A POSTERIORI ERROR ESTIMATION FOR PARABOLIC PROBLEMS USING ELLIPTIC RECONSTRUCTIONS. II: A THIRD-ORDER DISCONTINUOUS GALERKIN METHOD*

NATALIA KOPTEVA† AND TORSTEN LINSS‡

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Abstract. A semilinear second-order parabolic equation is considered in a regular and a singularly-perturbed regime. For this equation, we give computable a posteriori error estimates in the maximum norm. Semidiscrete and fully discrete versions of the discontinuous Galerkin method dG(1) are addressed; for the latter we employ elliptic reconstructions that are piecewise-quadratic in time. We also use certain bounds for the Green's function of the parabolic operator.

Key words. a posteriori error estimate, maximum norm, singular perturbation, elliptic reconstruction, discontinuous Galerkin method dG(1), parabolic equations, reaction-diffusion.

AMS subject classifications. 65M15 , 65M60.

1. Introduction.

Consider a semilinear parabolic equation in the form

$$\mathcal{M}u := \partial_t u + \mathcal{L}u + f(x,t,u) = 0 \qquad \text{for} \quad (x,t) \in Q := \Omega \times (0,T], \tag{1.1a}$$

with a second-order linear elliptic operator $\mathcal{L} = \mathcal{L}(t)$ in a spatial domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, subject to the initial and Dirichlet boundary conditions

$$u(x,0) = \varphi(x) \quad \text{for} \quad x \in \bar{\Omega}, \qquad u(x,t) = 0 \quad \text{for} \quad (x,t) \in \partial\Omega \times [0,T]. \tag{1.1b}$$

We assume that f is differentiable in the third argument and, for some positive constants γ and $\bar{\gamma}$, satisfies

$$0 \le \gamma^2 \le \partial_z f(x,t,z) \le \bar{\gamma}^2 \qquad \text{for} \quad (x,t,z) \in \bar{\Omega} \times [0,T] \times \mathbb{R}. \tag{1.2}$$

This paper is the second in the series of two giving computable a posteriori error estimates for fully discrete methods applied to problem (1.1). In the first paper [8], we considered the first-order Backward-Euler and the second-order Crank-Nicolson methods. Now a similar approach will be used to analyze the third-order discontinuous Galerkin discretization dG(1) in time.

The results will be applied to the model equation with $\mathcal{L} := -\varepsilon^2 \triangle = -\varepsilon^2 \sum_{i=1}^n \partial_{x_i}^2$:

$$\mathcal{M}u := \partial_t u - \varepsilon^2 \triangle u + f(x,t,u) = 0 \tag{1.3}$$

posed in a bounded polyhedral spatial domain $\Omega \subset \mathbb{R}^n$, with $n = 1, 2, 3$. This equation will be considered in the two regimes:

$$\text{(i)} \quad \varepsilon = 1, \;\; \gamma \ge 0; \qquad\qquad \text{(ii)} \quad \varepsilon \ll 1, \;\; \gamma > 0.$$

Note that regime (ii) yields a singularly perturbed reaction-diffusion equation, whose solutions may exhibit sharp layer phenomena. So it is important in this regime that

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†Department of Mathematics and Statistics, University of Limerick, Limerick, Ireland (natalia.kopteva@ul.ie).

‡Fakultät für Mathematik und Informatik, FernUniversität in Hagen, Lützowstr. 125, D-58097 Hagen, Germany (torsten.linss@fernuni-hagen.de).

a posteriori error estimates are robust in the sense that any dependence on the small perturbation parameter ε should be shown explicitly [11, 15].

We will give error estimates in the *maximum norm*, which is sufficiently strong to capture sharp layers and singularities that may occur, in particular, if problem (1.1) is of singularly-perturbed type. Our estimates will be of *interpolation type* in the sense that they will include certain terms that may be interpreted as approximating $\tau_j^3|\partial_t^3 u|$, where 3 is the discretization order and τ_j is the local step size in time (this is discussed in Remarks 3.4 and 3.6).

