A SECOND-ORDER OVERLAPPING SCHWARZ METHOD FOR A 2D SINGULARLY PERTURBED SEMILINEAR REACTION-DIFFUSION PROBLEM

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Abstract. An overlapping Schwarz domain decomposition is applied to a semilinear reaction-diffusion equation posed in a smooth two-dimensional domain. The problem may exhibit multiple solutions; its diffusion parameter ε^2 is arbitrarily small, which induces boundary layers. The Schwarz method invokes a boundary-layer subdomain and an interior subdomain, the narrow subdomain overlap being of width $O(\varepsilon|\ln h|)$, where h is the maximum side length of mesh elements, and the global number of mesh nodes does not exceed $O(h^{-2})$. We employ finite differences on layer-adapted meshes of Bakhvalov and Shishkin types in the boundary-layer subdomain, and lumped-mass linear finite elements on a quasiuniform Delaunay triangulation in the interior subdomain. For this iterative method, we present maximum norm error estimates for $\varepsilon \in (0, 1]$. It is shown, in particular, that when $\varepsilon \le C|\ln h|^{-1}$, one iteration is sufficient to get second-order convergence (with, in the case of the Shishkin mesh, a logarithmic factor) in the maximum norm uniformly in ε. Numerical results are presented to support our theoretical conclusions.

1. Introduction

Consider the singularly perturbed semilinear reaction-diffusion boundary-value problem

$$Fu := -\varepsilon^2 \Delta u + f(x, u) = 0, \qquad x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2, \tag{1.1a}$$

$$u(x) = g_0(x), \qquad x \in \partial\Omega, \tag{1.1b}$$

where ε is a small positive parameter, $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ is the Laplace operator, f and g_0 are sufficiently smooth functions, and Ω is a bounded two-dimensional domain whose boundary $\partial\Omega$ is sufficiently smooth.

We shall examine solutions of (1.1) that exhibit sharp boundary layers, which are narrow regions where solutions change rapidly (see Figure 1). To obtain reliable numerical approximations of layer solutions in an efficient way, one has to use locally refined meshes that are fine and anisotropic in layer regions and standard outside. When multidimensional meshes of different nature are introduced in different non-overlapping subdomains (e.g., in layer regions and outside), it might be rather

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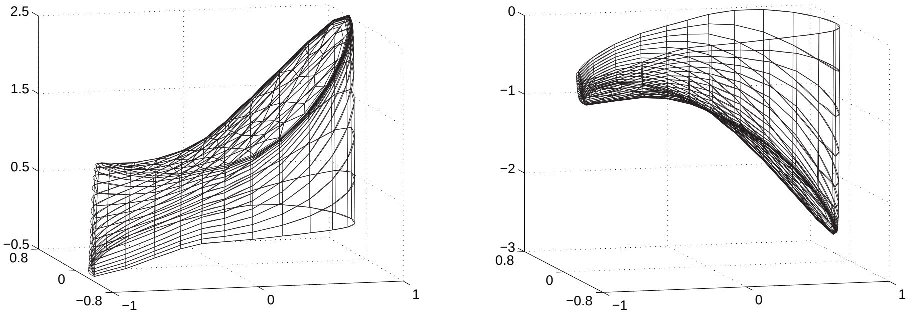

FIGURE 1. *Multiple boundary-layer solutions of model problem (5.1); in the interior subdomain $u(x) \approx z(x)$ (left) or $u(x) \approx -z(x)$ (right), where $\pm z(x)$ are stable solutions of the reduced problem (1.4).*

inconvenient to match them, while non-overlapping non-matching meshes require a special treatment (see, e.g., [6] for non-matching meshes used to solve a problem of type (1.1)). Furthermore, different discretizations of differential equations might be used in layer regions and outside, in which case they should be matched along the interface boundaries (see, e.g., [8]).

Handling non-overlapping non-matching meshes and matching different discretizations along the interface boundaries can be entirely avoided by invoking iterative overlapping domain decomposition methods of Schwarz-Chimera type; see, e.g., [19, §1.5]. Note that non-overlapping domain decomposition methods, at best, have conventional geometric rates of convergence when applied to singularly perturbed problems of type (1.1). In contrast, overlapping methods, with the subdomain overlap being as narrow as $O(\varepsilon|\ln h|)$, where h is the triangulation diameter, might enjoy much faster convergence. To be more precise, we prove in this paper that one iteration is sufficient to achieve second-order accurate computed solutions when $\varepsilon \le C|\ln h|^{-1}$, where the global number of mesh nodes does not exceed $O(h^{-2})$; see Theorems 3.9 and 4.4 for details.

We now present a continuous version of the discrete Schwarz method that we investigate. Define, for some $0 \le a < b$, subdomains of Ω:

$$\Omega_a := \{x \in \Omega \,:\, \mathrm{dist}(x, \partial\Omega) < a\}, \qquad \Omega_{[a,b]} := \{x \in \Omega \,:\, a < \mathrm{dist}(x, \partial\Omega) < b\},$$

so we have $\Omega_0 = \Omega$, $\Omega_{[a,b]} = \Omega_a \backslash \Omega_b$, and $\partial\Omega_{[a,b]} = \partial\Omega_a \cup \partial\Omega_b$. Consider the **overlapping subdomains** Ω_σ and $\Omega_{[0,2\sigma]} = \Omega \backslash \bar{\Omega}_{2\sigma}$, where $\sigma > 0$ is sufficiently small so that these subdomains are well-defined and smooth; see Figure 2 (left). Let u_σ and $u_{[0,2\sigma]}$ be solutions of the following boundary value problems

$$Fu_{[0,2\sigma]} = 0 \ \text{ for } x \in \Omega_{[0,2\sigma]}, \qquad \begin{array}{ll} u_{[0,2\sigma]}(x) = g_0(x) & \text{for } x \in \partial\Omega, \\ u_{[0,2\sigma]}(x) = g_{2\sigma}(x) & \text{for } x \in \partial\Omega_{2\sigma}, \end{array} \tag{1.2a}$$

$$Fu_\sigma = 0 \ \text{ for } x \in \Omega_\sigma, \qquad u_\sigma(x) = u_{[0,2\sigma]}(x) \ \text{ for } x \in \partial\Omega_\sigma. \tag{1.2b}$$

Here g_0 is from the boundary condition of our original problem (1.1), while $g_{2\sigma}$ is updated for each iteration by

$$g_{2\sigma}(x) = g_{2\sigma}^{[k]}(x) := \left\{ \begin{array}{ll} g_{2\sigma}^{[1]}(x) & \text{for } \ k = 1, \\ u^{[k-1]}(x) & \text{for } \ k = 2, 3, \ldots, \quad x \in \partial\Omega_{2\sigma}, \end{array} \right. \tag{1.3a}$$

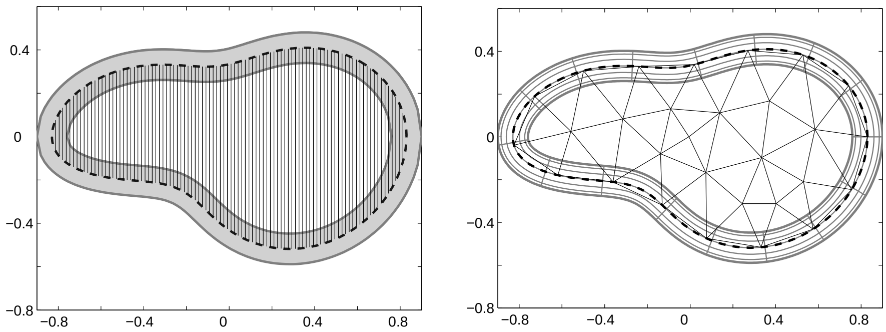

FIGURE 2. *Overlapping boundary-layer subdomain $\Omega_{[0,2\sigma]}$ and interior subdomain Ω_σ (left); layer-adapted tensor-product mesh in $\Omega_{[0,2\sigma]}$ and quasiuniform Delaunay triangulation in Ω_σ (right).*

with some suitable initial guess $g_{2\sigma}^{[1]}$. Successively solving problems (1.2a) and (1.2b) with $g_{2\sigma} = g_{2\sigma}^{[k]}$, for $k = 1, 2 \ldots$, we get the **kth-iteration approximations**:

$$u^{[k]}(x) := \begin{cases} u_{[0,2\sigma]}(x) & \text{for } x \in \bar{\Omega}_{[0,\sigma]} = \bar{\Omega} \backslash \Omega_\sigma, \\ u_\sigma(x) & \text{for } x \in \bar{\Omega}_\sigma. \end{cases} \tag{1.3b}$$

We discretize the domain $\Omega_{[0,2\sigma]}$ as in Figure 2 (right), using layer-adapted tensor-product meshes of Bakhvalov and Shishkin types whose number of mesh nodes does not exceed Ch^{-2}. We then solve problem (1.2a) in this domain using standard finite differences in curvilinear coordinates. For problem (1.2b) in the domain Ω_σ, we use lumped-mass linear finite elements on a quasiuniform Delaunay triangulation of diameter h.

When considering semilinear problems of type (1.1), it is frequently assumed in the numerical analysis literature (see, e.g., [3, 21]) that $f_u(x, u) > \gamma^2 > 0$ for all $(x, u) \in \bar{\Omega} \times \mathbb{R}$ and some positive constant γ. Under this assumption, our problem (1.1) and the associated reduced problem (1.1), i.e.

$$f(x, z(x)) = 0 \qquad \text{for } x \in \Omega, \tag{1.4}$$

defined by setting $\varepsilon = 0$ in (1.1), have unique solutions u and z. This global assumption is however rather restrictive. E.g., mathematical models of biological and chemical processes frequently involve problems related to (1.1) with $f(x, u)$ that is non-monotone with respect to u. Therefore, we examine problem (1.1) under the following weaker **assumptions** also used in [5, 17]:

- it has a *stable reduced solution*, i.e., there exists a sufficiently smooth solution z of (1.4) such that

$$f_u(x, z) > \gamma^2 > 0 \qquad \text{for all } x \in \bar{\Omega}; \tag{1.5a}$$

- the boundary condition g_0 on $\partial\Omega$, also denoted $\partial\Omega_0$, satisfies the assumption, with $d = 0$, that

$$\int_{z(x)}^{v} f(x, s)\, ds > 0 \qquad \text{for all } v \in \big(z(x), g_d(x)\big]', \qquad x \in \partial\Omega_d. \tag{1.5b}$$

Here the notation $(a, b]'$ is defined to be $(a, b]$ when $a < b$ and $[b, a)$ when $a > b$, while $(a, b]' = \emptyset$ when $a = b$.

Note that if $g_0(x) \approx u_0(x)$, then (1.5b) follows from (1.5a) combined with (1.4), while if $g_0(x) = u_0(x)$ at some point $x \in \partial\Omega$, then (1.5b) does not impose any restriction on g_0 at this point. (Problem (1.2a) should also satisfy (1.5b) with $d = 2\sigma$; otherwise the nonlinear problem (1.2a) may have no solutions. When σ is small, one can simply take $g_{2\sigma}^{[1]} \approx g_0$.)

Conditions (1.5) intrinsically arise from the asymptotic analysis of problem (1.1) and guarantee that there exists a boundary-layer solution u such that $u \approx u_0$ in the interior part of Ω, while the boundary layer is of width $O(\varepsilon|\ln\varepsilon|)$; see, e.g., [5, 17, 7]. Note that assumption (1.5a) is local, i.e. the reduced problem (1.4) is permitted to have more than one stable solution. Furthermore, if multiple stable solutions of the reduced problem satisfy (1.5), then problem (1.1) has multiple boundary-layer solutions.

The discrete Schwarz method that we consider is a domain decomposition version of the numerical method of [8], where problem (1.1), (1.5) was posed in a smooth two-dimensional domain, and it was shown that one gets second-order convergence in the discrete maximum norm under the condition $\varepsilon \le Ch$. Note that in the present paper we give convergence estimates for all $\varepsilon \in (0, 1]$. In one dimension, similar domain decomposition methods using layer-adapted meshes have been analyzed for linear [15, 23] and semilinear [10] equations of type (1.1); in particular, faster convergence of the algorithm for small values of ε was addressed in [10, 23]. The numerical analysis literature addressing problems of type (1.1) posed in various two-dimensional domains is discussed in [8]. In particular, the semilinear equation (1.1) under the condition $f_u > \gamma^2 > 0$ was considered in [3, 21], while linear equations of this type were considered in [1, 4, 13, 16].

The paper is organized as follows. In §2 we introduce independent meshes and discretizations in the subdomains $\Omega_{[0,2\sigma]}$ and Ω_σ, and then present a discrete version of the continuous Schwarz method (1.2), (1.3). The errors in the discrete Schwarz method are estimated in two regimes: for $\varepsilon \le Ch$ in §3 and $\varepsilon \ge Ch$ in §4. So throughout §3 we let $\varepsilon \le Ch$. Asymptotic properties of solutions in particular subdomains are discussed, and then appropriate sub- and super-solutions are constructed in §3.1. Errors in the continuous and discrete Swartz methods are estimated, respectively, in §3.2 and §§3.3–3.4. In particular, in §3.4 we employ discrete sub- and super-solutions, whose basic properties are sketched in §3.3. Throughout §4 we let $\varepsilon \ge Ch$ and make a simplifying assumption that $f_u > \gamma^2 > 0$. Then errors in the continuous and discrete Schwarz method are estimated, respectively, in §4.1 and §§4.2–4.3. In §4.4 we get an auxiliary stability result for the finite-difference discrete operator in $\Omega_{[0,2\sigma]}$, which is used to establish supraconvergence in this subdomain. In §4.5 we get another auxiliary result by extending a maximum norm error estimate for the standard finite element method [21] to its lumped-mass version. Finally, in §5, some numerical results illustrate our theoretical conclusions.

Notation. We let C denote a generic positive constant that may take different values in different formulas, but is independent of ε, h and the number of iterations taken by the Schwarz algorithm. A subscripted C (e.g., C_1) denotes a positive constant that is independent of ε, h and the number of iterations, but takes a fixed value. For any two quantities y and z, the notation $y = O(z)$ means $|y| \le Cz$.

