively, in §3.2 and §§3.3–3.4. In particular, in §3.4 we employ discrete sub- and super-solutions, whose basic properties are sketched in §3.3. Throughout §4 we let $\varepsilon \geq Ch$ and make a simplifying assumption that $f_u > \gamma^2 > 0$. Then errors in the continuous and discrete Schwarz method are estimated, respectively, in §4.1 and §§4.2–4.3. In §4.4 we get an auxiliary stability result for the finite-difference discrete operator in $\Omega_{[0,2\sigma]}$, which is used to establish supra-convergence in this subdomain. In §4.5 we get another auxiliary result by extending a maximum norm error estimate for the standard finite element method [23] to its lumped-mass version. Finally, in §5, some numerical results illustrate our theoretical conclusions.

*Notation.* We let $C$ denote a generic positive constant that may take different values in different formulas, but is independent of $\varepsilon$, $h$ and the number of iterations taken by the Schwarz algorithm. A subscripted $C$ (e.g., $C_1$) denotes a positive constant that is independent of $\varepsilon$, $h$ and the number of iterations, but takes a fixed value. For any two quantities $y$ and $z$, the notation $y = O(z)$ means $|y| \leq Cz$.

## 2. Discrete Schwarz method. Discretizations in particular subdomains

2.1. **Local curvilinear coordinates.** Let the boundary $\partial\Omega$ be parametrized by

$$x_1 = \varphi(l), \qquad x_2 = \psi(l), \qquad 0 \le l \le L,$$

where $(\varphi(0), \psi(0)) = (\varphi(L), \psi(L))$ and as $l$ increases, the domain remains on the left. Any functions that are defined for $l$ beyond $[0, L]$ should be understood as extended $L$-periodically. We shall use the magnitude $\tau > 0$ of the tangent vector $(\varphi', \psi')$ and the curvature $\kappa$ of the boundary at $(\varphi(l), \psi(l))$ that are defined by

$$\tau = \sqrt{\varphi'^2 + \psi'^2}, \qquad \kappa = \kappa(l) = \frac{\varphi'\psi'' - \psi'\varphi''}{\tau^3}.$$

In a narrow neighbourhood of $\partial\Omega$ that will be specified later, introduce the curvilinear local coordinates $(r, l)$ by

$$(2.1) \qquad\qquad x_1 = \varphi(l) + r n_1(l), \qquad x_2 = \psi(l) + r n_2(l),$$

where $(n_1, n_2)$ is the inward unit normal to $\partial\Omega$ at $(\varphi(l), \psi(l))$, i.e., it is orthogonal to the tangent vector $(\varphi', \psi')$ and is defined by

$$n_1 = \frac{-\psi'}{\tau}, \qquad n_2 = \frac{\varphi'}{\tau}.$$

Since $\partial\Omega$ is smooth, there exists a sufficiently small constant $C_1$ such that in the subdomain $\bar{\Omega}_{2C_1}$ the new coordinates are well-defined, the mapping $(r, l) \mapsto (x_1, x_2)$ is one-to-one and invertible, and, furthermore,

$$(2.2) \qquad\qquad \mathrm{dist}(x, \partial\Omega) = r \qquad \text{for all } x \in \bar{\Omega}_{2C_1}.$$

Throughout the paper we shall use a smooth positive cut-off function $\omega(x)$ that equals 1 for $r \le C_1$ and vanishes in $\bar{\Omega}\backslash\bar{\Omega}_{2C_1}$.

Note [8, Lemma 2.1] that the curvilinear coordinates (2.1) are orthogonal, and for the Laplace operator we have

$$(2.3) \qquad \triangle u = \eta^{-1}\frac{\partial}{\partial r}\left(\eta\frac{\partial u}{\partial r}\right) + \zeta\frac{\partial}{\partial l}\left(\zeta\frac{\partial u}{\partial l}\right), \quad \text{where } \eta := 1 - \kappa r, \ \zeta := (\tau\eta)^{-1}.$$

2.2. **Layer-adapted meshes.** To discretize the continuous Schwarz method in $(1.2), (1.3)$, we now introduce independent meshes in the overlapping subdomains $\Omega_{[0,2\sigma]}$ and $\Omega_\sigma$, to which we shall refer, respectively, as the boundary-layer subdomain and the interior subdomain; see Figure 2.

In the **interior subdomain** $\Omega_\sigma$ introduce a quasiuniform Delaunay triangulation of some small diameter $h \in (0, \frac{1}{2})$, i.e. the maximum side length of any triangle is at most $h$, the area of any triangle is bounded below by $Ch^2$, and the sum of the angles opposite to any edge is less than or equal to $\pi$ (while any angle opposite to $\partial\Omega_\sigma$ does not exceed $\pi/2$). Let the union of all the triangles define a polygonal domain $\Omega_\sigma^h$ whose boundary vertices lie on $\partial\Omega_\sigma$.

The **boundary-layer subdomain** $\Omega_{[0,2\sigma]}$ is the rectangle $(0, 2\sigma) \times [0, L]$ in the coordinates $(r, l)$. Hence in this subdomain introduce the tensor-product mesh $\{(r_i, l_j), i = 0, \ldots, 2N, j = 0, \ldots N_l + 1\}$, where $r_N = \sigma$ and, as usual, $r_0 = 0$, $r_{2N} = 2\sigma$, $l_0 = 0$, and $l_{N_l} = L$, while $l_{N_l+1} = l_1 + L$. Furthermore, let $\{l_j\}$ be a quasiuniform mesh on $[0, L]$, i.e., $C^{-1}h \le l_j - l_{j-1} \le Ch$. The choice of the

*layer-adapted* mesh $\{r_i\}$ on $[0, 2\sigma]$ is crucial and will be discussed later; see (a),(b). Now assume only that $r_i - r_{i-1} \leq h$ and

$$(2.4) \qquad C^{-1}h^{-1} \leq N \leq Ch^{-1}.$$

Note that we do **not require** that the interior and layer meshes have the same sets of nodes on $\partial\Omega_\sigma$. Thus information will be exchanged between $\Omega_{[0,2\sigma]}$ and $\Omega_\sigma$ using piecewise linear/bilinear computed solutions in these subdomains.

We focus on two particular choices of $\{r_i\}$:

**2.2(a) Shishkin mesh.** [24] Set $\sigma = \sigma_S := \min\{2\gamma^{-1}\varepsilon \ln N, \frac{1}{2}C_1\}$ and introduce a uniform mesh $\{r_i\}_{i=0}^{2N}$ on $[0, 2\sigma]$, i.e., $r_i - r_{i-1} = \sigma/N = 2\gamma^{-1}\varepsilon N^{-1}\ln N$.

**2.2(b) Bakhvalov mesh.** [2] Let $\rho := \varepsilon$ and $\sigma_B := 2\gamma^{-1}\varepsilon|\ln\rho|$. We now set $\sigma := \max\{\sigma_B, \sigma_S\}$ and $\bar\rho := (1-\rho) + [\sigma - \sigma_B]\rho/(2\gamma^{-1}\varepsilon)$, and define the meshpoints by $r_i := r\big([1 - \varepsilon]\,i/N\big)$ for $i = 0\ldots, 2N$, where $r(t) \in C^1[0, 2\bar\rho]$ is given by

$$r(t) := \begin{cases} -2\gamma^{-1}\varepsilon \ln(1-t) & \text{for } t \in [0, 1-\rho], \\ \sigma_B + [t - (1-\rho)](2\gamma^{-1}\varepsilon)/\rho & \text{for } t \in [1-\rho, \bar\rho], \\ 2\sigma - r(2\bar\rho - t) & \text{for } t \in [\bar\rho, 2\bar\rho], \end{cases}$$

so that the sub-mesh $\{r_i\}_{i=N}^{2N}$ reflects the sub-mesh $\{r_i\}_{i=1}^{N}$ in $r = \sigma$.

*Remark* 2.1. For the mesh §2.2(a) we always have $2\sigma \leq C_1$, so this mesh is always well-defined. The mesh §2.2(b) is well-defined provided that $\varepsilon \leq e^{-1}$ and $2\sigma_B \leq C_1$; otherwise we have $\varepsilon > C$ for some constant $C$, and, imitating [2], we extend the mesh definition §2.2(b) by using the mesh §2.2(a) with $\sigma := \frac{1}{2}C_1$. In general, when $\varepsilon > C$, i.e. our problem is not singularly perturbed, one can simply use linear finite elements on a quasiuniform Delaunay triangulation of the whole domain $\bar\Omega$ [23]. Note that one can replace $\ln N$ by $\ln(C'N)$ in the definition of $\sigma_S$ in §2.2(a), and can also use $\rho = C''\varepsilon$ for the mesh §2.2(b), with some arbitrary constants $C'$, $C''$.

*Remark* 2.2. In the mesh definitions §2.2(a) and §2.2(b) the constant $\gamma$ from (1.5a) can be replaced by an arbitrary constant $\tilde\gamma \in (0, \gamma_0)$, where $\gamma_0$ is from Lemma 3.1; see Remark 3.2.

2.3. **Discretization in the boundary-layer subdomain.** Recall that $\Omega_{[0,2\sigma]}$ is the rectangle $(0, 2\sigma) \times [0, L]$ in the coordinates $(r, l)$. Hence rewrite problem (1.2a) in the $(r, l)$ coordinates using (2.3), and then discretize it using the standard finite differences on the tensor-product mesh $\{(r_i, l_j)\}$ as follows. For $i = 1, \ldots, 2N - 1$, $j = 1, \ldots, N_l$, set

$$(2.5) \quad \begin{aligned} F_{[0,2\sigma]}^h U_{ij} &:= -\varepsilon^2 \eta_{ij}^{-1} D_r[\tilde\eta_{ij} D_r^- U_{ij}] - \varepsilon^2 \zeta_{ij} D_l[\tilde\zeta_{ij} D_l^- U_{ij}] + f(x_{ij}, U_{ij}) = 0, \\ U_{i,0} &= U_{i,N_l}, \quad U_{i,1} = U_{i,N_l+1}, \quad U_{0,j} = g_0(x_{0,j}), \quad U_{2N,j} = g_{2\sigma}(x_{2N,j}). \end{aligned}$$

Here $U_{ij}$ is the discrete computed solution at the mesh node $x_{ij} \in \bar\Omega_{[0,2\sigma]}$,

$$D_r^- v_{ij} := \frac{v_{ij} - v_{i-1,j}}{r_i - r_{i-1}}, \qquad D_r v_{ij} := \frac{v_{i+1,j} - v_{ij}}{(r_{i+1} - r_{i-1})/2},$$

$$D_l^- v_{ij} := \frac{v_{ij} - v_{i,j-1}}{l_j - l_{j-1}}, \qquad D_l v_{ij} := \frac{v_{i,j+1} - v_{ij}}{(l_{j+1} - l_{j-1})/2},$$

and

$$\eta_{ij} := \eta(r_i, l_j), \qquad \zeta_{ij} := \zeta(r_i, l_j), \qquad x_{ij} := x(r_i, l_j),$$
$$\tilde\eta_{ij} := \eta(r_{i-1/2}, l_j), \qquad \tilde\zeta_{ij} := \zeta(r_i, l_{j-1/2}).$$

The computed solution $U_{[0,2\sigma]}(x)$ for $x \in \bar{\Omega}_{[0,2\sigma]}$ is obtained by the standard bilinear interpolation of $U_{ij}$ in the $(r,l)$ coordinates on the tensor-product mesh $\{(r_i, l_j)\}$.

## 2.4. Discretization in the interior subdomain.

We discretize problem (1.2b) in $\Omega_\sigma$ using *lumped-mass* linear finite elements. Let $S^h \subset W_2^1(\Omega_\sigma^h)$ be the standard finite element space of continuous functions that are linear on each of the triangles of our mesh in $\bar{\Omega}_\sigma^h$. Let $\{q_i\}$ be the set of mesh nodes of the mesh in $\bar{\Omega}_\sigma^h$. Now we require the computed solution $U_\sigma \in S^h$ to satisfy $U_\sigma(q_j) = U_{[0,2\sigma]}(q_j)$ at each boundary mesh node $q_j \in \partial\Omega_\sigma^h$, and also

$$(2.6) \qquad F_\sigma^h U_\sigma(q_i) := \frac{\varepsilon^2}{\langle 1, \chi_i \rangle} \langle \nabla U_\sigma, \nabla \chi_i \rangle + f(q_i, U_\sigma(q_i)) = 0 \qquad \forall\, q_i \in \Omega_\sigma^h.$$

where $q_i$ is an interior mesh node in $\Omega_\sigma^h$, and $\chi_i \in S^h$ is the standard nodal basis function (i.e. $\chi_i(q_j)$ equals 1 if $i = j$, and 0 otherwise). The notation $\langle \cdot, \cdot \rangle$ is used for the inner product in $L_2(\Omega_\sigma^h)$. Note that the finite element method (2.6) uses the lumped-mass discretization of the integral involving $f$, which is more evident if (2.6) is multiplied by $\langle 1, \chi_i \rangle$. It is important to also note that as a Delaunay triangulation is used, the discretization of the operator $-\triangle$ in (2.6) is associated with an $M$-matrix (see, e.g., [30, §2]).