We employ the *elliptic reconstruction* technique, which was introduced in the recent papers [12, 9, 2] as a counterpart of the Ritz-projection in the a posteriori error estimation for parabolic problems. We also use certain bounds for the *Green's function* of the continuous parabolic operator, established in [8], in a manner similar to [2], only for a more general semilinear parabolic operator of (1.3) (compared to $\partial_t - \triangle$ of [2]). We also refer the reader to the papers [3, 13], which give a posteriori error estimates for discontinuous Galerkin time discretizations in other norms (for further details on [3], see Remark 5.6).

One distinctive feature of our analysis in [8] and here (compared, e.g., to [1, 2]) is that we use computed solutions and elliptic reconstructions that are piecewise-polynomial of degree $p-1$ in time, where p is the time discretization order. In [8], they were piecewise-constant in time when dealing with the first-order Backward-Euler method, and piecewise-linear in time when dealing with the second-order Crank-Nicolson method. In this paper, we consider a third-order method, so employ computed solutions and elliptic reconstructions that are piecewise-quadratic in time.

The paper is organized as follows. In Section 2, we introduce the Green's function and cite a stability lemma from [8], which is then used in Section 3 to obtain a posteriori error estimates for a semidiscrete version of the dG(1) method (with no spatial discretization). Next, in Section 4, we cite some elliptic a posteriori error estimates, which are used in Section 5 to derive a posteriori error estimates for a fully discrete dG(1) method. We conclude the paper by applying our results to the model problem (1.3) in Section 5.3, and briefly discussing the estimator computability in the final Section 5.4.

Notation. Throughout the paper, C, as well as c, denotes a generic positive constant that may take different values in different formulas, but is *independent of the diffusion coefficient* ε *and any mesh sizes.* We use $|x|$ for the Euclidian norm of $x \in \mathbb{R}^n$. The usual spaces $C(\bar{\Omega})$ and $H_0^1(\Omega)$ are used, as well as the spaces L_p, $1 \le p \le \infty$, with the norm $\|\cdot\|_{p,\Omega}$, while $\langle \phi, \psi \rangle = \int_\Omega \phi(x)\psi(x)\,\mathrm{d}x$ denotes the inner product in $L_2(\Omega)$. We also employ the standard piecewise-linear hat-functions in time $\{\phi_j(t)\}$ (related to the temporal mesh (2.3)) such that $\phi_j = \frac{t_{j+1}-t}{t_{j+1}-t_j}$ and $\phi_{j+1} = \frac{t-t_j}{t_{j+1}-t_j}$ for $t \in [t_j, t_{j+1}]$.
Distributions and left-continuity convention. Certain functions will be understood as distributions [6], which will in most cases be indicated. By contrast, if a certain function is Lebesgue-integrable in $\Omega \times (0, T)$, we shall refer to it as a regular function. Whenever we deal with a regular function, it will be understood *left-continuous* for all $t \in (0, T]$. In particular, this convention will be applied to all piecewise-continuous temporal derivatives.

2. The Green's function of the parabolic operator. In this section we define the Green's function $\mathcal{G}$ associated with the operator $\mathcal{M}$ of (1.1), and cite some related results from [8]. Our interest in the Green's function is in that it can be used to represent the error of a numerical approximation in terms of its residual.

For definitions and properties of fundamental solutions and Green's functions of second-order parabolic operators with variable coefficients, we refer the reader to [5, Chap. 1 and §7 of Chap. 3]. In particular, for fixed $(x,t)\in Q$, the Green's function $\mathcal{G}(x,t;\xi,s) =: \Gamma(\xi,s)$ solves the adjoint terminal-value problem

$$
\begin{aligned}
[-\partial_s - \mathcal{L}^* + a(\xi,s)]\,\Gamma(\xi,s) &= 0 && \text{for } (\xi,s)\in\Omega\times[0,t), && \text{(2.1a)}\\
\Gamma(\xi,t) &= \delta(\xi - x) && \text{for } \xi\in\Omega, && \text{(2.1b)}\\
\Gamma(\xi,s) &= 0 && \text{for } (\xi,s)\in\partial\Omega\times[0,t]. && \text{(2.1c)}
\end{aligned}
$$