2. Discrete Schwarz method. Discretizations in particular subdomains

2.1. Local curvilinear coordinates. Let the boundary $\partial\Omega$ be parametrized by

$$x_1 = \varphi(l), \qquad x_2 = \psi(l), \qquad 0 \le l \le L,$$

where $(\varphi(0), \psi(0)) = (\varphi(L), \psi(L))$ and as l increases, the domain remains on the left. Any functions that are defined for l beyond $[0, L]$ should be understood as extended L-periodically. We shall use the magnitude $\tau > 0$ of the tangent vector (φ', ψ') and the curvature κ of the boundary at $(\varphi(l), \psi(l))$ that are defined by

$$\tau = \sqrt{\varphi'^2 + \psi'^2}, \qquad \kappa = \kappa(l) = \frac{\varphi'\psi'' - \psi'\varphi''}{\tau^3}.$$

In a narrow neighbourhood of $\partial\Omega$ that will be specified later, introduce the curvilinear local coordinates (r, l) by

$$x_1 = \varphi(l) + rn_1(l), \qquad x_2 = \psi(l) + rn_2(l), \tag{2.1}$$

where (n_1, n_2) is the inward unit normal to $\partial\Omega$ at $(\varphi(l), \psi(l))$, i.e., it is orthogonal to the tangent vector (φ', ψ') and is defined by

$$n_1 = \frac{-\psi'}{\tau}, \qquad n_2 = \frac{\varphi'}{\tau}.$$

Since $\partial\Omega$ is smooth, there exists a sufficiently small constant C_1 such that in the subdomain $\bar\Omega_{2C_1}$ the new coordinates are well-defined, the mapping $(r, l) \mapsto (x_1, x_2)$ is one-to-one and invertible, and, furthermore,

$$\operatorname{dist}(x, \partial\Omega) = r \qquad \text{for all } \ x \in \bar\Omega_{2C_1}. \tag{2.2}$$

Throughout the paper we shall use a smooth positive cut-off function $\omega(x)$ that equals 1 for $r \le C_1$ and vanishes in $\bar\Omega \backslash \bar\Omega_{2C_1}$.

Note [8, Lemma 2.1] that the curvilinear coordinates (2.1) are orthogonal, and for the Laplace operator we have

$$\triangle u = \eta^{-1}\frac{\partial}{\partial r}\Big(\eta\frac{\partial u}{\partial r}\Big) + \zeta\frac{\partial}{\partial l}\Big(\zeta\frac{\partial u}{\partial l}\Big), \quad \text{where } \eta := 1 - \kappa r,\ \zeta := (\tau\eta)^{-1}. \tag{2.3}$$

2.2. Layer-adapted meshes. To discretize the continuous Schwarz method in (1.2), (1.3), we now introduce independent meshes in the overlapping subdomains $\Omega_{[0,2\sigma]}$ and Ω_σ, to which we shall refer as the interior subdomain and the boundary-layer subdomain respectively; see Figure 2.

In the **interior subdomain** Ω_σ introduce a quasiuniform Delaunay triangulation of some small diameter $h \in (0, \frac12)$, i.e. the maximum side length of any triangle is at most h, the area of any triangle is bounded below by Ch^2, and the sum of the angles opposite to any edge is less than or equal to π (while any angle opposite to $\partial\Omega_\sigma$ does not exceed $\pi/2$). Let the union of all the triangles define a polygonal domain Ω^h_σ whose boundary vertices lie on $\partial\Omega_\sigma$.

The **boundary-layer subdomain** $\Omega_{[0,2\sigma]}$ is the rectangle $(0, 2\sigma) \times [0, L]$ in the coordinates (r, l). Hence in this subdomain introduce the tensor-product mesh $\{(r_i, l_j), i = 0, \dots, 2N, j = 0, \dots N_l + 1\}$, where $r_N = \sigma$ and, as usual, $r_0 = 0$, $r_{2N} = 2\sigma$, $l_0 = 0$, and $l_{N_l} = L$, while $l_{N_l+1} = l_1 + L$. Furthermore, let $\{l_j\}$ be a quasiuniform mesh on $[0, L]$, i.e., $C^{-1}h \le l_j - l_{j-1} \le Ch$. The choice of the *layer-adapted* mesh $\{r_i\}$ on $[0, 2\sigma]$ is crucial and will be discussed later; see (a),(b). Now assume only that $r_i - r_{i-1} \le h$ and

$$C^{-1}h^{-1} \le N \le Ch^{-1}. \tag{2.4}$$

Note that we do **not require** that both the interior and layer meshes have the same sets of nodes on $\partial\Omega_\sigma$. Thus information will be exchanged between $\Omega_{[0,2\sigma]}$ and Ω_σ using piecewise linear/bilinear computed solutions (or their interpolants) in these subdomains.

We focus on two particular choices of $\{r_i\}$:

2.2(a) Shishkin mesh. [22] Set $\sigma = \sigma_S := \min\{2\gamma^{-1}\varepsilon \ln N, \frac{1}{2}C_1\}$ and introduce a uniform mesh $\{r_i\}_{i=0}^{2N}$ on $[0, 2\sigma]$, i.e., $r_i - r_{i-1} = \sigma/N = 2\gamma^{-1}\varepsilon N^{-1}\ln N$.

2.2(b) Bakhvalov mesh. [2] Let $\rho := \varepsilon$ and $\sigma_B := 2\gamma^{-1}\varepsilon|\ln\rho|$. We now set $\sigma := \max\{\sigma_B, \sigma_S\}$ and $\bar\rho := (1-\rho) + [\sigma - \sigma_B]\rho/(2\gamma^{-1}\varepsilon)$, and define the meshpoints by $r_i := r\big([1-\varepsilon]\, i/N\big)$ for $i = 0\ldots, 2N$, where $r(t) \in C^1[0, 2\bar\rho]$ is given by

$$r(t) := \begin{cases} -2\gamma^{-1}\varepsilon\ln(1-t) & \text{for } t \in [0, 1-\rho], \\ \sigma_B + [t - (1-\rho)](2\gamma^{-1}\varepsilon)/\rho & \text{for } t \in [1-\rho, \bar\rho], \\ 2\sigma - r(2\bar\rho - t) & \text{for } t \in [\bar\rho, 2\bar\rho], \end{cases}$$

so that the sub-mesh $\{r_i\}_{i=N}^{2N}$ reflects the sub-mesh $\{r_i\}_{i=1}^{N}$ in $r = \sigma$.

Remark 2.1. For the mesh §2.2(a) we always have $2\sigma \le C_1$, so this mesh is always well-defined. The mesh §2.2(b) is well-defined provided that $\varepsilon \le e^{-1}$ and $2\sigma_B \le C_1$; otherwise we have $\varepsilon > C$ for some constant C, and, imitating [2], we extend the mesh definition §2.2(b) by using the mesh §2.2(a) with $\sigma := \frac{1}{2}C_1$. In general, when $\varepsilon > C$, i.e. our problem is not singularly perturbed, one can simply use linear finite elements on a quasiuniform Delaunay triangulation of the whole domain $\bar\Omega$ [21]. Note that one can replace $\ln N$ by $\ln(C'N)$ in the definition of σ_S in §2.2(a), and can also use $\rho = C''\varepsilon$ for the mesh §2.2(b), with some arbitrary constants C', C''.

Remark 2.2. In the mesh definitions §2.2(a) and §2.2(b) the constant γ from (1.5a) can be replaced by an arbitrary constant $\tilde\gamma \in (0, \gamma_0)$, where γ_0 is from Lemma 3.1; see Remark 3.2.

2.3. **Discretization in the boundary-layer subdomain.** Recall that $\Omega_{[0,2\sigma]}$ is the rectangle $(0, 2\sigma) \times [0, L]$ in the coordinates (r, l). Hence rewrite problem (1.2a) in the (r, l) coordinates using (2.3), and then discretize it using the standard finite differences on the tensor-product mesh $\{(r_i, l_j)\}$ as follows. For $i = 1, \ldots, 2N-1$, $j = 1, \ldots, N_l$, set

$$\begin{aligned} &F^h_{[0,2\sigma]}U_{ij} := -\varepsilon^2\eta_{ij}^{-1}D_r[\tilde\eta_{ij}D_r^- U_{ij}] - \varepsilon^2\zeta_{ij}D_l[\tilde\zeta_{ij}D_l^- U_{ij}] + f(x_{ij}, U_{ij}) = 0, \\ &U_{i,0} = U_{i,N_l}, \quad U_{i,1} = U_{i,N_l+1}, \quad U_{0,j} = g_0(x_{0,j}), \quad U_{2N,j} = g_{2\sigma}(x_{2N,j}). \end{aligned} \tag{2.5}$$

Here U_{ij} is the discrete computed solution at the mesh node $x_{ij} \in \bar\Omega_{[0,2\sigma]}$,

$$D_r^- v_{ij} := \frac{v_{ij} - v_{i-1,j}}{r_i - r_{i-1}}, \qquad D_r v_{ij} := \frac{v_{i+1,j} - v_{ij}}{(r_{i+1} - r_{i-1})/2},$$
$$D_l^- v_{ij} := \frac{v_{ij} - v_{i,j-1}}{l_j - l_{j-1}}, \qquad D_l v_{ij} := \frac{v_{i,j+1} - v_{ij}}{(l_{j+1} - l_{j-1})/2},$$

and

$$\eta_{ij} := \eta(r_i, l_j), \qquad \zeta_{ij} := \zeta(r_i, l_j), \qquad x_{ij} := x(r_i, l_j),$$
$$\tilde\eta_{ij} := \eta(r_{i-1/2}, l_j), \qquad \tilde\zeta_{ij} := \zeta(r_i, l_{j-1/2}).$$

The computed solution $U_{[0,2\sigma]}(x)$ for $x \in \bar{\Omega}_{[0,2\sigma]}$ is obtained by the standard bilinear interpolation of U_{ij} in the (r,l) coordinates on the tensor-product mesh $\{(r_i, l_j)\}$.

2.4. **Discretization in the interior subdomain.** We discretize problem (1.2b) in Ω_σ using *lumped-mass* linear finite elements. Let $S^h \subset W_2^1(\Omega_\sigma^h)$ be the standard finite element space of continuous functions that are linear on each of the triangles of our mesh in $\bar{\Omega}_\sigma^h$. Let $\{q_i\}$ be the set of mesh nodes of the mesh in $\bar{\Omega}_\sigma^h$. Now we require the computed solution $U_\sigma \in S^h$ to satisfy $U_\sigma(q_j) = U_{[0,2\sigma]}(q_j)$ at each boundary mesh node $q_j \in \partial\Omega_\sigma^h$, and also

$$F_\sigma^h U_\sigma(q_i) := \frac{\varepsilon^2}{\langle 1, \chi_i \rangle} \langle \nabla U, \nabla \chi_i \rangle + f(q_i, U_\sigma(q_i)) = 0 \qquad \forall\, q_i \in \Omega_\sigma^h. \tag{2.6}$$

where q_i is a interior mesh node in Ω_σ^h, and $\chi_i \in S^h$ is the standard nodal basis functions (i.e. $\chi_i(q_j)$ equals 1 if $i = j$, and 0 otherwise). The notation $\langle \cdot, \cdot \rangle$ is used for the inner product in $L_2(\Omega_\sigma^h)$. Note that the finite element method (2.6) uses the lumped-mass discretization of the integral involving f, which is more evident if (2.6) is multiplied by $\langle 1, \chi_i \rangle$. Note also that as a a Delaunay triangulation is used, the discretization of the operator $-\triangle$ in (2.6) yields an M-matrix.

2.5. **Discrete Schwarz approximations.** We now imitate (1.3). The boundary condition $g_{2\sigma}$ in (2.5) is updated for each iteration by

$$g_{2\sigma}(x) = g_{2\sigma}^{[k]}(x) := \begin{cases} g_{2\sigma}^{[1]}(x) & \text{for } k = 1, \\ U^{[k-1]}(x) = U_\sigma(x) & \text{for } k = 2, 3, \ldots, \end{cases} \quad x \in \partial\Omega_{2\sigma}, \tag{2.7a}$$

with some suitable initial guess $g_{2\sigma}^{[1]}$. Successively solving problems (2.5) and (2.6) with $g_{2\sigma} = g_{2\sigma}^{[k]}$, for $k = 1, 2 \ldots$, we get the k**th-iteration approximations**:

$$U^{[k]}(x) := \begin{cases} U_{[0,2\sigma]}(x) & \text{for } x \in \bar{\Omega} \backslash \bar{\Omega}_\sigma^h, \\ U_\sigma(x) & \text{for } x \in \bar{\Omega}_\sigma^h. \end{cases} \tag{2.7b}$$

Strictly speaking, (2.7a) is well-defined for $k \geq 2$ only if $\Omega_{2\sigma} \subset \Omega_\sigma^h$, while we have $\Omega_{2\sigma} \subset \Omega_\sigma$, so some extrapolation of U_σ from $\bar{\Omega}_\sigma^h$ onto $\bar{\Omega}_\sigma \backslash \bar{\Omega}_\sigma^h$ may be employed. In practice, no extrapolation is required as $\text{dist}(\partial\Omega_{2\sigma}, \partial\Omega_\sigma) = \sigma \geq C\varepsilon \ln h$ and $\text{dist}(\partial\Omega_\sigma^h, \partial\Omega_\sigma) = O(h^2)$. Consequently, whenever $\varepsilon \geq Ch^2$, relation (2.7a) is well-defined; otherwise, as we shall show in Theorem 3.9, one iteration of the discrete Schwarz method is sufficient.

3. Maximum norm error analysis for $0 < \varepsilon \leq Ch$

3.1. **Continuous problems in particular subdomains. Sub- and super-solutions.** As our method involves the numerical solution of the differential equation (1.1a) in certain subdomains, we shall first consider this equation and asymptotic properties of its solutions in arbitrary particular subdomains Ω_a and $\Omega_{[a,b]}$.

Let $u_a(x)$ and $u_{[a,b]}(x)$ be solutions of the problems (compare with (1.2))

$$Fu_a = 0 \ \text{ for } x \in \Omega_a, \quad u_a(x) = g_a(x) \ \text{ for } x \in \partial\Omega_a, \tag{3.1}$$

$$Fu_{[a,b]} = 0 \ \text{ for } x \in \Omega_{[a,b]}, \quad \begin{array}{ll} u_{[a,b]}(x) = g_a(x) & \text{for } x \in \partial\Omega_a, \\ u_{[a,b]}(x) = g_b(x) & \text{for } x \in \partial\Omega_b. \end{array} \tag{3.2}$$

Here $0 \leq a < b \leq C_1$ so that the domains Ω_a and $\Omega_{[a,b]}$ are well-defined. Only to avoid considering cases, we assume that $g_d \geq u_0(d)$ for $d = a, b$.