## 2.5. Discrete Schwarz approximations.

We now imitate (1.3). The boundary condition $g_{2\sigma}$ in (2.5) is updated for each iteration by

$$(2.7a) \quad g_{2\sigma}(x) = g_{2\sigma}^{[k]}(x) := \begin{cases} g_{2\sigma}^{[1]}(x) & \text{for } k = 1, \\ U^{[k-1]}(x) = U_\sigma(x) & \text{for } k = 2, 3, \ldots, \end{cases} \quad x \in \partial\Omega_{2\sigma},$$

with some suitable initial guess $g_{2\sigma}^{[1]}$. Successively solving problems (2.5) and (2.6) with $g_{2\sigma} = g_{2\sigma}^{[k]}$, for $k = 1, 2\ldots$, we get the **$k$th-iteration approximations**:

$$(2.7b) \qquad\qquad U^{[k]}(x) := \begin{cases} U_{[0,2\sigma]}(x) & \text{for } x \in \bar{\Omega} \backslash \bar{\Omega}_\sigma^h, \\ U_\sigma(x) & \text{for } x \in \bar{\Omega}_\sigma^h. \end{cases}$$

Strictly speaking, (2.7a) is well-defined for $k \geq 2$ only if $\Omega_{2\sigma} \subset \Omega_\sigma^h$, while we have $\Omega_{2\sigma} \subset \Omega_\sigma$, so some extrapolation of $U_\sigma$ from $\bar{\Omega}_\sigma^h$ onto $\bar{\Omega}_\sigma \backslash \bar{\Omega}_\sigma^h$ may be employed. In practice, no extrapolation is required as $\text{dist}(\partial\Omega_{2\sigma}, \partial\Omega_\sigma) = \sigma \geq C\varepsilon \ln h$ and $\text{dist}(\partial\Omega_\sigma^h, \partial\Omega_\sigma) = O(h^2)$. Consequently, whenever $\varepsilon \geq Ch^2$, relation (2.7a) is well-defined; otherwise, as we shall show in Theorem 3.9, one iteration of the discrete Schwarz method is sufficient.

*Remark* 2.3. One advantage of the above domain decomposition method is related to the condition numbers of the associated linear systems. Note that the condition number (roughly, the ratio of the largest eigenvalue to the smallest) for a similar method without domain decomposition [8] is expected to be close to $O(h^{-2})$ (for a Shishkin mesh in one dimension, this is shown in [21]). For the finite differences in the boundary-layer subdomain $\Omega_{[0,2\sigma]}$, we expect a similar condition number (in view of the eigenvalues for a finite difference method obtained in [22, §II.3.2]). But for the finite elements in the interior subdomain $\Omega_\sigma$, in view of [26, Theorem 5.1], one expects a much smaller condition number of $O(\varepsilon^2 h^{-2} + 1)$.

### 3. Maximum norm error analysis for $0 < \varepsilon \le Ch$

3.1. **Continuous problems in particular subdomains. Sub- and super-solutions.** As our method involves the numerical solution of the differential equation (1.1a) in certain subdomains, we shall first consider this equation and asymptotic properties of its solutions in arbitrary particular subdomains $\Omega_a$ and $\Omega_{[a,b]}$.

Let $u_a(x)$ and $u_{[a,b]}(x)$ be solutions of the problems (compare with (1.2))

$$(3.1) \qquad Fu_a = 0 \text{ for } x \in \Omega_a, \quad u_a(x) = g_a(x) \text{ for } x \in \partial\Omega_a,$$

$$(3.2) \qquad Fu_{[a,b]} = 0 \text{ for } x \in \Omega_{[a,b]}, \qquad \begin{aligned} u_{[a,b]}(x) &= g_a(x) \quad \text{for } x \in \partial\Omega_a, \\ u_{[a,b]}(x) &= g_b(x) \quad \text{for } x \in \partial\Omega_b. \end{aligned}$$

Here $0 \le a < b \le C_1$ so that the domains $\Omega_a$ and $\Omega_{[a,b]}$ are well-defined. Only to avoid considering cases, we assume that $g_d \ge u_0(d)$ for $d = a, b$.

Then solutions $u_a$ and $u_{[a,b]}$ of problems (3.1) and (3.2) typically exhibit boundary layers, and their standard first-order **asymptotic expansions** $u_{\text{as};\,a}$ and $u_{\text{as};\,[a,b]}$ are given [5, 8, 18] by

$$(3.3\text{a}) \qquad u_{\text{as};\,a}(x) = z(x) + \big[v_{0;\,a}(\xi^+, l) + \varepsilon v_{1;\,a}(\xi^+, l)\big]\,\omega(x),$$

$$(3.3\text{b}) \qquad u_{\text{as};\,[a,b]}(x) = z(x) + \big[v_{0;\,a}(\xi^+, l) + \varepsilon v_{1;\,a}(\xi^+, l)\big]$$
$$\qquad\qquad\qquad\qquad + \big[v_{0;\,b}(\xi^-, l) + \varepsilon v_{1;\,b}(\xi^-, l)\big].$$

(Note that the cut-off function $\omega = 1$ in $\Omega_{[a,b]}$, so $u_{\text{as};\,[a,b]} - u_{\text{as};\,a} = [v_{0;\,b} + \varepsilon v_{1;\,b}]$.) Here the components $[v_{0;\,a} + \varepsilon v_{1;\,a}]$ and $[v_{0;\,b} + \varepsilon v_{1;\,b}]$ describe the boundary layers along $\partial\Omega_a$ and $\partial\Omega_b$, respectively. They use the stretched variables $\xi^+ = \xi_a^+ := \frac{r-a}{\varepsilon}$ and $\xi^- = \xi_b^- := \frac{b-r}{\varepsilon}$. More generally,

$$\xi_d^\pm := \pm(r - d)/\varepsilon.$$

When there is no ambiguity, as, e.g., in (3.3), the notation $\xi^\pm$ is used for $\xi_a^+$ and $\xi_b^-$. Note that $\xi_d^\pm = 0$ corresponds to $r = d$, and $\xi_d^+$ has the same positive direction as the $r$-axis, while $\xi_d^-$ has the opposite direction.

The boundary-layer functions $v_{0;\,d}(\xi^\pm, l)$ and $v_{1;\,d}(\xi^\pm, l)$ in (3.3), with $d = a, b$, satisfy the ordinary differential equations

$$(3.4\text{a}) \qquad -\big(\tfrac{\partial}{\partial\xi^\pm}\big)^2 v_{0;\,d} + f(\bar{x}_d, z(\bar{x}_d) + v_{0;\,d}) = 0,$$

$$(3.4\text{b}) \qquad \big[-\big(\tfrac{\partial}{\partial\xi^\pm}\big)^2 + f_u(\bar{x}_d, z(\bar{x}_d) + v_{0;\,d})\big] v_{1;\,d} = \mp Q_d(\xi^\pm, l),$$

with the boundary conditions

$$(3.4\text{c}) \qquad v_{0;\,d}(0, l) = g_d(\bar{x}_d) - z(\bar{x}_d), \quad v_{1;\,d}(0, l) = v_{0;\,d}(\infty, l) = v_{1;\,d}(\infty, l) = 0,$$

where the variable $l$ appears as a parameter, and

$$(3.4\text{d}) \qquad \bar{x}_d = \bar{x}_d(l) := \big(\varphi(l) + dn_1(l), \psi(l) + dn_2(l)\big) \in \partial\Omega_d,$$
$$Q_d(\xi^\pm, l) := \xi^\pm \tfrac{d}{dr} f(x, z(x) + s)\big|_{x:=\bar{x}_d;\, s=v_{0;\,d}} + \tfrac{\kappa}{1-\kappa d}\big(\tfrac{\partial}{\partial\xi^\pm} v_{0;\,d}\big).$$

Note that relations (3.4) either *all* use $\xi^+ = \xi_d^+$ and so define $v_{0;\,d}(\xi^+)$ and $v_{1;\,d}(\xi^+)$, or all use $\xi^- = \xi_d^-$ and then define $v_{0;\,d}(\xi^-)$ and $v_{1;\,d}(\xi^-)$. Note also that $Q_d$ in (3.4d) is obtained using $\eta^{-1}\tfrac{\partial\eta}{\partial r}\big|_{r=d} = \tfrac{\kappa}{1-\kappa d}$.

To construct sub- and super-solutions for problems (3.1) and (3.2), we need a perturbed version $\tilde{v}_{0;d} = \tilde{v}_{0;d}(\xi^{\pm}, l; p)$ of $v_{0;d}$, which, for $d = a, b$, is defined by generalizing equations (3.4a) with the boundary conditions (3.4c):

(3.5)
$$-\left(\tfrac{d}{d\xi^{\pm}}\right)^2 \tilde{v}_{0;d} + f(\bar{x}_d, z(\bar{x}_d) + \tilde{v}_{0;d}) = p\tilde{v}_{0;d},$$
$$\tilde{v}_{0;d}(0, l; p) = g_d(\bar{x}_d) - z(\bar{x}_d), \qquad \tilde{v}_{0;d}(\infty, l; p) = 0.$$

Clearly, we have $\tilde{v}_{0;d}(\xi^{\pm}, l; 0) = v_{0;d}(\xi^{\pm}, l)$ for $d = a, b$. Now, by replacing all $v_{0;d}$ by their perturbations $\tilde{v}_{0;d}$ and introducing a perturbation term $C_0 p$, we get **perturbed** versions $\beta_a$ and $\beta_{[a,b]}$ of the **asymptotic expansions** of (3.3):

(3.6a) $\qquad \beta_a(x; p) \;\; = \;\; z(x) + \left[\tilde{v}_{0;a}(\xi^+; p) + \varepsilon v_{1;a}(\xi^+)\right] \omega(x) + C_0 p,$

(3.6b) $\qquad \beta_{[a,b]}(x; p) \;\; = \;\; z(x) + \left[\tilde{v}_{0;a}(\xi^+; p) + \varepsilon v_{1;a}(\xi^+)\right]$
$$\qquad\qquad\qquad\qquad + \left[\tilde{v}_{0;b}(\xi^-; p) + \varepsilon v_{1;b}(\xi^-)\right] + C_0 p.$$

Here $p$ is a small real number that will be chosen later and is typically $o(h)$; for some small $p > 0$ the functions $\beta_{[a,b]}(x; \pm p)$ will serve as sub- and super-solutions.

The following lemma combines the results of [11, Lemma 2.1 and (2.15)]; the proof uses dynamical systems in the analysis of problems (3.4) and (3.5).

**Lemma 3.1.** *Set $\gamma_0^2 = \min\limits_{x \in \partial\Omega_a \cup \partial\Omega_b} f_u(x, z(x)) > \gamma^2$, where $\gamma > 0$ is from (1.5a). Given assumption (1.5b) with $d = a, b$, there exists $p_0 \in (0, \gamma_0^2)$ such that for all $|p| \leq p_0$, problems (3.4) and (3.5) have solutions $v_{0;a}(\xi^+, l)$, $v_{0;b}(\xi^-, l)$, $v_{1;a}(\xi^+, l)$, $v_{1;b}(\xi^-, l)$, $\tilde{v}_{0;a}(\xi^+, l; p)$ and $\tilde{v}_{0;b}(\xi^-, l; p)$. We also have*

(3.7) $\qquad v_{0;d}(\xi^{\pm}, l) \geq 0, \qquad \frac{\partial}{\partial p}\tilde{v}_{0;d}(\xi^{\pm}, l; p) \geq 0, \qquad where \quad d = a, b.$

*Furthermore, for an arbitrarily small but fixed $\delta \in (0, \gamma_0 - \sqrt{p_0})$, there is a positive constant $C_\delta$ such that*

(3.8) $\quad \left|\left(\tfrac{\partial}{\partial\xi^{\pm}}\right)^k \tilde{v}_{0;d}\right| + \left|\left(\tfrac{d}{d\xi^{\pm}}\right)^k v_{1;d}\right| + \left|\tfrac{\partial}{\partial p}\tilde{v}_{0;d}\right| \leq C_\delta\, e^{-(\gamma_0 - \sqrt{p_0} - \delta)\, \xi^{\pm}} \max\limits_{\partial\Omega_d} |g_d - z|$

*for $d = a, b$ and $\xi^{\pm} \geq 0$, $k = 0, 1, \ldots, 4$.*

*Remark* 3.2. As $\gamma_0 > \gamma$, choosing $p_0$ and $\delta$ in Lemma 3.1 sufficiently small, we can make $\gamma_0 - \sqrt{p_0} - \delta$ in (3.8) satisfy $\gamma_0 - \sqrt{p_0} - \delta > \gamma$, which then yields $e^{-(\gamma_0 - \sqrt{p_0} - \delta)\, \xi^{\pm}} \leq e^{-\gamma\, \xi^{\pm}}$ Consequently, we have

(3.9) $\qquad e^{-(\gamma_0 - \sqrt{p_0} - \delta)\, \xi_a^+} \leq e^{-\gamma\, (x-a)/\varepsilon}, \qquad e^{-(\gamma_0 - \sqrt{p_0} - \delta)\, \xi_b^-} \leq e^{-\gamma\, (b-x)/\varepsilon}.$

Similarly, we can choose $p_0$ and $\delta$ so that $\gamma_0 - \sqrt{p_0} - \delta > \tilde{\gamma}$ for any $\tilde{\gamma} < \gamma_0$, which then yields (3.9) with $\gamma$ replaced by $\tilde{\gamma}$.

Next we investigate the perturbed asymptotic expansions $\beta_a(x; p)$ and $\beta_{[a,b]}(x; p)$.