Here $\delta(\cdot)$ is the Dirac δ-distribution in $\mathbb{R}^n$ [6], and $\mathcal{L}^*$ is the adjoint operator to the linear operator $\mathcal{L}$. We set $a(x,t) := \partial_z f(x,t,z)$ if f is linear in the third argument, i.e. $f(x,t,z) = a(x,t)z + b(x,t)$. If f is nonlinear, we associate $a(x,t) := \int_0^1 \partial_z f(x,t,w + z[v-w])\,\mathrm{d}z$ with a pair of bounded functions v and w that vanish on $\partial\Omega$. As the standard linearization now yields $\mathcal{M}v - \mathcal{M}w = [\partial_t + \mathcal{L} + a(x,t)](v-w)$, so with the help of this Green's function, the difference $v-w$ is represented as

$$
\begin{aligned}
[v-w](x,t) = \int_\Omega \mathcal{G}(x,t;\xi,0)\,[v-w](\xi,0)\,\mathrm{d}\xi \\
+ \int_0^t \int_\Omega \mathcal{G}(x,t;\xi,s)\,[\mathcal{M}v - \mathcal{M}w](\xi,s)\;\mathrm{d}\xi\,\mathrm{d}s. \qquad (2.2)
\end{aligned}
$$

The analysis in this paper will be carried out under the following condition.

Condition 2.1. *There are constants κ_0, $\kappa_1 > 0$ and $\kappa_2 \ge 0$ such that the Green's function $\mathcal{G}$ of (2.1), (1.2) satisfies*

$$
\|\mathcal{G}(x,t;\cdot,s)\|_{1,\Omega} \le \kappa_0\,\mathrm{e}^{-\gamma^2(t-s)}, \qquad \int_0^{t-\tau} \|\partial_s\mathcal{G}(x,t;\cdot,s)\|_{1,\Omega}\,\mathrm{d}s \le \kappa_1\,\ell(\tau,t) + \kappa_2,
$$

where $x\in\Omega$, $\tau\in(0,t]$, $t\in(0,T]$, and $\ell(\tau,t) := \int_\tau^t s^{-1}\mathrm{e}^{-\frac{1}{2}\gamma^2 s}\,\mathrm{d}s \le \ln(t/\tau)$.

Note that our model problem satisfies this condition as follows.

Lemma 2.2 ([8, Lemma 2.2]). *Let $\varepsilon\in(0,1]$ and $\gamma\ge 0$. Under assumption (1.2), the model problem (1.3) satisfies Condition 2.1 with $\kappa_0 := 1$, $\kappa_1 := \frac{3^n}{2^{n/2+1}}$ and an ε-independent constant $\kappa_2 \ge 0$. If $f(x,t,z) = a(x)z + b(x,t)$, then $\kappa_2 = 0$. In general, $\kappa_2 = (\bar\gamma^2 - \gamma^2)\,\hat\kappa_2$, where $\hat\kappa_2 = \hat\kappa_2(\gamma)$ if $\gamma > 0$, and $\hat\kappa_2 = \hat\kappa_2(T)$ if $\gamma = 0$.*

The above Condition 2.1 will be employed by means of the following version of [8, Lemma 2.4], which plays a crucial role in our analysis here and in [8]. The lemma is formulated in the context of an arbitrary nonuniform mesh in the time direction

$$
0 = t_0 < t_1 < t_2 < \ldots < t_M = T, \quad \text{with } \tau_j := t_j - t_{j-1} \quad \text{for } j = 1,\ldots,M. \qquad (2.3)
$$

Lemma 2.3 ([8]). *Suppose the parabolic operator $\mathcal{M}$ of (1.1) satisfies (1.2) and Condition 2.1, and v, w are bounded in $\bar\Omega\times[0,T]$. Furthermore, let $v(\cdot,t), w(\cdot,t)\in H_0^1(\Omega)\cap C(\bar\Omega)$ for $t\in[0,T]$, and*

$$
\mathcal{M}v - \mathcal{M}w = \partial_t\mu + \vartheta \quad \text{in } Q, \qquad (2.4)
$$

where the function μ is continuous and bounded on $[t_0,t_1]$ and each $(t_{j-1},t_j]$, while $\partial_t\mu$ is continuous and bounded on $(t_{m-1},t_m]$ for some $1\le m\le M$, and $\|\vartheta(\cdot,s)\|_{\infty,\Omega}$