Then solutions u_a and $u_{[a,b]}$ of problems (3.1) and (3.2) typically exhibit boundary layers, and their standard first-order **asymptotic expansions** $u_{\mathrm{as}\,;\,a}$ and $u_{\mathrm{as}\,;\,[a,b]}$ are given [5, 8, 17] by

$$u_{\mathrm{as}\,;\,a}(x) = z(x) + \big[v_{0;\,a}(\xi^+,l) + \varepsilon v_{1;\,a}(\xi^+,l)\big]\,\omega(x), \tag{3.3a}$$

$$\begin{aligned} u_{\mathrm{as}\,;\,[a,b]}(x) &= z(x) + \big[v_{0;\,a}(\xi^+,l) + \varepsilon v_{1;\,a}(\xi^+,l)\big] \\ &\quad + \big[v_{0;\,b}(\xi^-,l) + \varepsilon v_{1;\,b}(\xi^-,l)\big]. \end{aligned} \tag{3.3b}$$

(Note that the cut-off function $\omega = 1$ in $\Omega_{[a,b]}$, so $u_{\mathrm{as}\,;\,[a,b]} - u_{\mathrm{as}\,;\,a} = [v_{0;\,b} + \varepsilon v_{1;\,b}]$.) Here the components $[v_{0;\,a} + \varepsilon v_{1;\,a}]$ and $[v_{0;\,b} + \varepsilon v_{1;\,b}]$ describe the boundary layers along $\partial\Omega_a$ and $\partial\Omega_b$, respectively. They use the stretched variables $\xi^+ = \xi_a^+ := \frac{r-a}{\varepsilon}$ and $\xi^- = \xi_b^- := \frac{b-r}{\varepsilon}$. More generally,

$$\xi_d^\pm := \pm(r-d)/\varepsilon.$$

When there is no ambiguity, as, e.g., in (3.3), the notation $\xi^\pm$ is used for ξ_a^+ and ξ_b^-. Note that $\xi_d^\pm = 0$ corresponds to $r = d$, and ξ_d^+ has the same positive direction as the r-axis, while ξ_d^- has the opposite direction.

The boundary-layer functions $v_{0;\,d}(\xi^\pm,l)$ and $v_{1;\,d}(\xi^\pm,l)$ in (3.3), with $d = a, b$, satisfy the ordinary differential equations

$$-\big(\tfrac{\partial}{\partial\xi^\pm}\big)^2 v_{0;\,d} + f(\bar{x}_d, z(\bar{x}_d) + v_{0;\,d}) = 0\,, \tag{3.4a}$$

$$\big[-\big(\tfrac{\partial}{\partial\xi^\pm}\big)^2 + f_u(\bar{x}_d, z(\bar{x}_d) + v_{0;\,d})\big]\, v_{1;\,d} = \mp Q_d(\xi^\pm,l)\,, \tag{3.4b}$$

with the boundary conditions

$$v_{0;\,d}(0,l) = g_d(\bar{x}_d) - z(\bar{x}_d), \quad v_{1;\,d}(0,l) = v_{0;\,d}(\infty,l) = v_{1;\,d}(\infty,l) = 0, \tag{3.4c}$$

where the variable l appears as a parameter, and

$$\begin{aligned} \bar{x}_d = \bar{x}_d(l) &:= \big(\varphi(l) + dn_1(l), \psi(l) + dn_2(l)\big) \in \partial\Omega_d, \\ Q_d(\xi^\pm,l) &:= \xi^\pm\ \tfrac{d}{dr} f(x, z(x)+s)\big|_{x:=\bar{x}_d;\ s=v_{0;d}} + \tfrac{\kappa}{1-\kappa d}\big(\tfrac{\partial}{\partial\xi^\pm} v_{0;\,d}\big)\,. \end{aligned} \tag{3.4d}$$

Note that relations (3.4) either *all* use $\xi^+ = \xi_d^+$ and so define $v_{0;\,d}(\xi^+)$ and $v_{1;\,d}(\xi^+)$, or all use $\xi^- = \xi_d^-$ and then define $v_{0;\,d}(\xi^-)$ and $v_{1;\,d}(\xi^-)$. Note also that Q_d in (3.4d) is obtained using $\eta^{-1}\frac{\partial\eta}{\partial r}\big|_{r=d} = \frac{\kappa}{1-\kappa d}$.

To construct sub- and super-solutions for problems (3.1) and (3.2), we need a perturbed version $\tilde{v}_{0;\,d} = \tilde{v}_{0;\,d}(\xi^\pm, l; p)$ of $v_{0;\,d}$, which, for $d = a, b$, is defined by generalizing equations (3.4a) with the boundary conditions (3.4c):

$$\begin{aligned} -\big(\tfrac{d}{d\xi^\pm}\big)^2 \tilde{v}_{0;\,d} + f(\bar{x}_d, z(\bar{x}_d) + \tilde{v}_{0;\,d}) &= p\tilde{v}_{0;\,l}, \\ \tilde{v}_{0;\,l}(0,l;p) = g_d(\bar{x}_d) - z(\bar{x}_d), \qquad \tilde{v}_{0;\,l}(\infty,l;p) &= 0. \end{aligned} \tag{3.5}$$

Clearly, we have $\tilde{v}_{0;\,d}(\xi^\pm,l;0) = v_{0;\,d}(\xi^\pm,l)$ for $d = a, b$. Now, by replacing $v_{0;\,d}$ by its perturbations $\tilde{v}_{0;\,d}$ and introducing a perturbation term $C_0 p$, we define **perturbed** versions β_a and $\beta_{[a,b]}$ of the **asymptotic expansions** of (3.3):

$$\beta_a(x;p) = z(x) + \big[\tilde{v}_{0;\,a}(\xi^+;p) + \varepsilon v_{1;\,a}(\xi^+)\big]\,\omega(x) + C_0 p, \tag{3.6a}$$

$$\begin{aligned} \beta_{[a,b]}(x;p) &= z(x) + \big[\tilde{v}_{0;\,a}(\xi^+;p) + \varepsilon v_{1;\,a}(\xi^+)\big] \\ &\quad + \big[\tilde{v}_{0;\,b}(\xi^-;p) + \varepsilon v_{1;\,b}(\xi^-)\big] + C_0 p. \end{aligned} \tag{3.6b}$$

Here p is a small real number that will be chosen later and is typically $o(h)$; for some small $p > 0$ the functions $\beta_{[a,b]}(x; \pm p)$ will serve as sub- and super-solutions.

The following lemma combines the results of [11, Lemma 2.1 and (2.15)]; the proof uses dynamical systems in the analysis of problems (3.4) and (3.5).

Lemma 3.1. *Set* $\gamma_0^2 = \min_{x\in\partial\Omega_a\cup\partial\Omega_b} f_u(x,z(x)) > \gamma^2$, *where* $\gamma > 0$ *is from (1.5a). Given assumption* (1.5b) *with* $d = a, b$, *there exists* $p_0 \in (0, \gamma_0^2)$ *such that for all* $|p| \le p_0$, *problems* (3.4) *and* (3.5) *have solutions* $v_{0;\,a}(\xi^+, l)$, $v_{0;\,b}(\xi^-, l)$, $v_{1;\,a}(\xi^+, l)$, $v_{1;\,b}(\xi^-, l)$, $\tilde v_{0;\,a}(\xi^+, l; p)$ *and* $\tilde v_{0;\,b}(\xi^-, l; p)$. *We also have*

$$v_{0;\,d}(\xi^\pm, l) \ge 0, \qquad \tfrac{\partial}{\partial p}\tilde v_{0;\,d}(\xi^\pm, l; p) \ge 0, \qquad \text{where} \quad d = a, b. \tag{3.7}$$

Furthermore, for an arbitrarily small but fixed $\delta \in (0, \gamma_0 - \sqrt{p_0})$, *there is a positive constant* C_δ *such that*

$$\bigl|\bigl(\tfrac{\partial}{\partial \xi^\pm}\bigr)^k \tilde v_{0;\,d}\bigr| + \bigl|\bigl(\tfrac{d}{d\xi^\pm}\bigr)^k v_{1;\,d}\bigr| + \bigl|\tfrac{\partial}{\partial p}\, \tilde v_{0;\,d}\bigr| \le C_\delta\, e^{-(\gamma_0-\sqrt{p_0}-\delta)\,\xi^\pm} \max_{\partial\Omega_d} |g_d - z| \tag{3.8}$$

for $d = a, b$ *and* $\xi^\pm \ge 0$, $k = 0, 1, \ldots, 4$.

Remark 3.2. As $\gamma_0 > \gamma$, choosing p_0 and δ in Lemma 3.1 sufficiently small, we can make $\gamma_0 - \sqrt{p_0} - \delta$ in (3.8) satisfy $\gamma_0 - \sqrt{p_0} - \delta > \gamma$, which then yields $e^{-(\gamma_0-\sqrt{p_0}-\delta)\,\xi^\pm} \le e^{-\gamma\,\xi^\pm}$ Consequently, we have

$$e^{-(\gamma_0-\sqrt{p_0}-\delta)\,\xi_a^+} \le e^{-\gamma\,(x-a)/\varepsilon}, \qquad e^{-(\gamma_0-\sqrt{p_0}-\delta)\,\xi_b^-} \le e^{-\gamma\,(b-x)/\varepsilon}. \tag{3.9}$$

Similarly, we can choose p_0 and δ so that $\gamma_0 - \sqrt{p_0} - \delta > \tilde\gamma$ for any $\tilde\gamma < \gamma_0$, which then yields (3.9) with γ replaced by $\tilde\gamma$.

Next we investigate the perturbed asymptotic expansions $\beta_a(x;p)$ and $\beta_{[a,b]}(x;p)$.

Lemma 3.3. *Under the assumptions of Lemma 3.1, the functions* $\beta_a(x;p)$ *and* $\beta_{[a,b]}(x;p)$ *of (3.6) satisfy*

$$F\beta_a(x;p) = C_0 p\, f_u(x,z) + [1 + C_0\lambda_a]\, p\, v_{0;\,a}(\xi^+, l) + O(\varepsilon^2 + p^2) \tag{3.10}$$

for $x \in \Omega_a$, *where* $\lambda_a = \lambda_a(x) := f_{uu}\bigl(x, z + \vartheta v_{0;\,a}\bigr)$ *and* $\vartheta = \vartheta(x) \in (0,1)$, *and*

$$\begin{aligned} F\beta_{[a,b]}(x;p) \;&=\; C_0 p\, f_u(x,z) + [1 + C_0\lambda_{[a,b]}]\, p\, [v_{0;\,a}(\xi^+, l) + v_{0;\,b}(\xi^-, l)] \\ &\qquad + O(\varepsilon^2 + p^2 + e^{-\gamma(b-a)/(2\varepsilon)}) \end{aligned} \tag{3.11}$$

for $x \in \Omega_{[a,b]}$, *where* $\lambda_{[a,b]} = \lambda_{[a,b]}(x) := f_{uu}\bigl(x, z + \hat\vartheta[v_{0;\,a} + v_{0;\,b}]\bigr)$, $\hat\vartheta = \hat\vartheta(x) \in (0,1)$.

Proof. The first assertion (3.10) (for $a = 0$) is given by [8, Lemma 2.8].

Next, consider $x \in \Omega_{[a,(a+b)/2]}$. In view of (3.10), to obtain the second assertion (3.11) in this case, it suffices to prove the bound $|F\beta_{[a,b]}(x;p) - F\beta_a(x;p)| \le Ce^{-\gamma\,(b-r)/\varepsilon} \le Ce^{-\gamma(b-a)/(2\varepsilon)}$ for all $r \in (a, (a+b)/2]$ and similar bounds for $|\lambda_{[a,b]} - \lambda_a|$ and $|v_{0;\,b}|$. In particular, the first of the required bounds follows from

$$F\beta_{[a,b]}(x;p) - F\beta_a(x;p) = -\bigl(\tfrac{d}{d\xi^+}\bigr)^2 [\beta_{[a,b]} - \beta_a] + O(\beta_{[a,b]} - \beta_a)$$

combined with $\beta_{[a,b]} - \beta_a = \tilde v_{0;\,b} + \varepsilon v_{1;\,b}$, for which we have (3.8), (3.9) and (2.2).

For $x \in \Omega_{[(a+b)/2,b]}$, estimate (3.11) is obtained similarly, but using a version of (3.10), in which β_a is replaced by $\beta_{[a,b]} - [\tilde v_{0;\,a} + \varepsilon v_{1;\,a}]$, so λ_a and $v_{0;\,a}(\xi^+, l)$ are replaced by $f_{uu}\bigl(x, z + \hat\vartheta v_{0;\,b}\bigr)$ and $v_{0;\,b}(\xi^-, l)$, respectively. □

Corollary 3.4. *Let $b-a \ge (4/\gamma)\varepsilon|\ln(Ch)|$. Then there exists positive C_0 and C_2 such that the functions $\beta_a(x;p)$ and $\beta_{[a,b]}(x;p)$ of (3.6), for all $0<|p|\le p_0$, satisfy*

$$\begin{aligned}(\operatorname{sgn} p)\, F\beta_a(x;p) &\ge C_0|p|\,\gamma^2 - C_2(\varepsilon^2+p^2),\\ (\operatorname{sgn} p)\, F\beta_{[a,b]}(x;p) &\ge C_0|p|\,\gamma^2 - C_2(\varepsilon^2+p^2+h^2),\end{aligned}$$

Proof. Recall (1.5a) and that $v_{0;\,a}(\xi^+,l)\ge 0$ and $v_{0;\,b}(\xi^-,l)\ge 0$, by (3.7). Now invoke Lemma 3.3 choosing a positive C_0 that does not exceed $\min_{x\in\bar\Omega_a}|\lambda_a(x)|^{-1}$ and $\min_{x\in\bar\Omega_{[a,b]}}|\lambda_{[a,b]}(x)|^{-1}$ so that $1+C_0\lambda_a(x)\ge 0$ and $1+C_0\lambda_{[a,b]}(x)\ge 0$. Finally note that $e^{-\gamma(b-a)/(2\varepsilon)}\le Ch^2$. □

Lemma 3.5. *Let $0\le a<b\le C_1$ and $b-a\ge(4/\gamma)\varepsilon|\ln(Ch)|$. Let f satisfy assumption (1.5a), and the boundary data g_d, where $d=a,b$, of problems* (3.1) *and* (3.2) *satisfy (1.5b). Then there is a sufficiently small positive constant $\tilde C_0$ such that if $\varepsilon\le\tilde C_0$, then problem* (3.1) *has a solution u_a, and if $\varepsilon+h\le\tilde C_0$, then problem* (3.2) *has a solution $u_{[a,b]}$, such that*

$$\big|(u_a-u_{\mathrm{as}\,;\,a})(x)\big| \le C\varepsilon^2 \quad \textit{for } x\in\bar\Omega_a, \tag{3.12a}$$

$$\big|(u_{[a,b]}-u_{\mathrm{as}\,;\,[a,b]})(x)\big| \le C(\varepsilon^2+h^2) \quad \textit{for } x\in\bar\Omega_{[a,b]}, \tag{3.12b}$$

where $u_{\mathrm{as}\,;\,a}$ and $u_{\mathrm{as}\,;\,[a,b]}$ are defined in (3.3). *Furthermore,*

$$|(u_{[a,b]}-u_a)(x)| \le C(\varepsilon^2+h^2) \quad \textit{for } x\in\bar\Omega_{[a,(a+b)/2]}, \tag{3.12c}$$

$$|(u_{[a,b]}-z)(x)| \le C(\varepsilon^2+h^2) \quad \textit{for } x\in\partial\Omega_{(a+b)/2}, \tag{3.12d}$$

$$|(u_a-z)(x)| \le C(\varepsilon^2+h^2) \quad \textit{for } x\in\bar\Omega_{(a+b)/2}. \tag{3.12e}$$

Proof. Existence of u_a (for $a=0$) and relation (3.12a) are established in [5, 17].