**Lemma 3.3.** *Under the assumptions of Lemma 3.1, the functions $\beta_a(x; p)$ and $\beta_{[a,b]}(x; p)$ of (3.6) satisfy*

(3.10) $\qquad F\beta_a(x; p) = C_0 p\, f_u(x, z) + [1 + C_0 \lambda_a]\, p\, v_{0;a}(\xi^+, l) + O(\varepsilon^2 + p^2)$

*for $x \in \Omega_a$, where $\lambda_a = \lambda_a(x) := f_{uu}(x, z + \vartheta v_{0;a})$ and $\vartheta = \vartheta(x) \in (0, 1)$, and*

(3.11) $\qquad F\beta_{[a,b]}(x; p) \;\; = \;\; C_0 p\, f_u(x, z) + [1 + C_0 \lambda_{[a,b]}]\, p\, [v_{0;a}(\xi^+, l) + v_{0;b}(\xi^-, l)]$
$$\qquad\qquad\qquad + O(\varepsilon^2 + p^2 + e^{-\gamma(b-a)/(2\varepsilon)})$$

*for $x \in \Omega_{[a,b]}$, where $\lambda_{[a,b]} = \lambda_{[a,b]}(x) := f_{uu}(x, z + \hat{\vartheta}[v_{0;a} + v_{0;b}])$, $\hat{\vartheta} = \hat{\vartheta}(x) \in (0, 1)$.*

*Proof.* The first assertion (3.10) (for $a = 0$) is given by [8, Lemma 2.8].

Next, consider $x \in \Omega_{[a,(a+b)/2]}$. In view of (3.10), to obtain the second assertion (3.11) in this case, it suffices to prove the bound $|F\beta_{[a,b]}(x;p) - F\beta_a(x;p)| \leq Ce^{-\gamma(b-r)/\varepsilon} \leq Ce^{-\gamma(b-a)/(2\varepsilon)}$ for all $r \in (a,(a+b)/2]$ and similar bounds for $|\lambda_{[a,b]} - \lambda_a|$ and $|v_{0;b}|$. In particular, the first of the required bounds follows from

$$F\beta_{[a,b]}(x;p) - F\beta_a(x;p) = -\left(\tfrac{d}{d\xi^+}\right)^2[\beta_{[a,b]} - \beta_a] + O(\beta_{[a,b]} - \beta_a)$$

combined with $\beta_{[a,b]} - \beta_a = \tilde{v}_{0;b} + \varepsilon v_{1;b}$, for which we have (3.8), (3.9) and (2.2).

For $x \in \Omega_{[(a+b)/2,b]}$, estimate (3.11) is obtained similarly, but using a version of (3.10), in which $\beta_a$ is replaced by $\beta_{[a,b]} - [\tilde{v}_{0;a} + \varepsilon v_{1;a}]$, so $\lambda_a$ and $v_{0;a}(\xi^+,l)$ are replaced by $f_{uu}(x, z + \vartheta v_{0;b})$ and $v_{0;b}(\xi^-,l)$, respectively. $\qquad \square$

**Corollary 3.4.** *Let $b - a \geq (4/\gamma)\varepsilon|\ln(Ch)|$. Then there exists positive $C_0$ and $C_2$ such that the functions $\beta_a(x;p)$ and $\beta_{[a,b]}(x;p)$ of (3.6), for all $0 < |p| \leq p_0$, satisfy*

$$\begin{aligned}
(\operatorname{sgn} p)\, F\beta_a(x;p) &\geq& C_0|p|\gamma^2 - C_2(\varepsilon^2 + p^2), \\
(\operatorname{sgn} p)\, F\beta_{[a,b]}(x;p) &\geq& C_0|p|\gamma^2 - C_2(\varepsilon^2 + p^2 + h^2),
\end{aligned}$$

*Proof.* Recall (1.5a) and that $v_{0;a}(\xi^+,l) \geq 0$ and $v_{0;b}(\xi^-,l) \geq 0$, by (3.7). Now invoke Lemma 3.3 choosing a positive $C_0$ that does not exceed $\min_{x \in \bar{\Omega}_a} |\lambda_a(x)|^{-1}$ and $\min_{x \in \bar{\Omega}_{[a,b]}} |\lambda_{[a,b]}(x)|^{-1}$ so that $1 + C_0\lambda_a(x) \geq 0$ and $1 + C_0\lambda_{[a,b]}(x) \geq 0$. Finally note that $e^{-\gamma(b-a)/(2\varepsilon)} \leq Ch^2$. $\qquad \square$

**Lemma 3.5.** *Let $0 \leq a < b \leq C_1$ and $b - a \geq (4/\gamma)\varepsilon|\ln(Ch)|$. Let $f$ satisfy assumption (1.5a), and the boundary data $g_d$, where $d = a, b$, of problems (3.1) and (3.2) satisfy (1.5b). Then there is a sufficiently small positive constant $\tilde{C}_0$ such that if $\varepsilon \leq \tilde{C}_0$, then problem (3.1) has a solution $u_a$, and if $\varepsilon + h \leq \tilde{C}_0$, then problem (3.2) has a solution $u_{[a,b]}$, such that*

$$(3.12a) \qquad |(u_a - u_{\mathrm{as};a})(x)| \quad \leq \quad C\varepsilon^2 \qquad\qquad for\ x \in \bar{\Omega}_a,$$

$$(3.12b) \qquad |(u_{[a,b]} - u_{\mathrm{as};[a,b]})(x)| \quad \leq \quad C(\varepsilon^2 + h^2) \quad for\ x \in \bar{\Omega}_{[a,b]},$$

*where $u_{\mathrm{as};a}$ and $u_{\mathrm{as};[a,b]}$ are defined in (3.3). Furthermore,*

$$(3.12c) \qquad |(u_{[a,b]} - u_a)(x)| \quad \leq \quad C(\varepsilon^2 + h^2) \qquad for\ x \in \bar{\Omega}_{[a,(a+b)/2]},$$

$$(3.12d) \qquad |(u_{[a,b]} - z)(x)| \quad \leq \quad C(\varepsilon^2 + h^2) \qquad for\ x \in \partial\Omega_{(a+b)/2},$$

$$(3.12e) \qquad |(u_a - z)(x)| \quad \leq \quad C(\varepsilon^2 + h^2) \qquad for\ x \in \bar{\Omega}_{(a+b)/2}.$$

*Proof.* Existence of $u_a$ (for $a = 0$) and relation (3.12a) are established in [5, 18].

For existence of $u_{[a,b]}$, set $\bar{p} := \frac{2C_2}{C_0\gamma^2}(\varepsilon^2 + h^2)$ so that $\frac{1}{2}C_0\bar{p}\gamma^2 \leq C_2(\varepsilon^2 + h^2)$. Then the choice $\tilde{C}_0^2 := \frac{C_0\gamma^2}{2C_2}\min\{p_0, \frac{C_0\gamma^2}{2C_2}\}$ provides $\bar{p} \leq p_0$ and $\frac{1}{2}C_0\bar{p}\gamma^2 \geq C_2\bar{p}^2$. So applying Corollary 3.4 yields $F\beta_{[a,b]}(x;-\bar{p}) \leq 0 \leq F\beta_{[a,b]}(x;\bar{p})$. Furthermore, since (3.7) implies that $\beta_{[a,b]}(x;p)$ is increasing in $p$, while $\beta_{[a,b]}(x;0) = u_{\mathrm{as};[a,b]}(x)$, we get

$$(3.13) \qquad \beta_{[a,b]}(x;-\bar{p}) \leq u_{\mathrm{as};[a,b]}(x) \leq \beta_{[a,b]}(x;\bar{p}).$$

Thus $\beta_{[a,b]}(x;-\bar{p})$ and $\beta_{[a,b]}(x;\bar{p})$ are ordered sub- and super-solutions for problem (3.2). Therefore this problem has a solution $u_{[a,b]}$ such that $\beta_{[a,b]}(x;-\bar{p}) \leq u_{[a,b]}(x) \leq \beta_{[a,b]}(x;\bar{p})$ and hence for this solution we obtain the desired bound (3.12b) from

$$(3.14) \qquad |u_{[a,b]}(x) - u_{\mathrm{as};[a,b]}(x)| \leq \beta_{[a,b]}(x;\bar{p}) - \beta_{[a,b]}(x;-\bar{p}) \leq C\bar{p}.$$

The final estimate here follows from $\beta_{[a,b]}(x;\bar{p}) - \beta_{[a,b]}(x;-\bar{p}) = 2\bar{p}\left(\frac{\partial}{\partial p}\tilde{v}_{0;a} + \frac{\partial}{\partial p}\tilde{v}_{0;b} + C_0\right)\big|_{p=\theta}$, where we used (3.6b) and (3.8). Thus we have (3.12b).

In view of (3.12a), (3.12b), it suffices to prove versions of (3.12c)–(3.12e), in which $u_{[a,b]}$ and $u_a$ are replaced by $u_{\mathrm{as};[a,b]}$ and $u_{\mathrm{as};a}$, respectively. They follow as (3.3) yields $u_{\mathrm{as};[a,b]} - u_{\mathrm{as};a} = I_b$ and $u_{\mathrm{as};[a,b]} - z = I_a + I_b$, while $u_{\mathrm{as};a} - z = I_a\omega$, where $I_a := \left[v_{0;a}(\xi_a^+, l) + \varepsilon v_{1;a}(\xi_a^+, l)\right]$ and $I_b := \left[v_{0;b}(\xi_b^-, l) + \varepsilon v_{1;b}(\xi_b^-, l)\right]$ so $|I_b| \le Ch^2$ for $x \in \bar{\Omega}_{[a,(a+b)/2]}$ and $|I_a| + |I_b| \le Ch^2$ for $x \in \partial\Omega_{(a+b)/2}$, while $|I_a| \le Ch^2$ for $x \in \bar{\Omega}_{(a+b)/2}$. Here the bounds for $I_{a,b}$ are obtained by combining (3.8), (3.9) with $\xi_b^- = \frac{b-r}{\varepsilon}$, $\xi_a^+ = \frac{r-a}{\varepsilon}$ and (2.2). $\qquad\square$

### 3.2. Error in the continuous Schwarz method.
We are now prepared to bound the error in the first iteration $u^{[1]}$ of the continuous Schwarz method (1.2), (1.3).

**Theorem 3.6.** *Let $(4/\gamma)\varepsilon\ln(Ch) \le 2\sigma \le C_1$ and $\varepsilon + h \le \tilde{C}_0$, where $\tilde{C}_0$ is from Lemma 3.5. Let the boundary data $g_0$ and $g_{2\sigma} = g_{2\sigma}^{[1]}$ of problems (1.1) and (1.2a) satisfy (1.5b) with $d = 0, 2\sigma$. Then there exist a solution $u$ of problem (1.1) and a first-iteration approximation $u^{[1]}$ defined by (1.2), (1.3) such that*
$$|(u - u^{[1]})(x)| \le C(\varepsilon^2 + h^2) \qquad \text{for } x \in \bar{\Omega}.$$

*Proof.* Applying Lemma 3.5, with $a := 0$ and $b := 2\sigma$, to problems (1.1) and (1.2a) immediately yields existence of their solutions $u_a = u$ and $u_{[a,b]} = u_{[0,2\sigma]}$. Furthermore, as $u^{[1]} = u_{[0,2\sigma]}$ in $\bar{\Omega}_{[0,\sigma]}$, estimate (3.12c) implies $u^{[1]} - u = O(\varepsilon^2 + h^2)$ in $\bar{\Omega}_{[0,\sigma]}$. Note also that (3.12e) yields $u - z = O(\varepsilon^2 + h^2)$ for $x \in \bar{\Omega}_\sigma$, while $u^{[1]} = u_\sigma$ in this subdomain. So, to complete the proof, it remains to show that there exists a solution $u_\sigma$ of problem (1.2b) such that

(3.15) $$u_\sigma - z = O(\varepsilon^2 + h^2) \qquad \text{for } x \in \bar{\Omega}_\sigma.$$

As the boundary condition in (1.2b) is $g_\sigma = u_{[0,2\sigma]}$ on $\partial\Omega_\sigma$, then (3.12d) yields $g_\sigma - z = O(\varepsilon^2 + h^2)$ on $\partial\Omega_\sigma$. So one can easily check that the boundary condition of problem (1.2b) satisfies assumption (1.5b) with $d = \sigma$. Now, Lemma 3.5, applied to problem (1.2b) as a particular case of (3.1) with $a = \sigma$, implies existence of a solution $u_\sigma$ such that $u_\sigma - u_{\mathrm{as};\sigma} = O(\varepsilon^2 + h^2)$. Furthermore, using (3.8) to estimate the boundary-layer components of $u_{\mathrm{as};\sigma}$, we observe that they do not exceed $C_\delta \max_{\partial\Omega_\sigma} |g_\sigma - z| = O(\varepsilon^2 + h^2)$. This yields $u_{\mathrm{as};\sigma} = z + O(\varepsilon^2 + h^2)$ and hence (3.15). $\qquad\square$

### 3.3. Z-fields.
We shall invoke the theory of $Z$-fields in our analysis of discretizations (2.5) in $\Omega_{[0,2\sigma]}$ and (2.6) in $\Omega_\sigma$.

**Definition.** An operator $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a $Z$-field if for all $i \ne j$ the mapping $x_j \mapsto \left(\mathcal{F}(x_1, x_2, \ldots, x_n)\right)_i$ is a monotonically decreasing function from $\mathbb{R}$ to $\mathbb{R}$ when $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n$ are fixed.