is integrable on $(0,t_m)$ *(possibly, in the sense of distributions). Then*

$$\begin{aligned}\big\|[v-w](\cdot,t_m)\big\|_{\infty,\Omega} &\\ \le \kappa_0 \mathrm{e}^{-\gamma^2 t_m}\big\|[v-w-\mu](\cdot,0)\big\|_{\infty,\Omega} &+ (\kappa_1\,\ell_m+\kappa_2)\sup_{s\in[0,t_{m-1}]}\big\|\mu(\cdot,s)\big\|_{\infty,\Omega}\\ + \kappa_0\,\big\|\mu(\cdot,t_{m-1}^+)\big\|_{\infty,\Omega} &+ \kappa_0\,\tau_m \sup_{s\in(t_{m-1},t_m]}\big\|\partial_s\mu(\cdot,s)\big\|_{\infty,\Omega}\\ + \kappa_0\int_0^{t_m} &\mathrm{e}^{-\gamma^2(t_m-s)}\,\big\|\vartheta(\cdot,s)\big\|_{\infty,\Omega}\,\mathrm{d}s,\end{aligned}$$

where $\ell_m=\ell_m(\gamma):=\int_{\tau_m}^{t_m} s^{-1}\mathrm{e}^{-\frac12\gamma^2 s}\,\mathrm{d}s\le\ln(t_m/\tau_m)$.

Proof. The desired assertion is obtained using (2.2); see [8] for details. □

REMARK 2.4. *The term* $\partial_t\mu$ *in the right-hand side of (2.4) is understood in the sense of distributions.*

REMARK 2.5. *One can easily check that if* $\gamma=0$*, then* $\ell_m=\ln(t_m/\tau_m)$*. Otherwise, if* $\gamma>0$*, one has* $\ell_m(\gamma)=E_1(\frac12\gamma^2\tau_m)-E_1(\frac12\gamma^2 t_m)$*, where* $E_1(t)=\int_t^\infty s^{-1}\mathrm{e}^{-s}\,\mathrm{d}s$*; so* $\ell_m(\gamma)\le|\ln(\frac12\gamma^2\tau_m)|$ *provided that* $\frac12\gamma^2\tau_m\le 0.67$ *(this is easily checked by finding the only root* ≈ 0.67 *of the equation* $E_1(s)=|\ln s|$ *on* $(0,1)$*). Note also that* $\ell_1=0$ *for any* $\gamma\ge 0$*.*

3. Semidiscrete Discontinuous Galerkin method dG(1) (no spatial discretization). Consider an arbitrary nonuniform mesh (2.3) in the time direction and discretize the abstract parabolic problem (1.1) in time using the third-order Discontinuous Galerkin method dG(1) (described, e.g., in [4, 16]) as follows.

Let $U^0:=\varphi$. Given an approximate solution $U^j\in H_0^1(\Omega)\cap C(\bar\Omega)$ associated with the time level t_j, we require an auxiliary approximate solution $y^j\in H_0^1(\Omega)\cap C(\bar\Omega)$ associated with the time level t_j^+ and an approximate solution $U^{j+1}\in H_0^1(\Omega)\cap C(\bar\Omega)$ associated with the time level t_{j+1} to satisfy

$$\big\langle y^j-U^j,v\big\rangle+\int_{t_j^+}^{t_{j+1}}\big\langle\partial_t U+\psi,\,v\phi_j+w\phi_{j+1}\big\rangle\,\mathrm{d}t=0\qquad\forall\,v,w\in H_0^1(\Omega),\tag{3.1}$$

where $\{\phi_j\}_{j=0}^M$ are the standard piecewise-linear hat-functions in time so $\phi_j=\frac{t_{j+1}-t}{\tau_{j+1}}$ and $\phi_{j+1}=\frac{t-t_j}{\tau_{j+1}}$ for $t\in[t_j,t_{j+1}]$, while

$$U:=y^j\,\phi_j+U^{j+1}\phi_{j+1},\qquad \psi:=\mathcal{L}(t)\,U+f(\cdot,t,U)\qquad\text{for }\ t\in(t_j,t_{j+1}].\tag{3.2}$$