For existence of $u_{[a,b]}$, set $\bar p := \frac{2C_2}{C_0\gamma^2}(\varepsilon^2+h^2)$ so that $\frac12 C_0\bar p\,\gamma^2 \le C_2(\varepsilon^2+h^2)$. Then the choice $\tilde C_0^2 := \frac{C_0\gamma^2}{2C_2}\min\{p_0,\frac{C_0\gamma^2}{2C_2}\}$ provides $\bar p\le p_0$ and $\frac12 C_0\bar p\gamma^2\ge C_2\bar p^2$. So applying Corollary 3.4 yields $F\beta_{[a,b]}(x;-\bar p)\le 0\le F\beta_{[a,b]}(x;\bar p)$. Furthermore, since (3.7) implies that $\beta_{[a,b]}(x;p)$ is increasing in p, while $\beta_{[a,b]}(x;0)=u_{\mathrm{as}\,;\,[a,b]}(x)$, we get

$$\beta_{[a,b]}(x;-\bar p)\le u_{\mathrm{as}\,;\,[a,b]}(x)\le\beta_{[a,b]}(x;\bar p). \tag{3.13}$$

Thus $\beta_{[a,b]}(x;-\bar p)$ and $\beta_{[a,b]}(x;\bar p)$ are ordered sub- and super-solutions for problem (3.2). Therefore this problem has a solution $u_{[a,b]}$ such that $\beta_{[a,b]}(x;-\bar p)\le u_{[a,b]}(x)\le\beta_{[a,b]}(x;\bar p)$ and hence for this solution we obtain the desired bound (3.12b) from

$$|u_{[a,b]}(x)-u_{\mathrm{as}\,;\,[a,b]}(x)|\le\beta_{[a,b]}(x;\bar p)-\beta_{[a,b]}(x;-\bar p)\le C\bar p. \tag{3.14}$$

The final estimate here follows from $\beta_{[a,b]}(x;\bar p)-\beta_{[a,b]}(x;-\bar p) = 2\bar p\,(\frac{\partial}{\partial p}\tilde v_{0;\,a}+\frac{\partial}{\partial p}\tilde v_{0;\,b}+C_0)\big|_{p=\theta}$, where we used (3.6b) and (3.8). Thus we have (3.12b).

In view of (3.12a), (3.12b), it suffices to prove versions of (3.12c)–(3.12e), in which $u_{[a,b]}$ and u_a are replaced by $u_{\mathrm{as}\,;\,[a,b]}$ and $u_{\mathrm{as}\,;\,a}$, respectively. They follow as (3.3) yields $u_{\mathrm{as}\,;\,[a,b]}-u_{\mathrm{as}\,;\,a}=I_b$ and $u_{\mathrm{as}\,;\,[a,b]}-z=I_a+I_b$, while $u_{\mathrm{as}\,;\,a}-z=I_a\omega$, where $I_a:=\big[v_{0;\,a}(\xi_a^+,l)+\varepsilon v_{1;\,a}(\xi_a^+,l)\big]$ and $I_b:=\big[v_{0;\,b}(\xi_b^-,l)+\varepsilon v_{1;\,b}(\xi_b^-,l)\big]$ so $|I_b|\le Ch^2$ for $x\in\bar\Omega_{[a,(a+b)/2]}$ and $|I_a|+|I_b|\le Ch^2$ for $x\in\partial\Omega_{(a+b)/2}$, while $|I_a|\le Ch^2$ for $x\in\bar\Omega_{(a+b)/2}$. Here the bounds for $I_{a,b}$ are obtained by combining (3.8), (3.9) with $\xi_b^-=\frac{b-r}{\varepsilon}$, $\xi_a^+=\frac{r-a}{\varepsilon}$ and (2.2). □

3.2. **Error in the continuous Schwarz method.** We are now prepared to bound the error in the first iteration $u^{[1]}$ of the continuous Schwarz method (1.2), (1.3).

Theorem 3.6. *Let* $(4/\gamma)\varepsilon\ln(Ch) \le 2\sigma \le C_1$ *and* $\varepsilon + h \le \tilde{C}_0$, *where* $\tilde{C}_0$ *is from Lemma 3.5. Let the boundary data* g_0 *and* $g_{2\sigma} = g_{2\sigma}^{[1]}$ *of problems* (1.1) *and* (1.2a) *satisfy (1.5b) with* $d = 0,\ 2\sigma$. *Then there exist a solution* u *of problem (1.1) and a first-iteration approximation* $u^{[1]}$ *defined by* (1.2), (1.3) *such that*

$$|(u - u^{[1]})(x)| \le C(\varepsilon^2 + h^2) \qquad \textit{for } x \in \bar{\Omega}.$$

Proof. Applying Lemma 3.5, with $a := 0$ and $b := 2\sigma$, to problems (1.1) and (1.2a) immediately yields existence of their solutions $u_a = u$ and $u_{[a,b]} = u_{[0,2\sigma]}$. Furthermore, as $u^{[1]} = u_{[0,2\sigma]}$ in $\bar{\Omega}_{[0,\sigma]}$, estimate (3.12c) implies $u^{[1]} - u = O(\varepsilon^2 + h^2)$ in $\bar{\Omega}_{[0,\sigma]}$. Note also that (3.12e) yields $u - z = O(\varepsilon^2 + h^2)$ for $x \in \bar{\Omega}_\sigma$, while $u^{[1]} = u_\sigma$ in this subdomain. So, to complete the proof, it remains to show that there exists a solution u_σ of problem (1.2b) such that

$$u_\sigma - z = O(\varepsilon^2 + h^2) \qquad \text{for } x \in \bar{\Omega}_\sigma. \tag{3.15}$$

As the boundary condition in (1.2b) is $g_\sigma = u_{[0,2\sigma]}$ on $\partial\Omega_\sigma$, then (3.12d) yields $g_\sigma - z = O(\varepsilon^2 + h^2)$ on $\partial\Omega_\sigma$. So one can easily check that the boundary condition of problem (1.2b) satisfies assumption (1.5b) with $d = \sigma$. Now, Lemma 3.5, applied to problem (1.2b) as a particular case of (3.1) with $a = \sigma$, implies existence of a solution u_σ such that $u_\sigma - u_{\mathrm{as}\,;\,\sigma} = O(\varepsilon^2 + h^2)$. Furthermore, using (3.8) to estimate the boundary-layer components of $u_{\mathrm{as}\,;\,\sigma}$, we observe that they do not exceed $C_\delta \max_{\partial\Omega_\sigma} |g_\sigma - z| = O(\varepsilon^2 + h^2)$. This yields $u_{\mathrm{as}\,;\,\sigma} = z + O(\varepsilon^2 + h^2)$ and hence (3.15). □

3.3. ***Z*-fields.** We shall invoke the theory of Z-fields in our analysis of discretizations (2.5) in $\Omega_{[0,2\sigma]}$ and (2.6) in Ω_σ.

Definition. An operator $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a Z-field if for all $i \ne j$ the mapping $x_j \mapsto \big(\mathcal{F}(x_1, x_2, \ldots, x_n)\big)_i$ is a monotonically decreasing function from $\mathbb{R}$ to $\mathbb{R}$ when $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n$ are fixed.

Lemma 3.7. [14] *Let* $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^n$ *be continuous and a* Z*-field. Let* $r \in \mathbb{R}^n$ *be given. Assume that there exist* $\alpha, \beta \in \mathbb{R}^n$ *such that* $\alpha \le \beta$ *and* $\mathcal{F}\alpha \le 0 \le \mathcal{F}\beta$. *(The inequalities are understood to hold true component-wise.) Then the equation* $\mathcal{F}y = 0$ *has a solution* $y \in \mathbb{R}^n$ *with* $\alpha \le y \le \beta$.

Proof. The proof can be found in Lorenz [14], and also in [11]. Alternatively, the desired result can be obtained by imitating the proof of [18, Theorem 3.1] (it is crucial in this argument that the discrete operator $\mathcal{F} + CI$ satisfies a discrete maximum principle, where I is the identity operator and C is an arbitrarily large but fixed positive constant). □

The functions α and β of Lemma 3.7 are called ordered *sub-* and *super-solutions* of the discrete problem $\mathcal{F}y = 0$.

Remark 3.8. The discrete operators $F^h_{[0,2\sigma]}$ of (2.5) and F^h_σ of (2.6) are Z-fields [8].

3.4. **Error in the discrete Schwarz method for $\varepsilon \le Ch$.** Throughout this subsection for any fixed positive constant C, we take

$$\varepsilon \le Ch. \tag{3.16}$$

This is not a practical restriction. Furthermore, in §4, we shall consider the case of $\varepsilon \ge Ch$.

Theorem 3.9. *Let* (3.16) *be satisfied, and the mesh* $\{r_i\}_{i=0}^{2N}$ *be one of the meshes in §2.2(a),(b). Let the boundary data* g_0 *and* $g_{2\sigma} = g_{2\sigma}^{[1]}$ *of problems* (1.1) *and* (1.2a) *satisfy (1.5b) with* $d = 0,\ 2\sigma$*. Then there exist a solution* u *of problem (1.1) and a first-iteration computed solution* $U^{[1]}$ *defined by* (2.5), (2.6), (2.7) *such that*

$$|(u - U^{[1]})(x)| \le Ch^2|\ln h|^m \qquad \text{for } x \in \bar\Omega,$$

where $m = 2$ *for the Shishkin mesh of §2.2(a) and* $m = 0$ *for the Bakhvalov mesh of §2.2(b).*

Proof. In view of Theorem 3.6 and definitions (1.3b) and (2.7b) of $u^{[1]}$ and $U^{[1]}$, it suffices to show that

$$\big|(U_{[0,2\sigma]} - u_{[0,2\sigma]})(x)\big| \ \le\ Ch^2|\ln h|^m \qquad \text{for } x \in \bar\Omega_{[0,2\sigma]}, \tag{3.17a}$$

$$\big|(U_\sigma - u_\sigma)(x)\big| \ \le\ Ch^2|\ln h|^m \qquad \text{for } x \in \bar\Omega_\sigma \tag{3.17b}$$

To prove (3.17a), note that problem (1.2a) is a particular case of (3.2), so we shall use some results of §3.1 setting $a := 0$ and $b := 2\sigma$. The corresponding function $\beta_{[0,2\sigma]}(x;p)$ is defined by (3.6b). We claim that for all $|p| \le p_0$ at all interior mesh nodes $x_{ij}, i = 1,\dots,N-1, j = 1,\dots,N_l$, we have

$$\big|F^h_{[0,2\sigma]}\beta_{[0,2\sigma]}(x_{ij};p) - F\beta_{[0,2\sigma]}(x_{ij};p)\big| \le Ch^2|\ln h|^m. \tag{3.18}$$

Note that the term $\varepsilon^2\zeta\frac{\partial}{\partial l}(\zeta\frac{\partial}{\partial l}\beta_{[0,2\sigma]})$ in $F\beta_{[0,2\sigma]}$ and its discretization in $F^h_{[0,2\sigma]}\beta_{[0,2\sigma]}$ are both $O(\varepsilon^2)$, so do not exceed Ch^2 by (3.16). The truncation error for the remaining term $\varepsilon^2\eta^{-1}\frac{\partial}{\partial r}(\eta\frac{\partial}{\partial r}\beta_{[0,2\sigma]})$ is bounded by $CN^{-2}\ln^m N$, as can be shown by imitating the argument of [11, Lemma 3.3 and §3.4.2]. Combining this with (2.4), we get (3.18).

Next, set $\bar p := \bar Ch^2|\ln h|^m$ and, using (3.18), choose $\bar C$ sufficiently large so that $|F^h_{[0,2\sigma]}\beta_{[0,2\sigma]} - F\beta_{[0,2\sigma]}| \le \frac12 C_0\bar p\gamma^2$ for all $|p| \le p_0$ including $p = \pm\bar p$. Now, by Corollary 3.4 and (3.16), for sufficiently small h, we have $\pm F\beta_{[0,2\sigma]}(x;\pm\bar p) \ge \frac12 C_0\bar p\gamma^2$. Consequently $\pm F^h_{[0,2\sigma]}\beta_{[0,2\sigma]}(x_{ij}\,;\pm\bar p) \ge 0$. Combining this with (3.13), we conclude that $\beta_{[0,2\sigma]}(x_{ij};\pm\bar p)$ are ordered discrete sub- and super-solutions. As discretization (2.5) is a Z-field (see Remark 3.8), an application of Lemma 3.7 yields existence of $U_{[0,2\sigma]}(x_{ij})$ between these sub- and super-solutions. Furthermore, a version of (3.14) for $U_{[0,2\sigma]}(x_{ij})$ implies $|(U_{[0,2\sigma]} - u_{\text{as}\,;[0,2\sigma]})(x_{ij})| \le C\bar p \le Ch^2|\ln h|^m$. Noting that, by (3.3b), (3.8), we also have $|(u^I_{\text{as}\,;[0,2\sigma]} - u_{\text{as}\,;[0,2\sigma]})(x_{ij})| \le Ch^2|\ln h|^m$, where $u^I_{\text{as}\,;[0,2\sigma]}$ is the bilinear interpolant of the computed solution on the mesh $\{(r_i,l_j)\}$, we get $|(U_{[0,2\sigma]} - u_{\text{as}\,;[0,2\sigma]})(x)| \le Ch^2|\ln h|^m$. Combining this with (3.12b) yields the desired estimate (3.17a).