**Lemma 3.7.** [15] *Let $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and a $Z$-field. Let $r \in \mathbb{R}^n$ be given. Assume that there exist $\alpha, \beta \in \mathbb{R}^n$ such that $\alpha \le \beta$ and $\mathcal{F}\alpha \le 0 \le \mathcal{F}\beta$. (The inequalities are understood to hold true component-wise.) Then the equation $\mathcal{F}y = 0$ has a solution $y \in \mathbb{R}^n$ with $\alpha \le y \le \beta$.*

*Proof.* The proof can be found in Lorenz [15], and also in [11]. Alternatively, the desired result can be obtained by imitating the proof of [19, Theorem 3.1] (it is crucial in this argument that the discrete operator $\mathcal{F} + CI$ satisfies a discrete

maximum principle, where $I$ is the identity operator and $C$ is an arbitrarily large but fixed positive constant). □

The elements $\alpha$ and $\beta$ of $\mathbb{R}^n$ that appear in Lemma 3.7, are called ordered *sub-* and *super-solutions* of the discrete problem $\mathcal{F}y = 0$.

*Remark* 3.8. The discrete operators $F^h_{[0,2\sigma]}$ of (2.5) and $F^h_\sigma$ of (2.6) are $Z$-fields [8].

3.4. **Error in the discrete Schwarz method for $\varepsilon \leq Ch$.** Throughout this subsection for any fixed positive constant $C$, we take

(3.16)                                   $\varepsilon \leq Ch.$

This is not a practical restriction. Furthermore, in §4, we shall consider the case of $\varepsilon \geq Ch$.

**Theorem 3.9.** *Let (3.16) be satisfied, and the mesh $\{r_i\}_{i=0}^{2N}$ be one of the meshes in §2.2(a),(b). Let the boundary data $g_0$ and $g_{2\sigma} = g^{[1]}_{2\sigma}$ of problems (1.1) and (1.2a) satisfy (1.5b) with $d = 0, 2\sigma$. Then there exist a solution $u$ of problem (1.1) and a first-iteration computed solution $U^{[1]}$ defined by (2.5), (2.6), (2.7) such that*

$$|(u - U^{[1]})(x)| \leq Ch^2 |\ln h|^m \qquad for\ x \in \bar{\Omega},$$

*where $m = 2$ for the Shishkin mesh of §2.2(a) and $m = 0$ for the Bakhvalov mesh of §2.2(b).*

*Proof.* In view of Theorem 3.6 and definitions (1.3b) and (2.7b) of $u^{[1]}$ and $U^{[1]}$, it suffices to show that

(3.17a)        $\left|(U_{[0,2\sigma]} - u_{[0,2\sigma]})(x)\right| \leq Ch^2 |\ln h|^m \qquad for\ x \in \bar{\Omega}_{[0,2\sigma]},$

(3.17b)                $\left|(U_\sigma - u_\sigma)(x)\right| \leq Ch^2 |\ln h|^m \qquad for\ x \in \bar{\Omega}_\sigma$

To prove (3.17a), note that problem (1.2a) is a particular case of (3.2), so we shall use some results of §3.1 setting $a := 0$ and $b := 2\sigma$. The corresponding function $\beta_{[0,2\sigma]}(x; p)$ is defined by (3.6b). We claim that for all $|p| \leq p_0$ at all interior mesh nodes $x_{ij}, i = 1, \ldots, N-1, j = 1, \ldots, N_l$, we have

(3.18)            $\left|F^h_{[0,2\sigma]}\beta_{[0,2\sigma]}(x_{ij}; p) - F\beta_{[0,2\sigma]}(x_{ij}; p)\right| \leq Ch^2 |\ln h|^m.$

Note that the term $\varepsilon^2 \zeta \frac{\partial}{\partial l}(\zeta \frac{\partial}{\partial l}\beta_{[0,2\sigma]})$ in $F\beta_{[0,2\sigma]}$ and its discretization in $F^h_{[0,2\sigma]}\beta_{[0,2\sigma]}$ are both $O(\varepsilon^2)$, so do not exceed $Ch^2$ by (3.16). The truncation error for the remaining term $\varepsilon^2 \eta^{-1} \frac{\partial}{\partial r}(\eta \frac{\partial}{\partial r}\beta_{[0,2\sigma]})$ is bounded by $CN^{-2} \ln^m N$, as can be shown by imitating the argument of [11, Lemma 3.3 and §3.4.2]. Combining this with (2.4), we get (3.18).

Next, set $\bar{p} := \bar{C}h^2 |\ln h|^m$ and, using (3.18), choose $\bar{C}$ sufficiently large so that $|F^h_{[0,2\sigma]}\beta_{[0,2\sigma]} - F\beta_{[0,2\sigma]}| \leq \frac{1}{2}C_0\bar{p}\gamma^2$ for all $|p| \leq p_0$ including $p = \pm\bar{p}$. Now, by Corollary 3.4 and (3.16), for sufficiently small $h$, we have $\pm F\beta_{[0,2\sigma]}(x; \pm\bar{p}) \geq \frac{1}{2}C_0\bar{p}\gamma^2$. Consequently $\pm F^h_{[0,2\sigma]}\beta_{[0,2\sigma]}(x_{ij}; \pm\bar{p}) \geq 0$. Combining this with (3.13), we conclude that $\beta_{[0,2\sigma]}(x_{ij}; \pm\bar{p})$ are ordered discrete sub- and super-solutions. As discretization (2.5) is a $Z$-field (see Remark 3.8), an application of Lemma 3.7 yields existence of $U_{[0,2\sigma]}(x_{ij})$ between these sub- and super-solutions. Furthermore, a version of (3.14) for $U_{[0,2\sigma]}(x_{ij})$ implies $|(U_{[0,2\sigma]} - u_{\text{as};[0,2\sigma]})(x_{ij})| \leq C\bar{p} \leq Ch^2 |\ln h|^m$. Noting that, by (3.3b), (3.8), we also have $|(u^I_{\text{as};[0,2\sigma]} - u_{\text{as};[0,2\sigma]})(x_{ij})| \leq Ch^2 |\ln h|^m$, where $u^I_{\text{as};[0,2\sigma]}$ is the bilinear interpolant of the computed solution on the mesh $\{(r_i, l_j)\}$,

we get $|(U_{[0,2\sigma]} - u_{\mathrm{as};[0,2\sigma]})(x)| \leq Ch^2 |\ln h|^m$. Combining this with (3.12b) yields the desired estimate (3.17a).

For (3.17b), in view of (3.15), it suffices to show that $|U_\sigma - z| \leq Ch^2 |\ln h|^m$. Note that, by (3.17a), at each mesh node $q_j \in \partial\Omega_\sigma$ we have $U_\sigma = u_{[0,2\sigma]} + O(h^2 |\ln h|^m)$, while $u_{[0,2\sigma]} = u_\sigma = z + O(h^2 |\ln h|^m)$ due to (1.2b) and (3.15). Consequently $|(U_\sigma - z)(q_j)| \leq \bar{C} h^2 |\ln h|^m$ at all $q_j \in \partial\Omega_\sigma$ for some sufficiently large $\bar{C}$.

Let $z^I$ be the piecewise linear interpolant of $z$ on the triangulation in $\bar{\Omega}_\sigma^h$. At any interior mesh node $q_i \in \Omega_\sigma^h$ one has

$$\langle \nabla z^I, \nabla \chi_i \rangle = \langle \nabla[z^I - z], \nabla \chi_i \rangle - \langle \triangle z, \chi_i \rangle.$$

Combining this with the interpolation error estimate $|\nabla(z^I - z)| \leq Ch$ and the standard quasiuniform-mesh properties $\langle 1, |\nabla \chi_i| \rangle \leq Ch$ and $\langle 1, \chi_i \rangle \geq Ch^2$, we conclude that $|\langle \nabla z^I, \nabla \chi_i \rangle| \leq C' \langle 1, \chi_i \rangle$. Now set $\bar{p} := \bar{C} h^2 |\ln h|^p$ with $\bar{C} \geq 2\gamma^{-2} C' C^2$, where $C$ is from (3.16), and let $h$ be sufficiently small so that, by (1.5a), we have $f_u(x, z \pm \bar{p}) \geq \frac{1}{2}\gamma^2$. A calculation shows that $\pm F_\sigma^h[z^I \pm p](q_i) \geq \frac{1}{2}\gamma^2 \bar{p} - C' \varepsilon^2 \geq 0$. Consequently $z^I \pm \bar{p}$ are sub- and super-solutions for the discrete problem (2.6). So, by Remark 3.8, an application of Lemma 3.7 yields existence of a solution $U_\sigma$ such that $z^I - \bar{p} \leq U_\sigma \leq z^I + \bar{p}$. Hence $|U_\sigma - z^I| \leq 2\bar{p}$. Combining this with $|z^I - z| \leq Ch^2$ implies (3.17b).                                                                                                          $\square$

The above Theorem 3.9 implies that if $\varepsilon \leq Ch$, one iteration of the discrete Schwarz method is sufficient to attain second-order convergence (with, in the case of the Shishkin mesh, a logarithmic factor) in the maximum norm uniformly in $\varepsilon$. In the next section we shall investigate the errors in the case of $\varepsilon \geq Ch$.

## 4. Maximum norm error analysis for $\varepsilon \geq Ch$

### 4.1. **Preliminaries. Error in the continuous Schwarz method.** Throughout this section, we make a simplifying assumption that

$$(4.1) \qquad\qquad f_u(x, u) > \gamma^2 > 0 \quad \text{for all } (x, u) \in \bar{\Omega} \times \mathbb{R}.$$

Under this assumption, problem (1.1) has a unique solution, and furthermore, applying the standard **linearization**, for any two functions $v$ and $w$ one gets

$$(4.2) \qquad Fv - Fw = \mathcal{L}[v - w], \quad \mathcal{L} := -\varepsilon^2 \triangle + p(x), \quad p(x) > \gamma^2 > 0 \text{ for } x \in \bar{\Omega}.$$

To be more precise, here the coefficient $p(x) = \int_0^1 f_u(x, w + s[v - w]) \, ds$, i.e. it involves the functions $v$ and $w$. Bearing this in mind, throughout this section, we let $p(x)$ denote a generic coefficient in $\mathcal{L}$, which in different places will involve different $v$ and $w$. Similarly, for the discrete operators $F_{[0,2\sigma]}^h$ of (2.5) and $F_\sigma^h$ of (2.6), we shall employ their linearized versions $\mathcal{L}_{[0,2\sigma]}^h$ and $\mathcal{L}_\sigma^h$ obtained using $f(x_{ij}, V_{ij}) - f(x_{ij}, W_{ij}) = p(x_{ij})[V_{ij} - W_{ij}]$ and $f(q_i, V(q_i)) - f(q_i, W(q_i)) = p(q_i)[V(q_i) - W(q_i)]$, respectively. In view of (4.1), the discrete operators $\mathcal{L}_{[0,2\sigma]}^h$ and $\mathcal{L}_\sigma^h$ satisfy the **discrete maximum principle**.

Under condition (4.1), it is not difficult to estimate the error in the continuous Schwarz method.

**Theorem 4.1.** *Let $u$ be a solution of problem (1.1) under condition (4.1), and $u^{[k]}$ be the kth iteration approximation (1.2), (1.3) obtained using $\sigma \in [C'\varepsilon, C_1]$ for some $C'$ and some $|g_{2\sigma}^{[1]}| \leq C$. There are positive constants $c_0$ and $\theta' \in (0, 1)$, independent*

*of $\varepsilon$, $\sigma$ and $k$, such that $|u^{[k]} - u| \leq C\theta^k$ in $\bar{\Omega}$, where $\theta \leq \min\{e^{-\sigma\gamma/\varepsilon}, \theta'\}$ for $\varepsilon \in (0, c_0]$, and $\theta \leq \theta'$ for $\varepsilon \in [c_0, 1]$. If $\Omega$ is convex, then $\theta' = \frac{1}{2}$.*

*Proof.* Set $\theta = \theta_\varepsilon(\sigma) := \max_{x \in \bar{\Omega}_{[0,\sigma]}} |\phi_\varepsilon(x)|$, where for each $\varepsilon \in (0, 1]$, $\sigma \in [C'\varepsilon, C_1]$, the auxiliary function $\phi_\varepsilon$ solves the problem

$$(4.3) \qquad \left[-\varepsilon^2 \triangle + \gamma_*^2\right]\phi_\varepsilon = 0 \ \text{in} \ \Omega_{[0,2\sigma]}, \quad \phi_\varepsilon = 0 \ \text{on} \ \partial\Omega, \quad \phi_\varepsilon = 1 \ \text{on} \ \partial\Omega_{2\sigma},$$

with some $\gamma_* > \gamma$. In view of (2.3), a calculation shows that the barrier functions $B_1(r) := e^{-\gamma(2\sigma-r)/\varepsilon}$ and $B_2(r) := r/(2\sigma)$ satisfy, respectively, $[-\varepsilon^2\triangle + \gamma_*^2]B_1 \geq (-\gamma^2 - \varepsilon\gamma C'' + \gamma_*^2)B_1$ for some $C''$, and $[-\triangle + \gamma_*^2]B_2 \geq \frac{\kappa/(2\sigma)}{1-\kappa r} + \gamma_*^2 B_2$. So, by the maximum principle, $\phi_\varepsilon \leq B_1$ if $\varepsilon \leq c_0 := (\gamma_*^2 - \gamma^2)/(\gamma C'')$, while $\phi_1 \leq B_2$ if $\kappa \geq 0$, i.e. if $\Omega$ is convex. These two observations imply that $\theta \leq e^{-\gamma\sigma/\varepsilon}$ for $\varepsilon \leq c_0$, and $\theta \leq \theta' := \frac{1}{2}$ if $\Omega$ is convex. To choose $\theta'$ for a non-convex $\Omega$, note that the maximum principle implies that $0 \leq \phi_\varepsilon < 1$ in $\bar{\Omega}_{[0,\sigma]}$, and also $\phi_\varepsilon \leq \phi_1$ in $\bar{\Omega}_{[0,2\sigma]}$ so $\theta_\varepsilon(\sigma) \leq \theta_1(\sigma) < 1$. Now, for $\varepsilon \geq c_0$ we have $\sigma \geq c_0 C'$ so $\theta \leq \theta'' := \max_{\sigma \in [c_0 C', C_1]} \theta_1(\sigma) < 1$, where $\theta''$ is independent of $\varepsilon$ and $\sigma$. Therefore $\theta \leq \theta' := \max\{e^{-\gamma C'}, \theta''\}$ for all $\varepsilon \in (0, 1]$. Thus, under our choice of $\theta'$, we have $\theta \leq \min\{e^{-\sigma\gamma/\varepsilon}, \theta'\}$ for $\varepsilon \in (0, c_0]$, and $\theta \leq \theta'$ for $\varepsilon \in [c_0, 1]$.