As $v,w\in H_0^1(\Omega)$ are arbitrary, the above relation (3.1) is equivalent to the system for y^j, $U^{j+1}\in H_0^1(\Omega)\cap C(\bar\Omega)$:

$$y^j-U^j+\int_{t_j^+}^{t_{j+1}}\big(\partial_t U+\psi\big)\,\phi_j\ \mathrm{d}t=0,\qquad \int_{t_j^+}^{t_{j+1}}\big(\partial_t U+\psi\big)\,\phi_{j+1}\ \mathrm{d}t=0.$$

Next, we take the sum and difference of these two equations. As $\phi_j+\phi_{j+1}=1$ and $\int_{t_j^+}^{t_{j+1}}\partial_t U\ \mathrm{d}t=U^{j+1}-y^j$, while $\int_{t_j^+}^{t_{j+1}}\partial_t U\,(\phi_j-\phi_{j+1})\ \mathrm{d}t=0$, so

$$U^{j+1}-U^j+\int_{t_j^+}^{t_{j+1}}\psi\ \mathrm{d}t=0,\tag{3.3a}$$

$$y^j-U^j+\int_{t_j^+}^{t_{j+1}}\psi\,(\phi_j-\phi_{j+1})\ \mathrm{d}t=0.\tag{3.3b}$$

Note that the system (3.3) is equivalent to (3.1), so the semidiscrete dG(1) method can be equivalently defined by (3.3) combined with (3.2).

Furthermore, applying the Radau quadrature to both integrals in (3.3), one gets the semidiscrete dG(1) method with quadrature:

$$U^{j+1} - U^j + \tfrac{1}{4}\,\tau_{j+1}\left(3\psi^{j+1/3} + \psi^{j+1}\right) = 0, \tag{3.4a}$$

$$y^j - U^j + \tfrac{1}{4}\,\tau_{j+1}\left(\psi^{j+1/3} - \psi^{j+1}\right) = 0, \tag{3.4b}$$

where

$$\psi^{j+1/3} := \psi(\cdot, t_{j+1/3}), \qquad \psi^{j+1} := \psi(\cdot, t_{j+1}), \tag{3.4c}$$

are computed using $U(\cdot, t_{j+1/3}) = \frac{2}{3}y^j + \frac{1}{3}U^{j+1}$ and $U(\cdot, t_{j+1}) = U^{j+1}$, and also $\mathcal{L}(t_{j+1/3})$ and $\mathcal{L}(t_{j+1})$, respectively, by virtue of (3.2).

Remark 3.1. *The semidiscrete dG(1) method with quadrature, defined by (3.4) combined with (3.2) is, in fact, an implicit two-stage Runge-Kutta method of order three with the Butcher Tableau:*

$$\begin{array}{c|cc} \frac{1}{3} & \frac{5}{12} & -\frac{1}{12} \\ 1 & \frac{3}{4} & \frac{1}{4} \\ \hline & \frac{3}{4} & \frac{1}{4} \end{array}$$

Remark 3.2. *The functions y^j and U^{j+1} obtained using the semidiscrete dG(1) method (3.3), as well as its version (3.4) with quadrature, respectively give a second-order approximation at the time level t_j and a third-order approximation at the time level t_{j+1}. Consequently, the more accurate approximate solutions $\{U^j\}_{j=1}^M$ are to be used, while the approximate solutions $\{y^j\}_{j=0}^{M-1}$ are auxilliary and may be discarded.*

3.1. A posteriori error estimate in the semidiscrete case. To formulate an a posteriori error estimate for the semidiscrete dG(1), we need to introduce some notation. Note that the computed solution interpolant U of (3.2) is only second-order accurate (as y^j is only second-order accurate, and also since U is a piecewise-linear interpolant). So we introduce a continuous piecewise-quadratic interpolant $\tilde{U}$ of the computed solution as follows:

$$\tilde{U} := U^j\phi_j + U^{j+1}\phi_{j+1} + \nu, \qquad \nu := 3\,\phi_j\,\phi_{j+1}\,\{y^j - U^j\} \qquad \text{on } [t_j, t_{j+1}]. \tag{3.5}$$