For (3.17b), in view of (3.15), it suffices to show that $|U_\sigma - z| \le Ch^2|\ln h|^m$. Note that, by (3.17a), at each mesh node $q_j \in \partial\Omega_\sigma$ we have $U_\sigma = u_{[0,2\sigma]} + O(h^2|\ln h|^m)$, while $u_{[0,2\sigma]} = u_\sigma = z + O(h^2|\ln h|^m)$ due to (1.2b) and (3.15). Consequently $|(U_\sigma - z)(q_j)| \le \bar Ch^2|\ln h|^m$ at all $q_j \in \partial\Omega_\sigma$ for some sufficiently large $\bar C$.

Let z^I be the piecewise linear interpolant of z on the triangulation in $\bar{\Omega}^h_\sigma$. At any interior mesh node $q_i \in \Omega^h_\sigma$ one has

$$\langle \nabla z^I, \nabla \chi_i \rangle = \langle \nabla [z^I - z], \nabla \chi_i \rangle - \langle \triangle z, \chi_i \rangle.$$

Combining this with the interpolation error estimate $|\nabla(z^I - z)| \le Ch$ and the standard quasiuniform-mesh properties $\langle 1, |\nabla \chi_i| \rangle \le Ch$ and $\langle 1, \chi_i \rangle \ge Ch^2$, we conclude that $|\langle \nabla z^I, \nabla \chi_i \rangle| \le C' \langle 1, \chi_i \rangle$. Now set $\bar{p} := \bar{C} h^2 |\ln h|^p$ with $\bar{C} \ge 2\gamma^{-2} C' C^2$, where C is from (3.16), and let h be sufficiently small so that, by (1.5a), we have $f_u(x, z \pm \bar{p}) \ge \frac{1}{2}\gamma^2$. A calculation shows that $\pm F^h_\sigma [z^I \pm p](q_i) \ge \frac{1}{2}\gamma^2 \bar{p} - C' \varepsilon^2 \ge 0$. Consequently $z^I \pm \bar{p}$ are sub- and super-solutions for the discrete problem (2.6). So, by Remark 3.8, an application of Lemma 3.7 yields existence of a solution $z^I - \bar{p} \le U_\sigma \le z^I + \bar{p}$ Hence $|U_\sigma - z^I| \le 2\bar{p}$. Combining this with $|z^I - z| \le Ch^2$ implies (3.17b). □

The above Theorem 3.9 implies that if $\varepsilon \le Ch$, one iteration of the discrete Schwarz method is sufficient to attain second-order convergence (with, in the case of the Shishkin mesh, a logarithmic factor) in the maximum norm uniformly in ε. In the next section we shall investigate the errors for $\varepsilon \ge Ch$.

4. Maximum norm error analysis for $\varepsilon \ge Ch$

4.1. **Preliminaries. Error in the continuous Schwarz method.** Throughout this section, we make a simplifying assumption that

$$f_u(x, u) > \gamma^2 > 0 \quad \text{for all } (x, u) \in \bar{\Omega} \times \mathbb{R}. \tag{4.1}$$

Under this assumption, problem (1.1) has a unique solution, and furthermore, applying the standard **linearization**, for any two functions v and w one gets

$$Fv - Fw = \mathcal{L}[v - w], \quad \mathcal{L} := -\varepsilon^2 \triangle + p(x), \quad p(x) > \gamma^2 > 0 \text{ for } x \in \bar{\Omega}. \tag{4.2}$$

To be more precise, here the coefficient $p(x) = \int_0^1 f_u(x, w + s[v - w])\, ds$, i.e. it involves the functions v and w. Bearing this in mind, throughout this section, we let $p(x)$ denote a generic coefficient in $\mathcal{L}$, which in different places will involve different v and w. Similarly, for the discrete operators $F^h_{[0,2\sigma]}$ of (2.5) and F^h_σ of (2.6), we shall employ their linearized versions $\mathcal{L}^h_{[0,2\sigma]}$ and $\mathcal{L}^h_\sigma$ obtained using $f(x_{ij}, V_{ij}) - f(x_{ij}, W_{ij}) = p(x_{ij})[V_{ij} - W_{ij}]$ and $f(q_i, V(q_i)) - f(q_i, W(q_i)) = p(q_i)[V(q_i) - W(q_i)]$, respectively. In view of (4.1), the discrete operators $\mathcal{L}^h_{[0,2\sigma]}$ and $\mathcal{L}^h_\sigma$ satisfy the **discrete maximum principle**.

Under condition (4.1), it is not difficult to estimate the error in the continuous Schwarz method.

Theorem 4.1. *Let u be a solution of problem (1.1) under condition (4.1), and $u^{[k]}$ be the kth iteration approximation (1.2), (1.3) obtained using some $|g^{[1]}_{2\sigma}| \le C$. There are positive constants c_0 and $\theta' \in (0, 1)$, independent of ε and k, such that $|u^{[k]} - u| \le C\theta^k$ in $\bar{\Omega}$, where $\theta \le \min\{e^{-\sigma\gamma/\varepsilon}, \theta'\}$ for $\varepsilon \in (0, c_0]$, and $\theta \le \theta'$ for $\varepsilon \in [c_0, 1]$. If Ω is convex, then $\theta' \le \frac{1}{2}$.*

Proof. Set $\theta = \theta_\varepsilon := \max_{x \in \bar{\Omega}_{[0,\sigma]}} |\phi_\varepsilon(x)|$, where for each $\varepsilon \in (0, 1]$, the auxiliary function ϕ_ε solves the problem

$$[-\varepsilon^2 \triangle + \gamma_*^2]\, \phi_\varepsilon = 0 \text{ in } \Omega_{[0,2\sigma]}, \quad \phi_\varepsilon = 0 \text{ on } \partial\Omega, \quad \phi_\varepsilon = 1 \text{ on } \partial\Omega_{2\sigma}, \tag{4.3}$$

with some $\gamma_* > \gamma$. By the maximum principle, we have $0 \le \phi_\varepsilon < 1$ in $\bar\Omega_{[0,\sigma]}$, and also $\phi_\varepsilon \le \phi_1$ in $\bar\Omega_{[0,2\sigma]}$. Then $\theta \le \theta' := \theta_1 \in (0,1)$, where θ' is independent of ε. In view of (2.3), a calculation shows that the barrier functions $B_1(r) := e^{-\gamma(2\sigma-r)/\varepsilon}$ and $B_2(r) := r/(2\sigma)$ satisfy $[-\varepsilon^2\triangle + \gamma_*^2]B_1 \ge (-\gamma^2 - \varepsilon\gamma C' + \gamma_*^2)B_1$ and $[-\triangle + \gamma_*^2]B_2 \ge \frac{\kappa/(2\sigma)}{1-\kappa r} + \gamma_*^2 B_2$. So, by the maximum principle, $\phi_\varepsilon \le B_1$ if $\varepsilon \le c_0 := (\gamma_*^2 - \gamma^2)/(\gamma C')$, while $\phi_1 \le B_2$ if $\kappa \ge 0$, i.e. if Ω is convex. These two observations imply that $\theta \le e^{-\gamma\sigma/\varepsilon}$ for $\varepsilon \le c_0$, and $\theta' \le \frac{1}{2}$ if Ω is convex.

Let $t^{[k]} := \max_{\partial\Omega_{2\sigma}} |g_{2\sigma}^{[k]} - u|$, where we clearly have $t^{[1]} \le C^*$. Now let $\gamma_* := \min_{\bar\Omega\times[-C',C']} f_u > \gamma^2$, where $C' := \gamma^{-2}\max_{\bar\Omega}|f(x,0)| + \max_{\partial\Omega}|g_0| + C^*$ is independent of ε and k. Consider the first iteration. In view of (1.1) and (1.2a), a linearization of type (4.2) yields $\mathcal{L}(u_{[0,2\sigma]} - u) = 0$ in $\Omega_{[0,2\sigma]}$, where $p(x) \ge \gamma_*^2$, subject to $u_{[0,2\sigma]} - u = 0$ on $\partial\Omega$ and $|u_{[0,2\sigma]} - u| \le t^{[1]}$ on $\partial\Omega_{2\sigma}$. So, using the maximum principle, we conclude that $|u_{[0,2\sigma]} - u| \le t^{[1]}\phi_\varepsilon$ in $\bar\Omega_{[0,2\sigma]}$. Therefore $|u^{[1]} - u| \le \theta t^{[1]}$ in $\bar\Omega_{[0,\sigma]}$ and consequently $|u_\sigma - u| \le \theta t^{[1]}$ on $\partial\Omega_\sigma$. Also, in view of (1.1) and (1.2b), a linearization of type (4.2) yields $\mathcal{L}(u_\sigma - u) = 0$ in Ω_σ. So, by the maximum principle, we get $|u^{[1]} - u| = |u_\sigma - u| \le \theta t^{[1]}$ in $\bar\Omega_\sigma$ as well. Thus we have shown that $|u^{[1]} - u| \le \theta t^{[1]}$ in $\bar\Omega$, which, by (1.3a), implies that $t^{[2]} \le \theta t^{[1]}$. Repeating this argument for further iterations and then noting that $|t^{[1]}| \le C$, we get the desired result. □

4.2. **Auxiliary computed solutions in $\mathbf{\Omega_{[0,2\sigma]}}$ and $\mathbf{\Omega_\sigma}$.** In this subsection we investigate auxiliary computed solutions $\tilde U_{[0,2\sigma]}$ and $\tilde U_\sigma$ defined by

$$\text{(4.4a)}\qquad F^h_{[0,2\sigma]}\tilde U_{[0,2\sigma]}(x_{ij}) = 0, \quad (\tilde U_{[0,2\sigma]} - g_0)(x_{0,j}), \quad (\tilde U_{[0,2\sigma]} - u)(x_{2N,j}) = 0,$$

with $\tilde U_{[0,2\sigma]}(x_{i,0}) = \tilde U_{[0,2\sigma]}(x_{i,N_l})$, $\tilde U_{[0,2\sigma]}(x_{i,1}) = U_{[0,2\sigma]}(x_{i,N_l+1})$, and $\tilde U_{[0,2\sigma]}$ in $\bar\Omega_{[0,2\sigma]}$ obtained by the bilinear interpolation of $\tilde U_{[0,2\sigma]}(x_{ij})$,

$$\text{(4.4b)}\qquad F^h_\sigma \tilde U_\sigma(q_i) = 0 \;\; \forall q_i \in \Omega^h_\sigma, \qquad \tilde U_\sigma(q_j) = u(q_j) \;\; \forall q_i \in \partial\Omega^h_\sigma.$$

Here $F^h_{[0,2\sigma]}$ and F^h_σ are the discrete operators that were used in problems (2.5), (2.6) for $U_{[0,2\sigma]}$ and U_σ. The only difference between these pairs of problems is in that we use the exact solution u of (1.1) in the boundary conditions for $\tilde U_{[0,2\sigma]}$ and $\tilde U_\sigma$. To estimate the errors of these auxiliary computed solutions, we need pointwise derivative estimates for the exact solution u.

Lemma 4.2. *Under condition (4.1), problem (1.1) has a unique solution u, and*

$$\text{(4.5a)}\qquad \Big|\frac{\partial^{j+m}}{\partial x_1^j \partial x_2^m}(u - z)\Big| \le C\varepsilon^{-(j+m)}\big[\varepsilon^2 + e^{-\gamma a/\varepsilon}\big] \quad \textit{for } x \in \bar\Omega_a,\; j+m = 0,1,2,$$

where $0 \le a \le C_1$ and z is a solution of (1.4) with $|\frac{\partial^{j+m}}{\partial x_1^j \partial x_2^m} z| \le C$, while

$$\text{(4.5b)}\qquad \Big|\frac{\partial^{j+m}}{\partial r^j \partial l^m} u\Big| \le C\big[1 + \varepsilon^{3-(j+m)} + \varepsilon^{-j}\, e^{-\gamma r/\varepsilon}\big] \quad \textit{for } x \in \bar\Omega_{[0,C_1]},\; j,m = 0,\dots,4.$$

Proof. We defer the proof of this lemma to Appendix A. □

We combine technical error estimates for $\tilde U_{[0,2\sigma]}$ and $\tilde U_\sigma$ in the following lemma.

Lemma 4.3. *Let u be a solution of (1.1) under condition (4.1), $\varepsilon \ge Ch$, and the mesh $\{r_i\}_{i=0}^{2N}$ be one of the meshes in §2.2(a),(b). Then for the solutions $\tilde U_{[0,2\sigma]}$*

and $\tilde U_\sigma$ *of problems in (4.4) we have*

$$|(\tilde U_{[0,2\sigma]} - u)(x)| \le Ch^2|\ln h|^m \quad \text{for } x\in\bar\Omega_{[0,2\sigma]}, \tag{4.6a}$$

$$|(\tilde U_\sigma - u)(x)| \le Ch^2\ln(C+\varepsilon/h) \quad \text{for } x\in\bar\Omega^h_\sigma, \tag{4.6b}$$

where $m=2$ *for the Shishkin mesh of §2.2(a) and* $m=0$ *for the Bakhvalov mesh of §2.2(b).*

In the proof of this lemma, for the finite element solution $\tilde U_\sigma$ we essentially use a maximum norm error estimate by Schatz and Wahlbin [21], which we generalize for the case of lumped-mass finite elements. For the finite difference solution $\tilde U_{[0,2\sigma]}$, a certain technical difficulty is due to the mesh $\{l_j\}$ being quasi-uniform, so the truncation error in the l direction is $O(h)$. We deal with this extending the one-dimensional supraconvergence analysis of [20].