We now focus on the error in the Schwarz method. Let $t^{[k]} := \max_{\partial\Omega_{2\sigma}} |g_{2\sigma}^{[k]} - u|$, where one has $t^{[1]} \leq C^*$ for some $C^*$. Now let $\gamma_* := \min_{\bar{\Omega} \times [-\bar{C}, \bar{C}]} f_u > \gamma^2$, where $\bar{C} := \gamma^{-2}\max_{\bar{\Omega}}|f(x,0)| + \max_{\partial\Omega}|g_0| + C^*$ is independent of $\varepsilon$, $\sigma$ and $k$. Consider the first iteration. In view of (1.1) and (1.2a), a linearization of type (4.2) yields $\mathcal{L}(u_{[0,2\sigma]} - u) = 0$ in $\Omega_{[0,2\sigma]}$, with $p(x) \geq \gamma_*^2$, subject to $u_{[0,2\sigma]} - u = 0$ on $\partial\Omega$ and $|u_{[0,2\sigma]} - u| \leq t^{[1]}$ on $\partial\Omega_{2\sigma}$. So, using the maximum principle, we conclude that $|u_{[0,2\sigma]} - u| \leq t^{[1]}\phi_\varepsilon$ in $\bar{\Omega}_{[0,2\sigma]}$. Therefore $|u^{[1]} - u| \leq \theta t^{[1]}$ in $\bar{\Omega}_{[0,\sigma]}$ and consequently $|u_\sigma - u| \leq \theta t^{[1]}$ on $\partial\Omega_\sigma$. Also, in view of (1.1) and (1.2b), a linearization of type (4.2) yields $\mathcal{L}(u_\sigma - u) = 0$ in $\Omega_\sigma$. So, by the maximum principle, we get $|u^{[1]} - u| = |u_\sigma - u| \leq \theta t^{[1]}$ in $\bar{\Omega}_\sigma$ as well. Thus we have shown that $|u^{[1]} - u| \leq \theta t^{[1]}$ in $\bar{\Omega}$, which, by (1.3a), implies that $t^{[2]} \leq \theta t^{[1]}$. Repeating this argument for further iterations and then noting that $|t^{[1]}| \leq C^*$, we get the desired result. $\qquad\square$

### 4.2. Auxiliary computed solutions in $\Omega_{[0,2\sigma]}$ and $\Omega_\sigma$.

In this subsection we investigate auxiliary computed solutions $\tilde{U}_{[0,2\sigma]}$ and $\tilde{U}_\sigma$. The first solution $\tilde{U}_{[0,2\sigma]}$ is obtained in $\bar{\Omega}_{[0,2\sigma]}$ by the bilinear interpolation of $\tilde{U}_{[0,2\sigma]}(x_{ij})$, where

$$(4.4a) \quad F_{[0,2\sigma]}^h \tilde{U}_{[0,2\sigma]}(x_{ij}) = 0, \quad (\tilde{U}_{[0,2\sigma]} - g_0)(x_{0,j}) = 0, \quad (\tilde{U}_{[0,2\sigma]} - u)(x_{2N,j}) = 0,$$

with $\tilde{U}_{[0,2\sigma]}(x_{i,0}) = \tilde{U}_{[0,2\sigma]}(x_{i,N_l})$ and $\tilde{U}_{[0,2\sigma]}(x_{i,1}) = U_{[0,2\sigma]}(x_{i,N_l+1})$. The other solution $\tilde{U}_\sigma \in S^h$ and satisfies

$$(4.4b) \qquad F_\sigma^h \tilde{U}_\sigma(q_i) = 0 \ \ \forall q_i \in \Omega_\sigma^h, \qquad \tilde{U}_\sigma(q_j) = u(q_j) \ \ \forall q_i \in \partial\Omega_\sigma^h.$$

Here $F_{[0,2\sigma]}^h$ and $F_\sigma^h$ are the discrete operators that were used in problems (2.5), (2.6) for $U_{[0,2\sigma]}$ and $U_\sigma$. The only difference between these pairs of problems is in that we use the exact solution $u$ of (1.1) in the boundary conditions for $\tilde{U}_{[0,2\sigma]}$ and $\tilde{U}_\sigma$. To estimate the errors of these auxiliary computed solutions, we need pointwise derivative estimates for the exact solution $u$.

**Lemma 4.2.** *Under condition (4.1), problem (1.1) has a unique solution $u$, and*

$$(4.5a) \qquad \left|\frac{\partial^{j+m}}{\partial x_1^j \partial x_2^m}(u - z)\right| \leq C\left[\varepsilon^2 + \varepsilon^{4-(j+m)} + \varepsilon^{-(j+m)}e^{-\gamma a/\varepsilon}\right] \quad \text{for } x \in \bar{\Omega}_a,$$

*where $0 \le a \le C_1$, $j+m = 0, \ldots, 4$, and $z$ is a solution of (1.4) with $|\frac{\partial^{j+m}}{\partial x_1^j \partial x_2^m} z| \le C$. Furthermore,*

(4.5b) $\quad |\frac{\partial^{j+m}}{\partial r^j \partial l^m} u| \le C [1 + \varepsilon^{4-(j+m)} + \varepsilon^{-j} e^{-\gamma r/\varepsilon}] \quad$ *for $x \in \bar{\Omega}_{[0,C_1]}$, $j, m = 0, \ldots, 4$.*

*Proof.* We defer the proof of this lemma to Appendix A. $\hspace{2cm} \square$

We combine technical error estimates for $\tilde{U}_{[0,2\sigma]}$ and $\tilde{U}_\sigma$ in the following lemma.

**Lemma 4.3.** *Let $u$ be a solution of (1.1) under condition (4.1), $\varepsilon \ge Ch$, and the mesh $\{r_i\}_{i=0}^{2N}$ be one of the meshes in §2.2(a),(b). Then for the solutions $\tilde{U}_{[0,2\sigma]}$ and $\tilde{U}_\sigma$ of problems in (4.4) we have*

(4.6a) $\qquad |(\tilde{U}_{[0,2\sigma]} - u)(x)| \quad \le \quad Ch^2 |\ln h|^m \qquad$ *for $x \in \bar{\Omega}_{[0,2\sigma]}$,*

(4.6b) $\qquad |(\tilde{U}_\sigma - u)(x)| \quad \le \quad Ch^2 \ln(C + \varepsilon/h) \quad$ *for $x \in \bar{\Omega}_\sigma^h$,*

*where $m = 2$ for the Shishkin mesh of §2.2(a) and $m = 0$ for the Bakhvalov mesh of §2.2(b).*

In the proof of this lemma, for the finite element solution $\tilde{U}_\sigma$ we essentially use a maximum norm error estimate by Schatz and Wahlbin [23], which we generalize for the case of lumped-mass finite elements.

For the finite difference solution $\tilde{U}_{[0,2\sigma]}$, a certain technical difficulty is due to the mesh $\{l_j\}$ being quasi-uniform, so the truncation error in the $l$ direction is $O(h)$, while, by (4.6a), the error is $O(h^2 |\ln h|^m)$. Thus the finite-difference method in $\Omega_{[0,2\sigma]}$ is supra-convergent (i.e. its error has a higher order of accuracy than may be expected from the local truncation error; the term supra-convergence was introduced in [12]). Note that the only maximum-norm supra-convergence error estimate in two dimensions of which we are aware is obtained in [28] by combining supra-convergence in the norm $H^1$ with a discrete Sobolev inequality. In comparison, our proof of (4.6a) extends the classical one-dimensional supra-convergence analysis presented in [22].

*Proof of (4.6a) (Supra-convergence of the Finite Difference Discretization).* Let $u_{ij} := u(x_{ij})$ and $\tilde{U}_{ij} = \tilde{U}_{[0,2\sigma]}(x_{ij})$. Using (1.1a), (4.4a) and then the definition (2.5) of $F_{[0,2\sigma]}^h$, we get

(4.7) $\qquad F_{[0,2\sigma]}^h \tilde{U}_{ij} - F_{[0,2\sigma]}^h u_{ij} = Fu(x_{ij}) - F_{[0,2\sigma]}^h u_{ij} = \varepsilon^2 (\eta_{ij}^{-1} R_1 + \zeta_{ij} R_2),$

where

$$R_1 = D_r[\tilde{\eta}_{ij} D_r^- u_{ij}] - \frac{\partial}{\partial r}(\eta \frac{\partial}{\partial r} u)\big|_{(x_{ij})}, \quad R_2 = D_l[\tilde{\zeta}_{ij} D_l^- u_{ij}] - \frac{\partial}{\partial l}(\zeta \frac{\partial}{\partial l} u)\big|_{(x_{ij})}.$$

For $R_1$, employing Taylor series expansions, one can show that

$$|R_1| \le C [(r_i - r_{i-1})^2 M_i^{(4)} + |r_{i+1} - 2r_i + r_{i-1}| M_i^{(3)}], \quad M_i^{(s)} := \sum_{n=0}^{s} \max_{\substack{r \in [r_{i-1}, r_{i+1}] \\ l \in [0,L]}} |\frac{\partial^n}{\partial r^n} u|.$$

For the Shishkin mesh of §2.2(a), combining $r_i - r_{i-1} \le 2\gamma^{-1} \varepsilon N^{-1} \ln N$ with (4.5b) yields $|\varepsilon^2 \eta^{-1} R_1| \le C N^{-2} \ln^2 N$. For the Bakhvalov mesh of §2.2(b), we only consider $i \le N/2$ (as the other case is similar). A calculation shows that

$$S_i := \varepsilon^{-2}(r_i - r_{i-1})^2 + \varepsilon^{-1}|r_{i+1} - 2r_i + r_{i-1}| \le \frac{CN^{-2}}{\max\{(1 - \frac{i+1}{N}), \rho\}^2} \le CN^{-2}\varepsilon^{-2}.$$

Consequently, by (4.5b), one has

$$|\varepsilon^2\eta^{-1}R_1| \leq CS_i\big[\varepsilon^4 M_i^{(4)} + \varepsilon^3 M_i^{(3)}\big] \leq C\big[N^{-2} + S_i e^{-\gamma r_{i-1}/\varepsilon}\big] \leq CN^{-2},$$

where we used $e^{-\gamma r_{i-1}/\varepsilon} \leq \max\{(1-\frac{i-1}{N}),\rho\}^2$ and $N^{-1} \leq Ch \leq C\varepsilon = C\rho$.

Note that $R_2$ is only $O(h)$ as $\{l_j\}$ is a general non-uniform mesh. To establish supra-convergence of our discretization, we imitate the truncation error analysis of [9, Lemma 3.1]; see also [22, Chap. III, §4]. Noting that

$$R_2 = D_l[\tilde{\zeta}_{ij}\hat{\mu}_{ij}] + D_l w_{i,j-1/2} - \tfrac{\partial}{\partial l}w\big|_{x_{ij}}, \;\; \hat{\mu}_{ij} := D_l^- u_{ij} - \tfrac{\partial}{\partial l}u\big|_{x_{i,j-1/2}}, \;\; w := \zeta\tfrac{\partial}{\partial l}u,$$

and then employing Taylor series expansions, one gets $R_2 = D_l[\mu(r_i,l_j)]+\hat{\nu}_{ij}$. Here

$$\mu(r,l_j) := (l_j-l_{j-1})^2\big(\tfrac{1}{24}\zeta\tfrac{\partial^3}{\partial l^3}u+\tfrac{1}{8}\tfrac{\partial^2}{\partial l^2}w\big)\big|_{(r,l_{j-1/2})}, \;\; |\hat{\nu}_{ij}| \leq Ch^2 \max_{\bar{\Omega}_{[0,2\sigma]}}(|\tfrac{\partial^3}{\partial l^3}w|+|\tfrac{\partial^4}{\partial l^4}u|),$$

so, by (4.5b), one has $|\hat{\nu}| \leq Ch^2$ and $|\tfrac{\partial^n}{\partial r^n}\mu| \leq C\varepsilon^{-n}h^2$ for $n = 0, 1, 2$.