Remark 3.3. *The truncation error analysis shows that $y^j - U^j \approx -\frac{1}{6}\tau_{j+1}^2\partial_t^2 u(\cdot, t_j)$, while U^j and U^{j+1} are third-order approximations of u at time levels t_j and t_{j+1}. So the piecewise-quadratic interpolant $\tilde{U}$ gives a third-order approximation to the exact solution u in the entire domain $\bar{Q}$.*

Furthermore, a comparison of (3.5) with (3.2) shows that

$$\tilde{U}(\cdot, t_{j+1/3}) = U(\cdot, t_{j+1/3}) = \tfrac{2}{3}\,y^j + \tfrac{1}{3}\,U^{j+1}, \quad \tilde{U}(\cdot, t_{j+1}) = U(\cdot, t_{j+1}) = U^{j+1}, \tag{3.6}$$

so $\frac{2}{3}\,y^j + \frac{1}{3}\,U^{j+1}$ gives a third-order approximation to $u(\cdot, t_{j+1/3})$.

Next, similarly to ψ of (3.2) , we define

$$\tilde{\psi} := \mathcal{L}(t)\,\tilde{U} + f(\cdot, t, \tilde{U}) \qquad \text{for } t \in [t_j, t_{j+1}]. \tag{3.7}$$

for which, (3.6) implies that

$$\tilde{\psi}(\cdot, t_{j+1/3}) = \psi^{j+1/3}, \qquad \tilde{\psi}(\cdot, t_{j+1}) = \psi^{j+1}.$$

Consequently, the quadratic interpolant $I_{2,t}\tilde{\psi}$ of $\tilde{\psi}$ in time, using the interpolation nodes t_j, $t_{j+1/3}$ and t_{j+1}, allows two equivalent representations:

$$\begin{aligned} I_{2,t}\tilde{\psi} &= \psi^j\,\phi_j + \psi^{j+1}\,\phi_{j+1} - \tfrac{1}{2}\,\phi_j\,\phi_{j+1}\,\chi^{j+1} \\ &= \psi^{j+1} - \tfrac{3}{2}\{\psi^{j+1} - \psi^{j+1/3}\}\,\phi_j - \tfrac{1}{2}\,\tau_{j+1}^{-1}(t - t_{j+1/3})\,\phi_{j+1}\,\chi^{j+1}, \end{aligned} \tag{3.8}$$

where $\chi^{j+1} = \partial_t^2(I_{2,t}\tilde{\psi})$ is given by

$$\chi^{j+1} := 3\,[2\psi^j - 3\psi^{j+1/3} + \psi^{j+1}]. \tag{3.9}$$

REMARK 3.4. *In view of Remark 3.3, the function $\tilde{\psi}$ of (3.7) approximates $-\partial_t u$, so χ^{j+1} of (3.9) approximates $-\tau_{j+1}^2\partial_t^3 u$.*

For the semidiscrete method with quadrature, we give the following result (a simplified version of this result for a t-independent elliptic operator $\mathcal{L}$ will be given in Section 3.2).

THEOREM 3.5. *Let u solve the problem (1.1) with the parabolic operator $\mathcal{M}$ satisfying (1.2) and Condition 2.1, and U^j solve the corresponding semidiscrete problem (3.4), (3.2). Then for $m = 1, \ldots, M$, one has*

$$\begin{aligned} &\big\|U^m - u(\cdot, t_m)\big\|_{\infty,\Omega} \\ &\quad\le \tfrac{2}{81}(\kappa_1\,\ell_m + \kappa_2) \max_{j=0,\ldots,m-2} \tau_{j+1}\big\|\chi^{j+1}\big\|_{\infty,\Omega} + \tfrac{1}{18}\kappa_0\,\tau_m\big\|\chi^m\big\|_{\infty,\Omega} \\ &\qquad + \kappa_0 \sum_{j=1}^{m} \int_{t_{j-1}}^{t_j} \mathrm{e}^{-\gamma^2(t_m - s)}\,\big\|[\tilde{\psi} - I_{2,t}\tilde{\psi}](\cdot, s)\big\|_{\infty,\Omega}\,\mathrm{d}s\,, \end{aligned} \tag{3.10}$$

with the notation (3.5)–(3.9).