Proof of (4.6a) (Supraconvegence of the Finite Difference Discretization). Let $u_{ij} := u(x_{ij})$ and $\tilde U_{ij} = \tilde U_{[0,2\sigma]}(x_{ij})$. Using (1.1a), (4.4a) and then the definition (2.5) of $F^h_{[0,2\sigma]}$, we get

$$F^h_{[0,2\sigma]}\tilde U_{ij} - F^h_{[0,2\sigma]}u_{ij} = Fu(x_{ij}) - F^h_{[0,2\sigma]}u_{ij} = \varepsilon^2(\eta^{-1}_{ij}R_1 + \zeta_{ij}R_2), \tag{4.7}$$

where

$$R_1 = D_r[\tilde\eta_{ij}D_r^- u_{ij}] - \tfrac{\partial}{\partial r}(\eta\tfrac{\partial}{\partial r}u)\big|_{(x_{ij})},\quad R_2 = D_l[\tilde\zeta_{ij}D_l^- u_{ij}] - \tfrac{\partial}{\partial l}(\zeta\tfrac{\partial}{\partial l}u)\big|_{(x_{ij})}.$$

For R_1, employing Taylor series expansions, one can show that

$$|R_1| \le C\big[(r_i-r_{i-1})^2M_i^{(4)} + |r_{i+1}-2r_i+r_{i-1}|M_i^{(3)}\big],\ M_i^{(s)} := \sum_{n=0}^{s}\max_{\substack{r\in[r_{i-1},r_{i+1}]\\ l\in[0,L]}}|\tfrac{\partial^n}{\partial r^n}u|.$$

For the Shishkin mesh of §2.2(a), combining $r_i - r_{i-1} \le 2\gamma^{-1}\varepsilon N^{-1}\ln N$ with (4.5b) yields $|\varepsilon^2\eta^{-1}R_1| \le CN^{-2}\ln^2 N$. For the Bakhvalov mesh of §2.2(b), we only consider $i\le N/2$ (as the other case is similar). A calculation shows that

$$S_i := \varepsilon^{-2}(r_i-r_{i-1})^2 + \varepsilon^{-1}|r_{i+1}-2r_i+r_{i-1}| \le \frac{CN^{-2}}{\max\{(1-\frac{i+1}{N}),\rho\}^2} \le CN^{-2}\varepsilon^{-2}.$$

Consequently

$$|\varepsilon^2\eta^{-1}R_1| \le CS_i\big[\varepsilon^4M_i^{(4)} + \varepsilon^3M_i^{(3)}\big] \le C\big[N^{-2} + S_ie^{-\gamma r_{i-1}/\varepsilon}\big] \le CN^{-2},$$

where we used $e^{-\gamma r_{i-1}/\varepsilon} \le \max\{(1-\frac{i-1}{N}),\rho\}^2$ and $N^{-1}\le Ch\le C\varepsilon = C\rho$.

Note that R_2 is only $O(h)$ as $\{l_j\}$ is a general non-uniform mesh. To establish supraconvergence of our discretization, we imitate the truncation error analysis of [9, Lemma 3.1]; see also [20, Chap. III, §4]. Noting that

$$R_2 = D_l[\tilde\zeta_{ij}\hat\mu_{ij}] + D_lw_{i,j-1/2} - \tfrac{\partial}{\partial l}w\big|_{(x_{ij})},\ \hat\mu_{ij} := D_l^-u_{ij} - \tfrac{\partial}{\partial l}u\big|_{(x_{i,j-1/2})},\ w := \zeta\tfrac{\partial}{\partial l}u,$$

and then employing Taylor series expansions, one gets $R_2 = D_l[\mu(r_i,l_j)] + \hat\nu_{ij}$. Here

$$\mu(r,l_j) := (l_j-l_{j-1})^2(\tfrac{1}{24}\zeta\tfrac{\partial^3}{\partial l^3}u + \tfrac18\tfrac{\partial^2}{\partial l^2}w)\big|_{(r,l_{j-1/2})},\quad |\hat\nu_{ij}| \le Ch^2\max_{\bar\Omega_{[0,2\sigma]}}(|\tfrac{\partial^3}{\partial l^3}w| + |\tfrac{\partial^4}{\partial l^4}u|),$$

so $|\hat\nu| \le Ch^2$ and $|\frac{\partial^n}{\partial r^n}\mu| \le C\varepsilon^{-n}h^2$ for $n=0,1,2$.

Linearizing (4.7) and combining our findings for R_1 and R_2, we conclude that

$$\mathcal{L}^h_{[0,2\sigma]}[\tilde U_{ij} - u_{ij}] = \varepsilon^2\zeta_{ij}D_l[\mu(r_i,l_j)] + \nu_{ij},\quad \varepsilon^n|\tfrac{\partial^n}{\partial r^n}\mu| \le Ch^2,\ \ |\nu| \le Ch^2|\ln h|^m.$$

Consequently, $|\tilde{U}(x_{ij}) - u(x_{ij})| \le Ch^2|\ln h|^m$ (this immediately follows from the stability result for the operator $\mathcal{L}^h_{[0,2\sigma]}$ given by Corollary 4.7 which we defer to §4.4). Combining this with the interpolation error bound $|u^I - u| \le Ch^2|\ln h|^m$ for the bilinear interpolant u^I of the exact solution u on the mesh $\{(r_i, l_j)\}$ (which is obtained again using (4.5b)), we get the desired estimate (4.6a). □

Proof of (4.6b) (Lumped-Mass Finite Element Error). We claim that

$$|(\tilde{U}_\sigma - u)(x)| \le Ch^2 \ln(C + \varepsilon/h)\,\big[\|u\|_{C^2(\bar{\Omega}^h_\sigma)} + \mathcal{E}_{\text{l.m.}}\big] \quad \text{for } x \in \bar{\Omega}^h_\sigma, \tag{4.8a}$$

where the error due to the lumped-mass discretization of $f(x, u)$ is described using

$$\mathcal{E}_{\text{l.m.}} := \|\Psi\|_{C^2(\bar{\Omega}^h_\sigma)} + \varepsilon^{-1}\|\Psi\|_{C^1(\bar{\Omega}^h_\sigma)}, \qquad \Psi(x) := f(x, u(x)). \tag{4.8b}$$

Estimate (4.8) is a generalization of a maximum norm error estimate [21] for the standard finite element method (for which estimate (4.8a) with $\mathcal{E}_{\text{l.m.}} = 0$ immediately follows from [21, Theorems 6.1 and 12.1]). We defer the proof of (4.8) to §4.5.

As the domain Ω is smooth, we note that $\Omega^h_\sigma \subset \Omega_{\tilde{\sigma}}$, for some $\tilde{\sigma} \le \sigma - Ch^2 \le \sigma - C\varepsilon$. Note also that, by (1.4), we have $\Psi(x) = [u - z]\int_0^1 f_u(x, z + s[u - z])\,ds$, so $\|\Psi\|_{C^n(\bar{\Omega}^h_\sigma)} \le C\|u - z\|_{C^n(\bar{\Omega}^h_\sigma)}$. Consequently, by (4.5a), a calculation shoes that $\|u\|_{C^2(\bar{\Omega}^h_\sigma)} \le C[1 + \varepsilon^{-2}e^{-\gamma\tilde{\sigma}/\varepsilon}]$ and $\|\Psi\|_{C^n(\bar{\Omega}^h_\sigma)} \le C[\varepsilon^{2-n} + \varepsilon^{-n}e^{-\gamma\tilde{\sigma}/\varepsilon}]$, so it remains to prove that $I := \varepsilon^{-2}e^{-\gamma\sigma/\varepsilon} \le C$. On both the Shishkin mesh and the Bakhvalov mesh of §2.2(a),(b), we have $\sigma \ge \sigma_S$, so a calculation yields $I \le \varepsilon^{-2}\max\{N^{-2}, e^{-\gamma C_1/(2\varepsilon)}\} \le C$, where we used (2.4) and $\varepsilon \ge Ch$. □

4.3. **Error in the discrete Schwarz method for $\varepsilon \ge Ch$.**

Theorem 4.4. *Let u be a solution of problem (1.1) under condition (4.1), $\varepsilon \ge Ch$, and $U^{[k]}$ be the discrete kth iteration approximation (2.5), (2.6), (2.7) obtained using some $|g^{[1]}_{2\sigma}| \le C$ and one of the meshes $\{r_i\}_{i=0}^{2N}$ in §2.2(a),(b). Set $m = 2$ for the Shishkin mesh of §2.2(a) and $m = 0$ for the Bakhvalov mesh of §2.2(b). There are constants c_0 and $\theta' \in (0, 1)$, independent of ε and k, such that*

$$|(U^{[k]} - u)(x)| \le C[\theta^k + h^2|\ln h|^m + h^2\ln(C + \varepsilon/h)] \quad \text{for } x \in \bar{\Omega}, \tag{4.9}$$

where $\theta \le \min\{e^{-\sigma\gamma/\varepsilon}, \theta'\}$ for $\varepsilon \in (0, c_0]$, and $\theta \le \theta'$ for $\varepsilon \in [c_0, 1]$. If Ω is convex, then $\theta' \le \frac{1}{2}$. If $\varepsilon \le \frac{1}{4}C_1\gamma(\ln N)^{-1}$, then $\theta \le Ch^2$.

Proof. We shall partly imitate the proof of Theorem 4.1. Applying the numerical method (2.5) to problem (4.3) for ϕ_ε, we get the computed solution Φ_ε, which satisfies a discrete equation of the type $[-\varepsilon^2\triangle^h_{[0,2\sigma]} + \gamma_*^2]\Phi_\varepsilon(x_{ij}) = 0$ in $\Omega_{[0,2\sigma]}$, subject to $\Phi_\varepsilon = 0$ on $\partial\Omega$ and $\Phi_\varepsilon = 1$ on $\partial\Omega_{2\sigma}$. Now set $\Theta := \max_{x_{ij}\in\bar{\Omega}\setminus\Omega^h_\sigma}|\Phi_\varepsilon|$. Note that a version of Lemma 4.2 can be obtained for the derivatives of of the exact solution ϕ_ε. Furthermore, imitating the proof of (4.6a), one can show that the error $|(\Phi_\varepsilon - \phi_\varepsilon)(x)| \le Ch^2|\ln h|^m$. Combining these two observations with $\text{dist}(\partial\Omega_\sigma, \partial\Omega^h_\sigma) \le Ch^2$, one concludes that $|\Theta - \theta| \le Ch^2|\ln h|^m$, where θ is from Theorem 4.1. Note also that $\varepsilon \le \frac{1}{4}C_1\gamma(\ln N)^{-1}$ implies $\sigma \ge \sigma_S = 2\gamma^{-1}\varepsilon\ln N$, so $\theta \le N^{-2} \le Ch^2$.

Next, introduce some notation using $g^{[k]}$ of (2.7a) and $\tilde{U}_{[0,2\sigma]}$ and $\tilde{U}_\sigma$ of (4.4):

$$T^{[k]} = \max_{\partial\Omega_{2\sigma}}|g^{[k]}_{2\sigma} - \tilde{U}_{[0,2\sigma]}|,\ \tilde{T}_\sigma = \max_{q_i\in\partial\Omega_\sigma}|(\tilde{U}_{[0,2\sigma]} - \tilde{U}_\sigma)(q_i)|,\ \tilde{T}_{2\sigma} = \max_{\partial\Omega_{2\sigma}}|\tilde{U}_{[0,2\sigma]} - \tilde{U}_\sigma|,$$

Note that $T^{[1]} \le t^{[1]} \le C^*$ (where $t^{[1]}$ is from the proof of Theorem 4.1). In view of (4.6), it suffices to estimate

$$E^{[k]} := \max_{x\in\bar\Omega\setminus\bar\Omega^h_\sigma} |(U^{[k]} - \tilde U_{[0,2\sigma]})(x)| + \max_{x\in\bar\Omega^h_\sigma} |(U^{[k]} - \tilde U_\sigma)(x)|.$$

Consider the first iteration. By (2.5) and (4.4a), a linearization of type (4.2) yields the discrete equation $\mathcal{L}^h_{[0,2\sigma]}(U_{[0,2\sigma]} - \tilde U_{[0,2\sigma]}) = 0$ in $\Omega_{[0,2\sigma]}$, where $p(x_{ij}) \ge \gamma_*^2$, subject to $U_{[0,2\sigma]} - \tilde U_{[0,2\sigma]} = 0$ on $\partial\Omega$ and $|U_{[0,2\sigma]} - \tilde U_{[0,2\sigma]}| \le T^{[1]}$ on $\partial\Omega_{2\sigma}$. So, using the discrete maximum principle, we conclude that $|U_{[0,2\sigma]} - \tilde U_{[0,2\sigma]}| \le T^{[1]}\Phi_\varepsilon$ in $\bar\Omega_{[0,2\sigma]}$. This immediately implies $|U^{[1]} - \tilde U_{[0,2\sigma]}| \le \Theta T^{[1]}$ in $\bar\Omega\setminus\bar\Omega^h_\sigma$. Furthermore $|(U_\sigma - \tilde U_\sigma)(q_j)| \le \Theta T^{[1]} + \tilde T_\sigma$ at any mesh node $q_j \in \partial\Omega_\sigma$. Combining this with $\mathcal{L}^h_\sigma(U_\sigma - \tilde U_\sigma)(q_i) = 0$ for all $q_i \in \Omega^h_\sigma$, which follows from (2.6) and (4.4b), and applying the discrete maximum principle, we get $|U^{[1]} - \tilde U_\sigma| = |U_\sigma - \tilde U_\sigma| \le \Theta T^{[1]} + \tilde T_\sigma$ in $\bar\Omega^h_\sigma$. Finally, by (2.7a), we have $T^{[2]} \le \Theta T^{[1]} + \tilde T_\sigma + \tilde T_{2\sigma}$. Noting that $\Theta T^{[1]} \le \theta T^{[1]} + |\Theta - \theta| C^*$, we summarize our findings for the first iteration as follows:

$$E^{[1]} + T^{[2]} \le \theta T^{[1]} + \lambda, \qquad \lambda := |\Theta - \theta| C^* + \tilde T_\sigma + \tilde T_{2\sigma}.$$

Next, by (4.6), we have

$$\lambda \le C[h^2 |\ln h|^m + h^2 \ln(C + \varepsilon/h)], \tag{4.10}$$

while $\theta \le \theta'$, with $\theta' \in (0,1)$ independent of ε and k. As $T^{[1]} \le C^*$, we also get $T^{[2]} \le C^*$ for sufficiently small h. Repeating the above argument for further iterations yields $E^{[k]} + T^{[k+1]} \le \theta T^{[k]} + \lambda$ and therefore $E^{[k]} \le \theta^k T^{[1]} + \lambda(1-\theta)^{-1}$. In view of (4.6) and (4.10), the desired estimate (4.9) follows. □

Corollary 4.5. *Under the conditions of Theorem 4.4, for $\varepsilon \le \frac14 C_1\gamma(\ln N)^{-1}$, we have*

$$|(U^{[1]} - u)(x)| \le C\,[h^2|\ln h|^m + h^2\ln(C + \varepsilon/h)] \quad \text{for } x \in \bar\Omega.$$

4.4. **Stability of the finite difference operator in the boundary-layer subdomain.** In this subsection we establish a stability result for the linearization $\mathcal{L}^h_{[0,2\sigma]}$ of the finite difference operator $F^h_{[0,2\sigma]}$ of (2.5). This result was used in the proof of (4.6a). We start with an auxiliary lemma for a related one-dimensional operator.