Linearizing (4.7) and combining our findings for $R_1$ and $R_2$, we conclude that

$$\mathcal{L}^h_{[0,2\sigma]}[\tilde{U}_{ij} - u_{ij}] = \varepsilon^2\zeta_{ij}D_l[\mu(r_i,l_j)] + \nu_{ij}, \;\; \varepsilon^n|\tfrac{\partial^n}{\partial r^n}\mu| \leq Ch^2, \;\; |\nu| \leq Ch^2|\ln h|^m.$$

Consequently, $|\tilde{U}(x_{ij}) - u(x_{ij})| \leq Ch^2|\ln h|^m$ (this immediately follows from the stability result for the operator $\mathcal{L}^h_{[0,2\sigma]}$ given by Corollary 4.7 which we defer to §4.4). Combining this with the interpolation error bound $|u^I - u| \leq Ch^2|\ln h|^m$ for the bilinear interpolant $u^I$ of the exact solution $u$ on the mesh $\{(r_i,l_j)\}$ (which is obtained again using (4.5b)), we get the desired estimate (4.6a). □

*Proof of (4.6b) (Lumped-Mass Finite Element Error).* We claim that

(4.8a) $\qquad |(\tilde{U}_\sigma - u)(x)| \leq Ch^2\ln(C + \varepsilon/h)\big[\|u\|_{C^2(\bar{\Omega}^h_\sigma)} + \mathcal{E}_{\text{l.m.}}\big] \;\;$ for $x \in \bar{\Omega}^h_\sigma$,

where the error due to the lumped-mass discretization of $f(x,u)$ is described using

(4.8b) $\qquad \mathcal{E}_{\text{l.m.}} := \|\Psi\|_{C^2(\bar{\Omega}^h_\sigma)} + \varepsilon^{-1}\|\Psi\|_{C^1(\bar{\Omega}^h_\sigma)}, \qquad \Psi(x) := f(x, u(x)).$

Estimate (4.8) is a generalization of a maximum norm error estimate [23] for the standard finite element method (for which estimate (4.8a) with $\mathcal{E}_{\text{l.m.}} = 0$ immediately follows from [23, Theorems 6.1 and 12.1]). We defer the proof of (4.8) to §4.5.

As the domain $\Omega$ is smooth, we note that $\Omega^h_\sigma \subset \Omega_{\tilde{\sigma}}$, for some $\tilde{\sigma} \leq \sigma - Ch^2 \leq \sigma - C\varepsilon$. Note also that, by (1.4), we have $\Psi(x) = [u - z]\int_0^1 f_u(x, z + s[u - z])\,ds$, so $\|\Psi\|_{C^n(\bar{\Omega}^h_\sigma)} \leq C\|u - z\|_{C^n(\bar{\Omega}^h_\sigma)}$. Consequently, by (4.5a), a calculation shows that $\|u\|_{C^2(\bar{\Omega}^h_\sigma)} \leq C[1 + \varepsilon^{-2}e^{-\gamma\tilde{\sigma}/\varepsilon}]$ and $\|\Psi\|_{C^n(\bar{\Omega}^h_\sigma)} \leq C[\varepsilon^2 + \varepsilon^{-n}e^{-\gamma\tilde{\sigma}/\varepsilon}]$ for $n = 1, 2$, so it remains to prove that $I := \varepsilon^{-2}e^{-\gamma\sigma/\varepsilon} \leq C$. On both the Shishkin mesh and the Bakhvalov mesh of §2.2(a),(b), we have $\sigma \geq \sigma_S$, so a calculation yields $I \leq \varepsilon^{-2}\max\{N^{-2}, e^{-\gamma C_1/(2\varepsilon)}\} \leq C$, where we used (2.4) and $\varepsilon \geq Ch$. □

### 4.3. Error in the discrete Schwarz method for $\varepsilon \geq Ch$.

**Theorem 4.4.** *Let $u$ be a solution of problem (1.1) under condition (4.1), $\varepsilon \geq Ch$, and $U^{[k]}$ be the discrete $k$th iteration approximation (2.5), (2.6), (2.7) obtained using some $|g^{[1]}_{2\sigma}| \leq C$ and one of the meshes $\{r_i\}_{i=0}^{2N}$ in §2.2(a),(b). Set $m = 2$ for the Shishkin mesh of §2.2(a) and $m = 0$ for the Bakhvalov mesh of §2.2(b). There are constants $c_0$ and $\theta' \in (0, 1)$, independent of $\varepsilon$ and $k$, such that*

(4.9) $\qquad |(U^{[k]} - u)(x)| \leq C[\theta^k + h^2|\ln h|^m + h^2\ln(C + \varepsilon/h)] \;\;$ *for $x \in \bar{\Omega}$,*

where $\theta \le \min\{e^{-\sigma\gamma/\varepsilon}, \theta'\}$ for $\varepsilon \le c_0$, and $\theta \le \theta'$ for $\varepsilon \ge c_0$. If $\Omega$ is convex, then $\theta' \le \frac{1}{2}$. If $\varepsilon \le \frac{1}{4}C_1\gamma(\ln N)^{-1}$, then $\theta \le Ch^2$.

*Proof.* We shall partly imitate the proof of Theorem 4.1. Note that the condition $\sigma \in [C'\varepsilon, C_1]$ of this theorem is satisfied if we take a sufficiently small $C' \le \frac{1}{2}C_1$. Applying the numerical method (2.5) to problem (4.3) for $\phi_\varepsilon$, we get the computed solution $\Phi_\varepsilon$, which satisfies a discrete equation of the type $[-\varepsilon^2 \triangle_{[0,2\sigma]}^h + \gamma_*^2]\Phi_\varepsilon(x_{ij}) = 0$ in $\Omega_{[0,2\sigma]}$, subject to the boundary conditions $\Phi_\varepsilon = 0$ on $\partial\Omega$ and $\Phi_\varepsilon = 1$ on $\partial\Omega_{2\sigma}$. Now set $\Theta := \max_{x_{ij} \in \bar\Omega \setminus \Omega_\sigma^h} |\Phi_\varepsilon|$. Note that a version of Lemma 4.2 can be obtained for the derivatives of the exact solution $\phi_\varepsilon$. Furthermore, imitating the proof of (4.6a), one can show that the error $|(\Phi_\varepsilon - \phi_\varepsilon)(x)| \le Ch^2|\ln h|^m$. Combining these two observations with $\mathrm{dist}(\partial\Omega_\sigma, \partial\Omega_\sigma^h) \le Ch^2$, one concludes that $|\Theta - \theta| \le Ch^2|\ln h|^m$, where $\theta$ is from Theorem 4.1. Note also that $\varepsilon \le \frac{1}{4}C_1\gamma(\ln N)^{-1}$ implies $\sigma \ge \sigma_S = 2\gamma^{-1}\varepsilon\ln N$, so $\theta \le N^{-2} \le Ch^2$.

Next, introduce some notation using $g^{[k]}$ of (2.7a) and $\tilde{U}_{[0,2\sigma]}$ and $\tilde{U}_\sigma$ of (4.4):

$$T^{[k]} = \max_{\partial\Omega_{2\sigma}} |g_{2\sigma}^{[k]} - \tilde{U}_{[0,2\sigma]}|, \quad \tilde{T}_\sigma = \max_{q_i \in \partial\Omega_\sigma} |(\tilde{U}_{[0,2\sigma]} - \tilde{U}_\sigma)(q_i)|, \quad \tilde{T}_{2\sigma} = \max_{\partial\Omega_{2\sigma}} |\tilde{U}_{[0,2\sigma]} - \tilde{U}_\sigma|.$$

Note that $T^{[1]} \le C^*$ for some $C^*$ (in fact, $T^{[1]} \approx t^{[1]}$, where $t^{[1]}$ is from the proof of Theorem 4.1). In view of (4.6), it suffices to estimate

$$E^{[k]} := \max\Big\{ \max_{x \in \bar\Omega \setminus \bar\Omega_\sigma^h} |(U^{[k]} - \tilde{U}_{[0,2\sigma]})(x)|, \ \max_{x \in \bar\Omega_\sigma^h} |(U^{[k]} - \tilde{U}_\sigma)(x)| \Big\}.$$

Consider the first iteration. By (2.5) and (4.4a), a linearization of type (4.2) yields the discrete equation $\mathcal{L}_{[0,2\sigma]}^h(U_{[0,2\sigma]} - \tilde{U}_{[0,2\sigma]}) = 0$ in $\Omega_{[0,2\sigma]}$, where $p(x_{ij}) \ge \gamma_*^2$, subject to $U_{[0,2\sigma]} - \tilde{U}_{[0,2\sigma]} = 0$ on $\partial\Omega$ and $|U_{[0,2\sigma]} - \tilde{U}_{[0,2\sigma]}| \le T^{[1]}$ on $\partial\Omega_{2\sigma}$. So, using the discrete maximum principle, we conclude that $|U_{[0,2\sigma]} - \tilde{U}_{[0,2\sigma]}| \le T^{[1]}\Phi_\varepsilon$ in $\bar\Omega_{[0,2\sigma]}$. This immediately implies $|U^{[1]} - \tilde{U}_{[0,2\sigma]}| \le \Theta T^{[1]}$ in $\bar\Omega \setminus \bar\Omega_\sigma^h$. Furthermore, $|(U_\sigma - \tilde{U}_\sigma)(q_j)| \le \Theta T^{[1]} + \tilde{T}_\sigma$ at any mesh node $q_j \in \partial\Omega_\sigma$. Combining this with $\mathcal{L}_\sigma^h(U_\sigma - \tilde{U}_\sigma)(q_i) = 0$ for all $q_i \in \Omega_\sigma^h$, which follows from (2.6) and (4.4b), and applying the discrete maximum principle, we get $|U^{[1]} - \tilde{U}_\sigma| = |U_\sigma - \tilde{U}_\sigma| \le \Theta T^{[1]} + \tilde{T}_\sigma$ in $\bar\Omega_\sigma^h$. Finally, by (2.7a), we have $T^{[2]} \le \Theta T^{[1]} + \tilde{T}_\sigma + \tilde{T}_{2\sigma}$. Noting that $\Theta T^{[1]} \le \theta T^{[1]} + |\Theta - \theta|C^*$, we summarize our findings for the first iteration as follows:

$$\max\{E^{[1]}, T^{[2]}\} \le \theta T^{[1]} + \lambda, \qquad \lambda := |\Theta - \theta|C^* + \tilde{T}_\sigma + \tilde{T}_{2\sigma}.$$

Next, by (4.6), we have

$$(4.10) \qquad \lambda \le C[h^2|\ln h|^m + h^2\ln(C + \varepsilon/h)],$$

while $\theta \le \theta'$, with $\theta' \in (0,1)$ independent of $\varepsilon$ and $k$. As $T^{[1]} \le C^*$, we also get $T^{[2]} \le C^*$ for sufficiently small $h$. Repeating the above argument for further iterations yields $\max\{E^{[k]}, T^{[k+1]}\} \le \theta T^{[k]} + \lambda$ and so $E^{[k]} \le \theta^k T^{[1]} + \lambda(1-\theta)^{-1}$. In view of (4.6) and (4.10), the desired estimate (4.9) follows. $\square$

**Corollary 4.5.** *Under the conditions of Theorem 4.4, for $\varepsilon \le \frac{1}{4}C_1\gamma(\ln N)^{-1}$, we have*

$$|(U^{[1]} - u)(x)| \le C[h^2|\ln h|^m + h^2\ln(C + \varepsilon/h)] \quad \text{for } x \in \bar\Omega.$$

4.4. **Stability of the finite difference operator in the boundary-layer sub-domain.** In this subsection we establish a stability result for the linearization $\mathcal{L}^h_{[0,2\sigma]}$ of the finite difference operator $F^h_{[0,2\sigma]}$ of (2.5). This result was crucial in the proof of the supra-convergence estimate (4.6a). We start with an auxiliary lemma for a related one-dimensional operator.

**Lemma 4.6.** *Let the function $W(r,l_j)$, for $r \in [0,2\sigma]$, $j = 1,\ldots,N_l$, satisfy*

$$(4.11) \qquad \mathcal{M}W(r,l_j) := -D_l[\,\tilde{\zeta}(r,l_j)\,D_l^- W(r,l_j)\,] + W(r,l_j) = D_l[\,\mu(r,l_j)\,],$$

*subject to periodicity conditions $W(r,l_0) = W(r,l_{N_l})$ and $W(r,l_1) = W(r,l_{N_l+1})$, where $\tilde{\zeta}(r,l_j) := \zeta(r,l_{j-1/2})$ and $\mu(r,l_1) = \mu(r,l_{N_l+1})$. Then we have*

$$(4.12) \qquad |\tfrac{\partial^m}{\partial r^m} W(r,l_j)| \le C \sum_{n=0}^{m} \max_{\substack{r \in [0,2\sigma] \\ l=1,\ldots,N_l}} |\tfrac{\partial^n}{\partial r^n} \mu(r,l_j)| \quad for \ \ m = 0,1,2.$$

*Proof.* Differentiating (4.11) in $r$, we get, with the notation $W_m(r,l_j) := \frac{\partial^m}{\partial r^m} W(r,l_j)$, $\zeta_m(r,l_j) := \frac{\partial^m}{\partial r^m}\tilde{\zeta}(r,l_j)$ and $\mu_m(r,l_j) := \frac{\partial^m}{\partial r^m}\mu(r,l_j)$,

$$\mathcal{M}W_1(r,l_j) = D_l[\,\zeta_1(r,l_j)D_l^- W(r,l_j) + \mu_1(r,l_j)\,],$$

$$\mathcal{M}W_2(r,l_j) = D_l[\,\zeta_2(r,l_j)D_l^- W(r,l_j) + 2\zeta_1(r,l_j)D_l^- W_1(r,l_j) + \mu_2(r,l_j)\,].$$

Note that $\tilde{\zeta}(r,l_j) \ge C > 0$, so problem (4.11) is well posed. Define two discrete $L_2(0,L)$ norms by $\|y\|_h^2 = \sum_{j=1}^{N_l} y_j^2(l_j - l_{j-1})$ and $\|y\|_{h,*}^2 = \sum_{j=1}^{N_l} \frac{1}{2}y_j^2(l_{j+1} - l_{j-1})$. Now, applying the method of energy inequalities [22, Chap. II, §3.5] to (4.11), one can show that $\|D_l^- W(r,\cdot)\|_h + \|W(r,\cdot)\|_{h,*} \le C\|\mu(r,\cdot)\|_h$. Furthermore, we get

$$\|D_l^- W_1(r,\cdot)\|_h + \|W_1(r,\cdot)\|_{h,*} \le C\|\zeta_1 D_l^- W + \mu_1\|_h \le C(\|\mu\|_h + \|\mu_1\|_h)$$

and a similar estimate for $W_2$. Thus we have

$$\|D_l^- W_m(r,\cdot)\|_h + \|W_m(r,\cdot)\|_{h,*} \le C \sum_{n=0}^{m} \|\mu_n(r,\cdot)\|_h \quad for \ \ m = 0,1,2.$$

The desired result follows as for all $r \in [0,2\sigma]$ we have $\|\mu_m\|_h \le C \max_j |\mu_m(r,l_j)|$ and $\max_j |W_m(r,l_j)| \le C(\|D_l^- W_m\|_h + \|W_m\|_{h,*})$ (the former estimate is a discrete version of a Sobolev imbedding theorem). $\qquad\square$

The main result of this section is as follows.