Lemma 4.6. *Let the function $W(r, l_j)$, for $r \in [0, 2\sigma]$, $j = 1, \ldots, N_l$, satisfy*

$$\mathcal{M}W(r,l_j) := -D_l[\,\tilde\zeta(r,l_j)\,D_l^- W(r,l_j)\,] + W(r,l_j) = D_l[\,\mu(r,l_j)\,], \tag{4.11}$$

subject to periodicity conditions $W(r, l_0) = W(r, l_{N_l})$ and $W(r, l_1) = W(r, l_{N_l+1})$, where $\tilde\zeta(r, l_j) := \zeta(r, l_{j-1/2})$ and $\mu(r, l_1) = \mu(r, l_{N_l+1})$. Then we have

$$|\tfrac{\partial^m}{\partial r^m} W(r,l_j)| \le C \sum_{n=0}^{m} \max_{\substack{r\in[0,2\sigma]\\ l=1,\ldots,N_l}} |\tfrac{\partial^n}{\partial r^n}\mu(r,l_j)| \quad \text{for } m = 0,1,2. \tag{4.12}$$

Proof. Differentiating (4.11) in r, we get, with the notation $W_m(r,l_j) := \frac{\partial^m}{\partial r^m} W(r,l_j)$, $\zeta_m(r,l_j) := \frac{\partial^m}{\partial r^m}\tilde\zeta(r,l_j)$ and $\mu_m(r,l_j) := \frac{\partial^m}{\partial r^m}\mu(r,l_j)$,

$$\mathcal{M}W_1(r,l_j) = D_l[\,\zeta_1(r,l_j)D_l^- W(r,l_j) + \mu_1(r,l_j)\,],$$

$$\mathcal{M}W_2(r,l_j) = D_l[\,\zeta_2(r,l_j)D_l^- W(r,l_j) + 2\zeta_1(r,l_j)D_l^- W_1(r,l_j) + \mu_2(r,l_j)\,].$$

Note that $\tilde{\zeta}(r, l_j) \geq C > 0$, so problem (4.11) is well posed. Define two discrete $L_2(0, L)$ norms by $\|y\|_h^2 = \sum_{j=1}^{N_l} y_j^2(l_j - l_{j-1})$ and $\|y\|_{h,*}^2 = \sum_{j=1}^{N_l} \frac{1}{2} y_j^2(l_{j+1} - l_{j-1})$. Now, applying the method of energy inequalities [20, Chap. II, §3.5] to (4.11), one can show that $\|D_l^- W(r,\cdot)\|_h + \|W(r,\cdot)\|_{h,*} \leq C\|\mu(r,\cdot)\|_h$. Furthermore, we get

$$\|D_l^- W_1(r,\cdot)\|_h + \|W_1(r,\cdot)\|_{h,*} \leq C\|\zeta_1 D_l^- W + \mu_1\|_h \leq C(\|\mu\|_h + \|\mu_1\|_h)$$

and a similar estimate for W_2. Thus we have

$$\|D_l^- W_m(r,\cdot)\|_h + \|W_m(r,\cdot)\|_{h,*} \leq C \sum_{n=0}^{m} \|\mu_n(r,\cdot)\|_h \quad \text{for } m = 0, 1, 2.$$

The desired result follows as for all $r \in [0, 2\sigma]$ we have $\|\mu_m\|_h \leq C \max_j |\mu_m(r, l_j)|$ and $\max_j |W_m(r, l_j)| \leq C(\|D_l^- W_m\|_h + \|W_m\|_{h,*})$ (the former estimate is a discrete version of the Sobolev imbedding theorem). □

The main result of this section is as follows.

Corollary 4.7. *Let $\mathcal{L}^h_{[0,2\sigma]}$ be linearization of type (4.2) of the finite difference operator $F^h_{[0,2\sigma]}$ in (2.5). Let V_{ij}, for $i = 1, \ldots, 2N-1$, $j = 1, \ldots, N_l$, satisfy*

$$\mathcal{L}^h_{[0,2\sigma]} V_{ij} = \varepsilon^2 \zeta_{ij} D_l[\mu(r_i, l_j)] + \nu_{ij},$$

subject to $V_{i,0} = V_{i,N_l}$, $V_{i,1} = V_{i,N_l+1}$ and $V_{0,j} = V_{2N,j} = 0$, where we also have $\mu(r, l_1) = \mu(r, l_{N_l+1})$ and $\nu_{i,0} = \nu_{i,N_l}$, $\nu_{i,1} = \nu_{i,N_l+1}$. Then

$$\max_{i,j} |V_{ij}| \leq C\Big(\sum_{n=0}^{2} \varepsilon^n \max_{\substack{r\in[0,2\sigma]\\ l=1,\ldots,N_l}} |\tfrac{\partial^n}{\partial r^n}\mu| + \max_{ij} |\nu_{ij}|\Big). \tag{4.13}$$

Proof. A calculation using $W_{ij} := W(r_i, l_j)$ from Lemma 4.6 shows that

$$\mathcal{L}^h_{[0,2\sigma]}(V_{ij} - W_{ij}) = \nu_{ij} + \varepsilon^2 \eta_{ij}^{-1} D_r[\tilde{\eta}_{ij} D_r^- W_{ij}] + (\varepsilon^2 - p_{ij}) W_{ij}.$$

This implies that

$$|\mathcal{L}^h_{[0,2\sigma]}(V_{ij} - W_{ij})| \leq C\Big(\max_{ij} |\nu_{ij}| + \sum_{n=0}^{2} \varepsilon^n \max_{\substack{r\in[0,2\sigma]\\ l=1,\ldots,N_l}} |\tfrac{\partial^n}{\partial r^n} W|\Big).$$

Now, $|V_{ij}| \leq |V_{ij} - W_{ij}| + |W_{ij}|$, while, by the discrete maximum principle, we have $|V_{ij} - W_{ij}| \leq C \max |\mathcal{L}^h_{[0,2\sigma]}(V_{ij} - W_{ij})|$. Combining this with (4.12), we get the desired estimate (4.13). □

4.5. **Proof of the lumped-mass finite element error estimate (4.8).** In this subsection, we generalize a maximum norm error estimate for the standard finite element method [21] to its lumped mass version. Note that the energy arguments are not suitable in estimation of the lumped-mass error for singularly perturbed equations of type (1.1), as they result in the error constants involving negative powers of the small parameter ε.

We use the **notation** of §2.4, and also the space $\mathring{S}^h := \{\chi \in S^h,\ \chi = 0 \text{ on } \partial\Omega^h_\sigma\}$, and the forms

$$\begin{aligned} a(v, w) &:= \varepsilon^2 \langle \nabla v, \nabla w \rangle + \langle f(x, v), w \rangle, \\ a_h(v, w) &:= \varepsilon^2 \langle \nabla v, \nabla w \rangle + \langle f(x, v), w \rangle_h, \qquad \langle \varphi, w \rangle_h := \int_{\Omega^h_\sigma} (\varphi w)^I, \end{aligned}$$

where $(\varphi w)^I$ is the standard piecewise linear interpolant of the function φw. Then the lumped mass solution $\tilde U_\sigma \in S^h$ of (4.4b) using the operator F^h_σ of (2.6), and the standard finite element solution $u_h \in S^h$ satisfy $a_h(\tilde U_\sigma, \chi) = 0$ and $a(u_h, \chi) = 0$ for all $\chi \in \mathring{S}^h$. We shall also use the form

$$\delta_h(v, w) := a(v, w) - a_h(v, w), \tag{4.14}$$

and the discrete function $r_h \in \mathring{S}^h$ such that

$$a(u_h + r_h, \chi) - a(u_h, \chi) = \delta_h(u, \chi) \quad \forall\, \chi \in \mathring{S}^h. \tag{4.15}$$

Note that for any v, w and any nodal basis function function χ_i, a calculation yields

$$|\delta_h(v, \chi_i) - \delta_h(w, \chi_i)| \le C\, \langle 1, \chi_i\rangle \max_{\bar\Omega^h_\sigma} |v - w|. \tag{4.16}$$

Our proof is in **two steps**. First, we shall show that

$$|\tilde U_\sigma - u_h| \le\; C\big(\max_{\bar\Omega^h_\sigma} |u_h - u| + \max_{\bar\Omega^h_\sigma} |r_h|\big). \tag{4.17}$$

For all $\chi \in \mathring{S}^h$ we have $a_h(\tilde U_\sigma, \chi) = a(u_h, \chi)$, so, invoking (4.14) and (4.15), we get

$$a_h(u_h + r_h, \chi) - a_h(\tilde U_\sigma, \chi) = a_h(u_h + r_h, \chi) - a(u_h, \chi) = \delta_h(u, \chi) - \delta_h(u_h + r_h, \chi).$$

Next, by (4.16),

$$|a_h(u_h + r_h, \chi_i) - a_h(\tilde U_\sigma, \chi_i)| \le \langle 1, \chi_i\rangle \max_{\bar\Omega^h_\sigma}(|u_h - u| + |r_h|),$$

which can be rewritten in terms the linearization $\mathcal{L}^h_\sigma$ of F^h_σ as

$$|\mathcal{L}^h_\sigma(u_h + r_h - \tilde U_\sigma)| \le C \max_{\bar\Omega^h_\sigma}(|u_h - u| + |r_h|).$$

Now, by the discrete maximum principle, $|u_h + r_h - \tilde U_\sigma| \le C \max |\mathcal{L}^h_\sigma(u_h + r_h - \tilde U_\sigma)|$, which immediately yields (4.17).

It remains to estimate r_h. Linearizing (4.15), we get $A(r_h, \chi) = \delta_h(u, \chi)$ for all $\chi \in \mathring{S}^h$, where the symmetric bilinear form $A(\cdot, \cdot)$ is given by

$$A(w, \chi) = \varepsilon^2 \langle \nabla w, \nabla \chi\rangle + (pw, \chi), \qquad p(x) := \int_0^1 f_u(x, u_h + s r_h)\, ds.$$

Consider an arbitrary point $x_* \in \tau_*$, where τ_* is some triangle of our triangulation in Ω^h_σ. Then, imitating the proof of [21, Theorem 6.1] we first use an inverse inequality and then the dual argument to get

$$|r_h(x_*)| \le C h^{-1} \|r_h\|_{L_2(\tau_*)} = C h^{-1} \sup_{\phi \in C_0^\infty(\tau_*),\, \|\phi\|_{L_2(\tau_*)} = 1} \langle r_h, \phi\rangle. \tag{4.18}$$

For any such ϕ, we introduce $v_h \in \mathring{S}^h$ such that $A(v_h, \chi) = \langle \phi, \chi\rangle$ for all $\chi \in \mathring{S}^h$. The solution v of the corresponding continuous problem will be employed as well. Then

$$h^{-1}\langle r_h, \phi\rangle = h^{-1} A(r_h, v_h) = h^{-1}\delta_h(u, v_h) \tag{4.19}$$

Note that an inspection of the analysis of [21, §6] yields

$$h^{-1}\|v_h\|_{L_1(\Omega^h_\sigma)} \le C, \qquad h^{-1}\|\nabla v_h\|_{L_1(\Omega^h_\sigma)} \le C\varepsilon^{-1}(1 + h/\varepsilon) \ln(C + \varepsilon/h). \tag{4.20}$$

Indeed, the first bound in (4.20) follows from [21, (6.21)]. The second bound in (4.20) is obtained as follows. First, note that [21, (6.17)] yields $\|\nabla(v - v_h)\|_{L_1(\Omega^h_\sigma)} \le C(h/\varepsilon)^2 \ln(C + \varepsilon/h)$. Then $\|\nabla v\|_{L_1(\Omega^h_\sigma)} \le (h/\varepsilon) \ln(C + \varepsilon/h)$ is obtained employing

[21, (2.7),(6.8)] by imitating the estimation in [21, (6.12),(6.13)]. Combining these observations with $\varepsilon \geq Ch$, we get (4.20).

Now we are ready to estimate the right-hand side in (4.19). In view of (4.14), setting $\Psi(x) := f(x, u(x))$, we get

$$|\delta_h(u, v_h)| = |\langle \Psi, v_h \rangle - \langle \Psi, v_h \rangle_h| \leq C(\|\Psi\|_{C^2} \|v_h\|_{L_1} + \|\Psi\|_{C^1} \|\nabla v_h\|_{L_1}) \cdot h^2,$$

where we used a version of [25, Lemma 3.1]. Combining this with (4.20) immediately yields

$$h^{-1}|\delta_h(u, v_h)| \leq C\,\mathcal{E}_{\text{l.m.}} h^2 \ln(C + \varepsilon/h),$$

where the quantity $\mathcal{E}_{\text{l.m.}}$ is defined in (4.8b). In view of (4.18), (4.19), we now arrive at $|r_h(x_*)| \leq C\,\mathcal{E}_{\text{l.m.}} h^2 \ln(C+\varepsilon/h)$ for all $x^* \in \bar{\Omega}^h_\sigma$. Combining this with (4.17) and noting that [21, Theorems 6.1 and 12.1]) imply $|u_h - u| \leq Ch^2 \ln(C+\varepsilon/h)\|u\|_{C^2(\bar{\Omega}^h_\sigma)}$, we get the desired lumped-mass finite element error estimate (4.8). □

5. Numerical results

Our model problem (see [8]) is posed in the domain Ω shown on Figure 2, whose boundary $\partial\Omega$ is parameterized by $x_1 = \varphi(l) := R\cos\theta$ and $x_2 = \psi(l) := R\sin\theta$, where $l \in [0, 2\pi]$,

$$R = R(l) = 0.4 + \cos^2(l/2), \qquad \theta = \theta(l) = l + e^{(l-5)/2}\sin(l/2)\sin l.$$

In this domain, we consider (1.1) with

$$b(x, u) = \big(u - z(x)\big)u\big(u + z(x)\big), \qquad z(x) = x_1^2 + x_1 + 1. \tag{5.1}$$

Thus $\pm z(x)$ are two stable solutions and 0 is an unstable solution of the corresponding reduced problem. The boundary condition $g_0(x) = (x_1 - x_1^2)/3$ satisfies (1.5b) for both $\pm z$; see Figure 1. We present numerical results only for the solution u near z; see Figure 1 (left); the results for the solution near $-z$ are similar.