**Corollary 4.7.** *Let $\mathcal{L}^h_{[0,2\sigma]}$ be linearization of type (4.2) of the finite difference operator $F^h_{[0,2\sigma]}$ in (2.5). Let $V_{ij}$, for $i = 1,\ldots,2N-1$, $j = 1,\ldots,N_l$, satisfy*

$$\mathcal{L}^h_{[0,2\sigma]}V_{ij} = \varepsilon^2 \zeta_{ij} D_l[\mu(r_i,l_j)] + \nu_{ij},$$

*subject to $V_{i,0} = V_{i,N_l}$, $V_{i,1} = V_{i,N_l+1}$ and $V_{0,j} = V_{2N,j} = 0$, where we also have $\mu(r,l_1) = \mu(r,l_{N_l+1})$ and $\nu_{i,0} = \nu_{i,N_l}$, $\nu_{i,1} = \nu_{i,N_l+1}$. Then*

$$(4.13) \qquad \max_{i,j} |V_{ij}| \le C\Big(\sum_{n=0}^{2} \varepsilon^n \max_{\substack{r \in [0,2\sigma] \\ l=1,\ldots,N_l}} |\tfrac{\partial^n}{\partial r^n}\mu| + \max_{ij} |\nu_{ij}|\Big).$$

*Proof.* A calculation using $W_{ij} := W(r_i, l_j)$ from Lemma 4.6 shows that

$$\mathcal{L}^h_{[0,2\sigma]}(V_{ij} - W_{ij}) = \nu_{ij} + \varepsilon^2 \eta_{ij}^{-1} D_r[\tilde{\eta}_{ij} D_r^- W_{ij}] + (\varepsilon^2 - p_{ij})W_{ij}.$$

This implies that

$$|\mathcal{L}^h_{[0,2\sigma]}(V_{ij} - W_{ij})| \le C\big(\max_{ij}|\nu_{ij}| + \sum_{n=0}^2 \varepsilon^n \max_{\substack{r \in [0,2\sigma] \\ l=1,\dots,N_l}} |\tfrac{\partial^n}{\partial r^n}W|\big).$$

Now, $|V_{ij}| \le |V_{ij} - W_{ij}| + |W_{ij}|$, while, by the discrete maximum principle, we have $|V_{ij} - W_{ij}| \le C \max |\mathcal{L}^h_{[0,2\sigma]}(V_{ij} - W_{ij})|$. Combining this with (4.12), we get the desired estimate (4.13). $\qquad\square$

### 4.5. Proof of the lumped-mass finite element error estimate (4.8).

In this subsection, we generalize a maximum norm error estimate for the standard finite element method [23] to its lumped mass version. Note that the energy arguments are not suitable in estimation of the lumped-mass error for singularly perturbed equations of type (1.1), as they result in the error constants involving negative powers of the small parameter $\varepsilon$.

We use the **notation** of §2.4, and also the space $\mathring{S}^h := \{\chi \in S^h, \ \chi = 0 \text{ on } \partial\Omega_\sigma^h\}$, and the forms

$$\begin{aligned}
a(v,w) &:= \varepsilon^2\langle \nabla v, \nabla w\rangle + \langle f(x,v), w\rangle, \\
a_h(v,w) &:= \varepsilon^2\langle \nabla v, \nabla w\rangle + \langle f(x,v), w\rangle_h, \qquad \langle\varphi, w\rangle_h := \int_{\Omega_\sigma^h}(\varphi w)^I,
\end{aligned}$$

where $(\varphi w)^I$ is the standard piecewise linear interpolant of the function $\varphi w$. Then the lumped mass solution $\tilde{U}_\sigma \in S^h$ of (4.4b) using the operator $F_\sigma^h$ of (2.6), and the standard finite element solution $u_h \in S^h$ satisfy $a_h(\tilde{U}_\sigma, \chi) = 0$ and $a(u_h, \chi) = 0$ for all $\chi \in \mathring{S}^h$. We shall also use the form

$$(4.14) \qquad\qquad \delta_h(v,w) := a(v,w) - a_h(v,w),$$

and the discrete function $r_h \in \mathring{S}^h$ such that

$$(4.15) \qquad a(u_h + r_h, \chi) - a(u_h, \chi) = \delta_h(u, \chi) \quad \forall\, \chi \in \mathring{S}^h.$$

Note that for any $v, w$ and any nodal basis function function $\chi_i$, a calculation yields

$$(4.16) \qquad |\delta_h(v, \chi_i) - \delta_h(w, \chi_i)| \le C\,\langle 1, \chi_i\rangle \max_{\bar{\Omega}_\sigma^h}|v - w|.$$

Our proof is in **two steps**. First, we shall show that

$$(4.17) \qquad\qquad |\tilde{U}_\sigma - u_h| \le C\big(\max_{\bar{\Omega}_\sigma^h}|u_h - u| + \max_{\bar{\Omega}_\sigma^h}|r_h|\big).$$

For all $\chi \in \mathring{S}^h$ we have $a_h(\tilde{U}_\sigma, \chi) = a(u_h, \chi)$, so, invoking (4.14) and (4.15), we get

$$a_h(u_h + r_h, \chi) - a_h(\tilde{U}_\sigma, \chi) = a_h(u_h + r_h, \chi) - a(u_h, \chi) = \delta_h(u, \chi) - \delta_h(u_h + r_h, \chi).$$

Next, by (4.16),

$$|a_h(u_h + r_h, \chi_i) - a_h(\tilde{U}_\sigma, \chi_i)| \le \langle 1, \chi_i\rangle \max_{\bar{\Omega}_\sigma^h}(|u_h - u| + |r_h|),$$

which can be rewritten in terms the linearization $\mathcal{L}_\sigma^h$ of $F_\sigma^h$ as

$$|\mathcal{L}_\sigma^h(u_h + r_h - \tilde{U}_\sigma)| \le C\max_{\bar{\Omega}_\sigma^h}(|u_h - u| + |r_h|).$$

Now, by the discrete maximum principle, $|u_h + r_h - \tilde{U}_\sigma| \leq C \max |\mathcal{L}_\sigma^h(u_h + r_h - \tilde{U}_\sigma)|$, which immediately yields (4.17).

It remains to estimate $r_h$. Linearizing (4.15), we get $A(r_h, \chi) = \delta_h(u, \chi)$ for all $\chi \in \mathring{S}^h$, where the symmetric bilinear form $A(\cdot, \cdot)$ is given by

$$A(w, \chi) = \varepsilon^2 \langle \nabla w, \nabla \chi \rangle + (pw, \chi), \qquad p(x) := \int_0^1 f_u(x, u_h + sr_h)\, ds.$$

Consider an arbitrary point $x_* \in \tau_*$, where $\tau_*$ is some triangle of our triangulation in $\Omega_\sigma^h$. Then, imitating the proof of [23, Theorem 6.1] we first use an inverse inequality and then the dual argument to get

$$(4.18) \qquad |r_h(x_*)| \leq Ch^{-1} \|r_h\|_{L_2(\tau_*)} = Ch^{-1} \sup_{\phi \in C_0^\infty(\tau_*),\, \|\phi\|_{L_2(\tau_*)}=1} \langle r_h, \phi \rangle.$$

For any such $\phi$, we introduce $v_h \in \mathring{S}^h$ such that $A(v_h, \chi) = \langle \phi, \chi \rangle$ for all $\chi \in \mathring{S}^h$. The solution $v$ of the corresponding continuous problem will be employed as well. Then

$$(4.19) \qquad h^{-1} \langle r_h, \phi \rangle = h^{-1} A(r_h, v_h) = h^{-1} \delta_h(u, v_h)$$

Note that an inspection of the analysis of [23, §6] yields

$$(4.20) \quad h^{-1} \|v_h\|_{L_1(\Omega_\sigma^h)} \leq C, \qquad h^{-1} \|\nabla v_h\|_{L_1(\Omega_\sigma^h)} \leq C\varepsilon^{-1}(1 + h/\varepsilon) \ln(C + \varepsilon/h).$$

Indeed, the first bound in (4.20) follows from [23, (6.21)]. The second bound in (4.20) is obtained as follows. First, note that [23, (6.17)] yields $\|\nabla(v - v_h)\|_{L_1(\Omega_\sigma^h)} \leq C(h/\varepsilon)^2 \ln(C + \varepsilon/h)$. Then $\|\nabla v\|_{L_1(\Omega_\sigma^h)} \leq (h/\varepsilon) \ln(C + \varepsilon/h)$ is obtained employing [23, (2.7),(6.8)] by imitating the estimation in [23, (6.12),(6.13)]. Combining these observations with $\varepsilon \geq Ch$, we get (4.20).

Now we are ready to estimate the right-hand side in (4.19). In view of (4.14), setting $\Psi(x) := f(x, u(x))$, we get

$$|\delta_h(u, v_h)| = |\langle \Psi, v_h \rangle - \langle \Psi, v_h \rangle_h| \leq C(\|\Psi\|_{C^2}\|v_h\|_{L_1} + \|\Psi\|_{C^1}\|\nabla v_h\|_{L_1}) \cdot h^2,$$

where we used a version of [29, Lemma 3.1]. Combining this with (4.20) immediately yields

$$h^{-1}|\delta_h(u, v_h)| \leq C\,\mathcal{E}_{\mathrm{l.m.}} h^2 \ln(C + \varepsilon/h),$$

where the quantity $\mathcal{E}_{\mathrm{l.m.}}$ is defined in (4.8b). In view of (4.18), (4.19), we now arrive at $|r_h(x_*)| \leq C\,\mathcal{E}_{\mathrm{l.m.}} h^2 \ln(C + \varepsilon/h)$ for all $x^* \in \bar{\Omega}_\sigma^h$. Combining this with (4.17) and noting that [23, Theorems 6.1 and 12.1]) imply $|u_h - u| \leq Ch^2 \ln(C + \varepsilon/h)\|u\|_{C^2(\bar{\Omega}_\sigma^h)}$, we get the desired lumped-mass finite element error estimate (4.8). $\qquad\square$

## 5. NUMERICAL RESULTS

Our model problem (see [8]) is posed in the domain $\Omega$ shown on Figure 2, whose boundary $\partial\Omega$ is parameterized by $x_1 = \varphi(l) := R\cos\theta$ and $x_2 = \psi(l) := R\sin\theta$, where $l \in [0, 2\pi]$,

$$R = R(l) = 0.4 + \cos^2(l/2), \qquad \theta = \theta(l) = l + e^{(l-5)/2}\sin(l/2)\sin l.$$

In this domain, we consider (1.1) with

$$(5.1) \qquad b(x, u) = (u - z(x))u(u + z(x)), \qquad z(x) = x_1^2 + x_1 + 1.$$

Thus $\pm z(x)$ are two stable solutions and 0 is an unstable solution of the corresponding reduced problem. The boundary condition $g_0(x) = (x_1 - x_1^2)/3$ satisfies

TABLE 1. Errors $\max_{\bar{\Omega}} |U^{[1]} - U_{\mathrm{ref}}|$ and maximum nodal errors $|u - U_{\mathrm{ref}}|$

|  | $N$ | Shishkin mesh | | | Bakhvalov mesh | | |
|---|---|---|---|---|---|---|---|
|  | | $\varepsilon = 10^{-2}$ | $\varepsilon = 10^{-4}$ | $\varepsilon = 10^{-8}$ | $\varepsilon = 10^{-2}$ | $\varepsilon = 10^{-4}$ | $\varepsilon = 10^{-8}$ |
| $|U^{[1]}-U_{\mathrm{ref}}|$ | 32 | 4.344e-3 | 6.849e-3 | 6.900e-3 | 3.390e-3 | 1.171e-3 | 1.172e-3 |
| | 64 | 7.903e-4 | 1.191e-3 | 1.201e-3 | 9.683e-4 | 3.004e-4 | 2.999e-4 |
| | 128 | 2.825e-4 | 2.065e-4 | 2.131e-4 | 2.827e-4 | 7.658e-5 | 7.644e-5 |
| $|u-U_{\mathrm{ref}}|$ | 32 | 2.248e-2 | 2.273e-2 | 2.274e-2 | 3.658e-3 | 3.841e-3 | 3.843e-3 |
| | 64 | 8.959e-3 | 9.039e-3 | 9.039e-3 | 9.156e-4 | 9.532e-4 | 9.536e-4 |
| | 128 | 3.215e-3 | 3.232e-3 | 3.232e-3 | 2.311e-4 | 2.387e-4 | 2.388e-4 |

TABLE 2. Case of $\varepsilon = 0.1$: number of iterations $k$, value of $\max_{x_{ij} \in \partial\Omega_{2\sigma}} |(g_{2\sigma}^{[k+1]} - g_{2\sigma}^{[k]})(x_{ij})|$ triggering the stopping criterion, errors $\max_{\bar{\Omega}} |U^{[k]} - U_{\mathrm{ref}}|$ and maximum nodal errors $|u - U_{\mathrm{ref}}|$

|  | Shishkin & Bakhvalov meshes, $\varepsilon = 0.1$ | | |
|---|---|---|---|
|  | $N = 32$ | $N = 64$ | $N = 128$ |
| k | 13 | 16 | 19 |
| $|g_{2\sigma}^{[k+1]} - g_{2\sigma}^{[k]}|$ | 9.480e-4 | 2.360e-4 | 5.824e-5 |
| $|U^{[k]} - U_{\mathrm{ref}}|$ | 1.094e-2 | 2.859e-3 | 8.336e-4 |
| $|u - U_{\mathrm{ref}}|$ | 9.684e-3 | 2.656e-3 | 7.489e-4 |

(1.5b) for both $\pm z$; see Figure 1. We present numerical results only for the solution $u$ near $z$; see Figure 1 (left); the results for the solution near $-z$ are similar.