This model problem was solved by the discrete discrete Schwarz method (2.5), (2.6), (2.7) with $g^{[1]}_{2\sigma}\big|_{(r=2\sigma,l)} := g_0\big|_{(r=0,l)}$. In the boundary-layer subdomain $\Omega_{[0,2\sigma]}$, we used the Shishkin and Bakhvalov meshes $\{r_i\}$ of §2.2(a),(b) with $\gamma := 0.8\gamma_0$, where $\gamma_0 = 3\sqrt{2}/4$ (see Remark 2.2), and $C_1 := 0.2$, $\sigma_S := \min\{2\gamma^{-1}\varepsilon\ln(N/2), \frac{1}{2}C_1\}$, $\rho := 2\varepsilon$ (see Remark 2.1). The mesh $\{l_j\}$, with $N_l := 4N$, was chosen so that the arc-length between any two consecutive boundary mesh nodes was (almost) constant. In the interior subdomain Ω_σ, we required the diameter of quasiuniform Delaunay triangulations to not exceed N^{-1}. In (2.5), we set $\tilde{\zeta}_{ij} := 2/[\zeta^{-1}_{i,j-1} + \zeta^{-1}_{ij}]$ which is $\zeta(r_i, l_{j-1/2}) + O(h^2)$ (so all our theoretical results remain valid for this modification). The discrete nonlinear problems (2.5) and (2.6) were solved by Newton's method.

In Tables 1 and 2, we compare the kth-interation Schwarz approximation $U^{[k]}$ with the reference computed solution U_{ref} obtained using the numerical method [8] on the mesh that coincides with the triangulation for the corresponding $U^{[k]}$ in $\bar{\Omega}_\sigma$ and the matching-tensor product mesh $\{(r_i, \tilde{l}_j), i = 0, \ldots, N, j = 0, \ldots, \tilde{N}_l\}$ in $\bar{\Omega}_{[0,\sigma]}$. Note that for $\varepsilon \leq Ch$, the error $U_{\text{ref}} - u$ of this method was shown to be $O(h^2|\ln h|^m)$ in the discrete maximum norm [8]. In both tables, we also give the maximum nodal values of the errors $U_{\text{ref}} - u$ computed as described in [11, §4] (by employing an auxiliary computed solution obtained after bisecting the tensor-product mesh in $\Omega_{[0,\sigma]}$ in both directions and dividing each triangle of the Delaunay triangulation in Ω_σ into four triangles of the same shape).

TABLE 1. Errors $\max_{\bar\Omega}|U^{[1]}-U_{\rm ref}|$ and maximum nodal errors $|u-U_{\rm ref}|$

		Shishkin mesh			Bakhvalov mesh		
	N	$\varepsilon=10^{-2}$	$\varepsilon=10^{-4}$	$\varepsilon=10^{-8}$	$\varepsilon=10^{-2}$	$\varepsilon=10^{-4}$	$\varepsilon=10^{-8}$
$\lvert U^{[1]}-U_{\rm ref}\rvert$	32	4.344e-3	6.849e-3	6.900e-3	3.390e-3	1.171e-3	1.172e-3
	64	7.903e-4	1.191e-3	1.201e-3	9.683e-4	3.004e-4	2.999e-4
	128	2.825e-4	2.065e-4	2.131e-4	2.827e-4	7.658e-5	7.644e-5
$\lvert u-U_{\rm ref}\rvert$	32	2.248e-2	2.273e-2	2.274e-2	3.658e-3	3.841e-3	3.843e-3
	64	8.959e-3	9.039e-3	9.039e-3	9.156e-4	9.532e-4	9.536e-4
	128	3.215e-3	3.232e-3	3.232e-3	2.311e-4	2.387e-4	2.388e-4

TABLE 2. Case of $\varepsilon=0.1$: number of iterations k, value of $\max_{x_{ij}\in\partial\Omega_{2\sigma}}|(g_{2\sigma}^{[k+1]}-g_{2\sigma}^{[k]})(x_{ij})|$ triggering the stopping criterion, errors $\max_{\bar\Omega}|U^{[k]}-U_{\rm ref}|$ and maximum nodal errors $|u-U_{\rm ref}|$

	Shishkin & Bakhvalov meshes, $\varepsilon=0.1$		
	$N=32$	$N=64$	$N=128$
k	13	16	19
$\lvert g_{2\sigma}^{[k+1]}-g_{2\sigma}^{[k]}\rvert$	9.480e-4	2.360e-4	5.824e-5
$\lvert U^{[k]}-U_{\rm ref}\rvert$	1.094e-2	2.859e-3	8.336e-4
$\lvert u-U_{\rm ref}\rvert$	9.684e-3	2.656e-3	7.489e-4

Table 1 describes errors of the first-interation approximation $U^{[1]}$ for $\varepsilon\le 10^{-2}$, and thus illustrates Theorem 3.9 and Corollary 4.5. We observe that the maximum errors $|U^{[1]}-U_{\rm ref}|$ are close to or much smaller than the maximum errors $|u-U_{\rm ref}|$.

In Table 2, we focus on $\varepsilon=0.1$. As for our domain Ω, the sub-domain $\Omega_{[0,C_1]}$ is well-defined for a relatively small $C_1=0.2$, the condition $\varepsilon\le\frac14 C_1\gamma(\ln N)^{-1}$ of Corollary 4.5 is violated (note that in this case $\sigma=\frac12 C_1$ and the Bakhvalov mesh coincides with the Shishkin mesh). So, in view of Theorem 4.4, a number of iterations is required for our method to produce an accurate computed solution. We used the stopping criterion $\max_{x_{ij}\in\partial\Omega_{2\sigma}}|(g_{2\sigma}^{[k+1]}-g_{2\sigma}^{[k]})(x_{ij})|\le N^{-2}$. We again note that the maximum errors $|U^{[k]}-U_{\rm ref}|$ are close to the maximum errors $|u-U_{\rm ref}|$.

In summary, the above numerical results agree with the theoretical conclusions of Theorems 3.9, 4.4 and Corollary 4.5.

APPENDIX A. PROOF OF LEMMA 4.2

Proof. We decompose the solution u into a regular component v and a boundary-layer function w as follows. By imitating the argument of [4, §2], where a linear equation of type (1.1a) was considered in a rectangular domain, one can smoothly extend the function f into some extended domain $\Omega^*\times\mathbb{R}$ such that $\Omega\subset\Omega^*$ and $\mathrm{dist}(\partial\Omega,\partial\Omega^*)>1$. Then one can show that there exists a regular function v such that $Fv=0$ in Ω^*, with $|\frac{\partial^{j+m}}{\partial x_1^j\partial x_2^m}v|\le C[1+\varepsilon^{3-(j+m)}]$ and $|\frac{\partial^{j+m}}{\partial x_1^j\partial x_2^m}(v-z)|\le C\varepsilon^{2-(j+m)}$ in $\bar\Omega$ for $j,m=0,\dots,4$.

Thus it remains to prove our assertions (4.5) for $w := u - v$, for which the standard linearization yields $\mathcal{L}w = -\varepsilon^2 \triangle w + p(x)w = 0$ in Ω, with $p(x) = \int_0^1 f_u(x, v + sw)\, ds$, while $w = g_w := g_0 - v$ on $\partial\Omega$, so g_w is a sufficiently smooth regular function. In fact, it suffices to prove that w satisfies (4.5a), and that

$$\text{(A.1)}\quad |\tfrac{\partial^j}{\partial r^j}(\tfrac{\partial^m}{\partial l^m} w)| \le C[\varepsilon^{-j} e^{-\gamma_m r/\varepsilon} + \varepsilon^{(7-m)-j}] \quad \text{for } x \in \bar\Omega_{[0,C_1]},\ j, m = 0, \dots, 4,$$

where $\gamma < \gamma_4 < \ldots < \gamma_0 < \gamma_* := \min_{x\in\bar\Omega} \sqrt{p(x)}$. Note that (A.1) immediately implies (4.5a), as $\varepsilon^{(7-m)-j} \le \varepsilon^{3-j}$ and $e^{-\gamma_m r/\varepsilon} \le e^{-\gamma r/\varepsilon}$.

Consider the barrier function $B_0(x) := \omega(x)e^{-r\gamma_0/\varepsilon} + C'\varepsilon^7$, where ω is the cut-off function from §2.1, and C' is a sufficiently large constant. We claim that $\mathcal{L}B_0 \ge 0$ in Ω. Indeed, in Ω_{2C_1}, where $\omega = 0$, this follows from $\mathcal{L}[\varepsilon^7] \ge p(x)\varepsilon^7 \ge \gamma_0^2\varepsilon^7$. Next, in $\Omega_{[0,C_1]}$, where $\omega = 1$, using (2.3), we get $\mathcal{L}[e^{-\gamma_0 r/\varepsilon}] \ge [-\gamma_0^2 - \varepsilon\gamma_0 C'' + p(x)]e^{-\gamma_0 r/\varepsilon}$ (with $C'' = \max_{x\in\bar\Omega_{[0,C_1]}} |\eta^{-1} \frac{\partial}{\partial r}\eta|$). So for $\varepsilon \le c_0 := \min(\gamma_*^2 - \gamma_0^2)/(\gamma_0 C'')$, we have $\mathcal{L}[e^{-\gamma_0 r/\varepsilon}] \ge 0$ and hence again $\mathcal{L}B_0 \ge 0$. (Note that for $\varepsilon \ge c_0$, problem (1.1) is not singularly perturbed so the desired bounds (4.5) follow from the Schauder-type estimates.) Finally, in $\Omega_{[C_1,2C_1]}$, where $0 < \omega < 1$, one has $r > C_1$ and therefore $|\mathcal{L}[e^{-\gamma_0 r/\varepsilon}]| \le Ce^{-\gamma_0 C_1/\varepsilon} \le C'\gamma_0^2\varepsilon^7$. So we get $\mathcal{L}B_0 \ge 0$ in $\Omega_{[C_1,2C_1]}$ and thus in the entire domain Ω.

Now an application of the maximum/comparison principle yields $|w| \le CB_0(x)$ so we get (4.5a) and (A.1) for $j = m = 0$. To estimate the derivatives of w, note that the stretching transformation $\hat x = x/\varepsilon$ maps any domain Ω_a into the domain $\hat\Omega_a$ and, using the notation $\hat w(\hat x) = w(x)$ and $\hat p(\hat x) = p(x)$, we get $-\triangle\hat w + \hat p\hat w = 0$. Next, using the interior Schauder-type estimates [12, p. 110, (1.12)] for any interior subdomain $\hat\Omega_a$ with $a \in [\varepsilon, C_1]$ and $\operatorname{dist}(\partial\hat\Omega_{a-\varepsilon}, \partial\hat\Omega_a) = 1$, and then rewriting the result in the original variables $x = (x_1, x_2)$, we get $\max_{\bar\Omega_a} |\frac{\partial^{j+m}}{\partial x_1^j \partial x_2^m} w| \le C\varepsilon^{-(j+m)} \max_{\bar\Omega_{a-\varepsilon}} |B_0|$. This implies that

$$\text{(A.2)}\qquad |\tfrac{\partial^{j+m}}{\partial x_1^j \partial x_2^m} w| \le C\varepsilon^{-(j+m)}[e^{-\gamma_0 a/\varepsilon} + \varepsilon^7] \quad \text{for } x \in \bar\Omega_a,\ \ j, m = 0, \dots, 4,$$

where $a \in [\varepsilon, C_1]$. For $a \in [0, \varepsilon]$, bound (A.2) is obtained in a similar way, but using the global Schauder-type estimates [12, p. 110, (1.12)] in the domain $\hat\Omega$. So we have (A.2) for all $a \in [0, C_1]$. This immediately yields (4.5a). Furthermore, restricting (A.2) to $x \in \partial\Omega_a$ and then setting $a := r$ yields (A.1) for $m = 0$.

It remains to prove (A.1) for $m > 0$. To do this, we first need to show that $|\frac{\partial^m}{\partial l^m} p| \le C(1 + \varepsilon^{3-m})$ in $\Omega_{[0,C_1]}$. As the definition of p involves the regular function v and the boundary-layer function $w = u - v$, it suffices to show that $|\frac{\partial^m}{\partial l^m} u| \le C(1 + \varepsilon^{3-m})$ in $\Omega_{[0,C_1]}$, which is the rectangle $[0, C_1] \times [0, L]$ in the variables (r, l). Note that, by (1.1b) and (4.5a), we have $|\frac{\partial^m}{\partial l^m} u| \le C(1 + \varepsilon^{3-m})$ on $\partial\Omega \cup \partial\Omega_{C_1}$. Now, differentiating equation (1.1a) in l, and then using (2.3) and the crude estimate $|\frac{\partial^{j+m}}{\partial x_1^j \partial x_2^m} u| \le C\varepsilon^{-(j+m)}$ to deal with the term $\frac{\partial}{\partial l}(\varepsilon^2\triangle u)$, one gets $|[-\varepsilon^2\triangle + f_u(x, u)]\frac{\partial}{\partial l} u| \le C$. So applying the maximum/comparison principle, we conclude that $|\frac{\partial}{\partial l} u| \le C$ and hence $|\frac{\partial}{\partial l} p| \le C$. In a similar manner, differentiating equation (1.1a) m times in l and applying the maximum/comparison principle to estimate $\frac{\partial^m}{\partial l^m} u$, one can show that indeed $|\frac{\partial^m}{\partial l^m} p| \le C(1 + \varepsilon^{3-m})$ in $\Omega_{[0,C_1]}$ for $m = 1, \dots, 4$.

We are now ready to establish (A.1) for $m > 0$. Each of $\frac{\partial^m}{\partial l^m} w$, for $m = 1, \dots, 4$, will be estimated using the barrier function $B_m(x) := e^{-r\gamma_m/\varepsilon} + C'\varepsilon^{7-m}$. Imitating the argument that was used to estimate $\mathcal{L}B_0$, one can show that $\mathcal{L}B_m \geq CB_m$. Note also that, by (A.2) with $a := C_1$, we have $|\frac{\partial^m}{\partial l^m} w| \leq CB_m$ on $\partial\Omega \cup \partial\Omega_{C_1}$. For $m = 1$, a calculation shows that $|\mathcal{L}(\frac{\partial}{\partial l} w)| \leq CB_0 \leq CB_1$, so an application of the maximum/coparison principle yields $|\frac{\partial}{\partial l} w| \leq CB_1$ in $\bar{\Omega}_{[0,C_1]}$. Furthermore, imitating the argument that was used to prove (A.2), one gets $|\frac{\partial^{j+m}}{\partial x_1^j \partial x_2^m}(\frac{\partial}{\partial l} w)| \leq C\varepsilon^{-(j+m)}[e^{-\gamma_1 a/\varepsilon} + \varepsilon^6]$, which implies (A.1) for $m = 1$. For $m = 2, 3, 4$, estimate (A.1) is obtained similarly. □

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