This model problem was solved by the discrete discrete Schwarz method (2.5), (2.6), (2.7) with $g_{2\sigma}^{[1]}|_{(r=2\sigma,l)} := g_0|_{(r=0,l)}$. In the boundary-layer subdomain $\Omega_{[0,2\sigma]}$, we used the Shishkin and Bakhvalov meshes $\{r_i\}$ of §2.2(a),(b) with $\gamma := 0.8\gamma_0$, where $\gamma_0 = 3\sqrt{2}/4$ (see Remark 2.2), and $C_1 := 0.2$, $\sigma_S := \min\{2\gamma^{-1}\varepsilon \ln(N/2), \frac{1}{2}C_1\}$, $\rho := 2\varepsilon$ (see Remark 2.1). The mesh $\{l_j\}$, with $N_l := 4N$, was chosen so that the arc-length between any two consecutive boundary mesh nodes was (almost) constant. In the interior subdomain $\Omega_\sigma$, we required the diameter of quasiuniform Delaunay triangulations to not exceed $N^{-1}$. In (2.5), we set $\tilde{\zeta}_{ij} := 2/[\zeta_{i,j-1}^{-1} + \zeta_{ij}^{-1}]$ which is $\zeta(r_i, l_{j-1/2}) + O(h^2)$ (so all our theoretical results remain valid for this modification). The discrete nonlinear problems (2.5) and (2.6) were solved by Newton's method.

In Tables 1 and 2, we compare the $k$th-interation Schwarz approximation $U^{[k]}$ with the reference computed solution $U_{\mathrm{ref}}$ obtained using the numerical method [8] on the mesh that coincides with the triangulation for the corresponding $U^{[k]}$ in $\bar{\Omega}_\sigma$ and the matching-tensor product mesh $\{(r_i, \tilde{l}_j), i = 0, \ldots, N, j = 0, \ldots, \tilde{N}_l\}$ in $\bar{\Omega}_{[0,\sigma]}$. Note that for $\varepsilon \leq Ch$, the error $U_{\mathrm{ref}} - u$ of this method was shown to be $O(h^2|\ln h|^m)$ in the discrete maximum norm [8]. In both tables, we also give the maximum nodal values of the errors $U_{\mathrm{ref}} - u$ computed as described in [11, §4]

(by employing an auxiliary computed solution obtained after bisecting the tensor-product mesh in $\Omega_{[0,\sigma]}$ in both directions and dividing each triangle of the Delaunay triangulation in $\Omega_\sigma$ into four triangles of the same shape).

Table 1 describes errors of the first-interation approximation $U^{[1]}$ for $\varepsilon \leq 10^{-2}$, and thus illustrates Theorem 3.9 and Corollary 4.5. We observe that the maximum errors $|U^{[1]} - U_{\mathrm{ref}}|$ are close to or much smaller than the maximum errors $|u - U_{\mathrm{ref}}|$.

In Table 2, we focus on $\varepsilon = 0.1$. As for our domain $\Omega$, the sub-domain $\Omega_{[0,C_1]}$ is well-defined for a relatively small $C_1 = 0.2$, the condition $\varepsilon \leq \frac{1}{4}C_1\gamma(\ln N)^{-1}$ of Corollary 4.5 is violated (note that in this case $\sigma = \frac{1}{2}C_1$ and the Bakhvalov mesh coincides with the Shishkin mesh). So, in view of Theorem 4.4, a number of iterations is required for our method to produce an accurate computed solution. We used the stopping criterion $\max_{x_{ij} \in \partial\Omega_{2\sigma}} |(g_{2\sigma}^{[k+1]} - g_{2\sigma}^{[k]})(x_{ij})| \leq N^{-2}$. We again note that the maximum errors $|U^{[k]} - U_{\mathrm{ref}}|$ are close to the maximum errors $|u - U_{\mathrm{ref}}|$.

In summary, the above numerical results agree with the theoretical conclusions of Theorems 3.9, 4.4 and Corollary 4.5.

## Appendix A. Proof of Lemma 4.2

*Proof.* We decompose the solution $u$ into a regular component $v$ and a boundary-layer function $w$ as follows. By imitating the argument of [4, §2], where a linear equation of type (1.1a) was considered in a rectangular domain, one can smoothly extend the function $f$ into some extended domain $\Omega^* \times \mathbb{R}$ such that $\Omega \subset \Omega^*$ and $\mathrm{dist}(\partial\Omega, \partial\Omega^*) > 1$. Then one can show that there exists a regular function $v$ such that $Fv = 0$ in $\Omega^*$, with $|\frac{\partial^{j+m}}{\partial x_1^j \partial x_2^m}v| \leq C[1 + \varepsilon^{4-(j+m)}]$ and $|\frac{\partial^{j+m}}{\partial x_1^j \partial x_2^m}(v - z)| \leq C[\varepsilon^2 + \varepsilon^{4-(j+m)}]$ in $\bar{\Omega}$ for $j, m = 0, \ldots, 4$.

Thus it remains to prove our assertions (4.5a) and (4.5b) with $u - z$ and $u$, respectively, replaced by $w := u - v$. The standard linearization yields $\mathcal{L}w = -\varepsilon^2\triangle w + p(x)w = 0$ in $\Omega$, with $p(x) = \int_0^1 f_u(x, v + sw)\, ds$, while $w = g_w := g_0 - v$ on $\partial\Omega$, so $g_w$ is a sufficiently smooth regular function.

Consider the barrier function $B_0(x) := \omega(x)e^{-\gamma r/\varepsilon} + C'\varepsilon^4$, where $\omega$ is the cut-off function from §2.1, and $C'$ is a sufficiently large constant. We claim that $\mathcal{L}B_0 \geq 0$ in $\Omega$. Indeed, in $\Omega_{2C_1}$, where $\omega = 0$, this follows from $\mathcal{L}[\varepsilon^4] \geq p(x)\varepsilon^4 \geq \gamma^2\varepsilon^4$. Next, in $\Omega_{[0,C_1]}$, where $\omega = 1$, using (2.3), we get $\mathcal{L}[e^{-\gamma r/\varepsilon}] \geq [-\gamma^2 - \varepsilon\gamma C'' + p(x)]e^{-\gamma r/\varepsilon}$ (with $C'' = \max_{x \in \bar{\Omega}_{[0,C_1]}} |\eta^{-1}\frac{\partial}{\partial r}\eta|$). So for $\varepsilon \leq c_0 := \min(\gamma_*^2 - \gamma^2)/(\gamma C'')$, where $\gamma_* := \min_{x \in \bar{\Omega}} \sqrt{p(x)}$, we have $\mathcal{L}[e^{-\gamma r/\varepsilon}] \geq 0$ and hence again $\mathcal{L}B_0 \geq 0$. (Note that for $\varepsilon \geq c_0$, problem (1.1) is not singularly perturbed so the desired bounds (4.5) follow from the Schauder-type estimates.) Finally, in $\Omega_{[C_1,2C_1]}$, where $0 < \omega < 1$, one has $r > C_1$ and therefore $|\mathcal{L}[e^{-\gamma r/\varepsilon}]| \leq Ce^{-\gamma C_1/\varepsilon} \leq C'\gamma^2\varepsilon^4$. So we get $\mathcal{L}B_0 \geq 0$ in $\Omega_{[C_1,2C_1]}$ and thus in the entire domain $\Omega$.

Now an application of the maximum/comparison principle yields $|w| \leq CB_0(x)$ so we get (4.5a) and (4.5b) for $j = m = 0$. To estimate the derivatives of $w$, note that the stretching transformation $\hat{x} = x/\varepsilon$ maps any domain $\Omega_a$ into the domain $\hat{\Omega}_a$ and, using the notation $\hat{w}(\hat{x}) = w(x)$ and $\hat{p}(\hat{x}) = p(x)$, we get $\triangle\hat{w} = \hat{p}\hat{w}$. Next, using the interior Schauder-type estimates [13, p. 110, (1.12)] for any interior subdomain $\hat{\Omega}_a$ with $a \in [\varepsilon, C_1]$ and $\mathrm{dist}(\partial\hat{\Omega}_{a-\varepsilon}, \partial\hat{\Omega}_a) = 1$, and then rewriting the result in the original variables $x = (x_1, x_2)$, we get $\max_{\bar{\Omega}_a} |\frac{\partial^{j+m}}{\partial x_1^j \partial x_2^m}w| \leq$

$C\varepsilon^{-(j+m)} \max_{\bar{\Omega}_{a-\varepsilon}} |B_0|$. This implies that

$$(A.1) \qquad |\tfrac{\partial^{j+m}}{\partial x_1^j \partial x_2^m} w| \le C\varepsilon^{-(j+m)}[e^{-\gamma a/\varepsilon} + \varepsilon^4] \quad \text{for } x \in \bar{\Omega}_a, \ j,m = 0,\dots,4,$$

where $a \in [\varepsilon, C_1]$. For $a \in [0,\varepsilon]$, bound (A.1) is obtained in a similar way, but using the global Schauder-type estimates [13, p. 110, (1.12)] in the domain $\hat{\Omega}$. So we have (A.1) for all $a \in [0, C_1]$. This immediately yields (4.5a). Furthermore, restricting (A.1) to $x \in \partial\Omega_a$ and then setting $a := r$ yields (4.5b) for $m = 0$.

It remains to prove (4.5b) for $m = 1,\dots,4$. To do this, we first need to show that $|\tfrac{\partial^m}{\partial l^m} p| \le C$ in $\Omega_{[0,C_1]}$. As the definition of $p$ involves the regular function $v$ and the boundary-layer function $w = u - v$, it suffices to show that $|\tfrac{\partial^m}{\partial l^m} u| \le C$ in $\Omega_{[0,C_1]}$, which is the rectangle $[0,C_1] \times [0,L]$ in the variables $(r,l)$. Note that, by (1.1b) and (4.5a), we have $|\tfrac{\partial^m}{\partial l^m} u| \le C$ on $\partial\Omega \cup \partial\Omega_{C_1}$. Now, differentiating equation (1.1a) in $l$, and then using (2.3) and the crude estimate $|\tfrac{\partial^{j+m}}{\partial x_1^j \partial x_2^m} u| \le C\varepsilon^{-(j+m)}$ to deal with the term $\Upsilon := \tfrac{\partial}{\partial l}(\triangle u) - \triangle(\tfrac{\partial}{\partial l} u)$, one gets $|[-\varepsilon^2\triangle + f_u(x,u)]\tfrac{\partial}{\partial l} u| = |\varepsilon^2\Upsilon - \nabla_x f(x,u)\cdot\tfrac{\partial x}{\partial l}| \le C$. So applying the maximum/comparison principle, we conclude that $|\tfrac{\partial}{\partial l} u| \le C$ and hence $|\tfrac{\partial}{\partial l} p| \le C$. In a similar manner, differentiating equation (1.1a) $m$ times in $l$ and applying the maximum/comparison principle to estimate $\tfrac{\partial^m}{\partial l^m} u$, one can show that indeed $|\tfrac{\partial^m}{\partial l^m} p| \le C$ in $\Omega_{[0,C_1]}$ for $m = 1,\dots,4$.

We are now ready to establish (4.5b) for $m > 0$. Each of $\tfrac{\partial^m}{\partial l^m} w$, for $m = 1,\dots,4$, will be estimated using the barrier function $B_m(x) := e^{-r\gamma/\varepsilon} + C'\varepsilon^{4-m}$. Imitating the argument that was used to estimate $\mathcal{L}B_0$, one can show that $\mathcal{L}B_m \ge CB_m$. Note also that, by (A.1) with $a := C_1$, we have $|\tfrac{\partial^m}{\partial l^m} w| \le CB_m$ on $\partial\Omega \cup \partial\Omega_{C_1}$. For $m = 1$, a calculation shows that $|\mathcal{L}(\tfrac{\partial}{\partial l} w)| \le CB_0 \le CB_1$, so an application of the maximum/coparison principle yields $|\tfrac{\partial}{\partial l} w| \le CB_1$ in $\bar{\Omega}_{[0,C_1]}$. Furthermore, imitating the argument that was used to prove (A.1), one gets $|\tfrac{\partial^{j+m}}{\partial x_1^j \partial x_2^m}(\tfrac{\partial}{\partial l} w)| \le C\varepsilon^{-(j+m)}[e^{-\gamma a/\varepsilon} + \varepsilon^3]$, which implies (4.5b) for $m = 1$. For $m = 2,3,4$, estimate (4.5b) is obtained similarly. $\qquad\square$

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