REMARK 3.6. *The truncation error analysis (see Remarks 3.3 and 3.4) shows that the right-hand side in the estimate (3.10) is $\mathcal{O}(\max_{j=1,\ldots,m}\tau_j^3)$. So (3.10) gives a third-order a posteriori error estimate.*

Proof of Theorem 3.5. In view of (3.4a) combined with $\partial_t(U^j\phi_j + U^{j+1}\phi_{j+1}) = (U^{j+1} - U^j)/\tau_{j+1}$, for $\tilde{U}$ of (3.5) one gets

$$\partial_t\tilde{U} = \partial_t\nu - \tfrac{1}{4}\big(3\psi^{j+1/3} + \psi^{j+1}\big) \quad \text{for } t \in (t_j, t_{j+1}]. \tag{3.11}$$

Next, note that for any linear function $w = w(t)$

$$w(t) - w(t_{j+1/2}) = (t - t_{j+1/2})\,w' = \partial_t[-\tfrac{1}{2}\tau_{j+1}^2\,\phi_j\phi_{j+1}\,w'].$$

So for the function $\psi^{j+1} - \frac{3}{2}\{\psi^{j+1} - \psi^{j+1/3}\}\,\phi_j$, linear for $t \in [t_j, t_{j+1}]$, one gets

$$\begin{aligned} &\big(\psi^{j+1} - \tfrac{3}{2}\{\psi^{j+1} - \psi^{j+1/3}\}\,\phi_j\big) - \tfrac{1}{4}\big(3\psi^{j+1/3} + \psi^{j+1}\big) \\ &\qquad\qquad = \partial_t\big[-\tfrac{3}{4}\tau_{j+1}\,\phi_j\phi_{j+1}\,\{\psi^{j+1} - \psi^{j+1/3}\}\big] = -\partial_t\nu, \end{aligned}$$

where we also used (3.4b) and the definition of ν in (3.5). Also, a calculation yields

$$\tfrac{1}{2}\,\tau_{j+1}^{-1}(t - t_{j+1/3})\,\phi_{j+1}\,\chi^{j+1} = \partial_t\big[-\tfrac{1}{6}\tau_{j+1}\,\phi_j^2\,\phi_{j+1}\,\chi^{j+1}\big].$$

So, taking the difference of the above two relations and recalling (3.8), one gets

$$I_{2,t}\tilde{\psi} - \tfrac{1}{4}\big(3\psi^{j+1/3} + \psi^{j+1}\big) = \partial_t\mu - \partial_t\nu, \qquad \mu := \tfrac{1}{6}\tau_{j+1}\,\phi_j^2\,\phi_{j+1}\,\chi^{j+1}. \tag{3.12}$$

For this function μ, a calculation shows that

$$|\mu| \le \tfrac{2}{81}\tau_{j+1}|\chi^{j+1}|, \quad \tau_{j+1}\,|\partial_t\mu| \le \tfrac{1}{18}\tau_{j+1}|\chi^{j+1}|, \quad \text{for } t\in(t_j,t_{j+1}]. \tag{3.13}$$

Now, combining (3.11) with (3.12), we arrive at

$$\partial_t\tilde{U} = \partial_t\mu - I_{2,t}\tilde{\psi} \qquad \text{for } t\in(t_j,t_{j+1}].$$

As $\mathcal{M}\tilde{U} = \partial_t\tilde{U} + \tilde{\psi}$ and $\mathcal{M}u = 0$, so

$$\mathcal{M}\tilde{U} - \mathcal{M}u = \partial_t\mu + \big(\tilde{\psi} - I_{2,t}\tilde{\psi}\big) \qquad \text{for } t\in(t_j,t_{j+1}].$$

The desired bound for $U^m - u(\cdot,t_m) = [\tilde{U}-u](\cdot,t_m)$ is then obtained by an application of Lemma 2.3 with μ of (3.12) and $\vartheta := \tilde{\psi} - I_{2,t}\tilde{\psi}$, using (3.13) combined with $\mu(\cdot,t_{m-1}^+) = 0$ and the observation that that $[\tilde{U} - u - \mu](\cdot,0) = U^0 - \varphi = 0$. This completes the proof. □

Remark 3.7 (Computability). *The computation of the right-hand side in the estimate (3.10) involves computing χ^{j+1} of (3.9) for $j < m$. Note that the terms $\psi^{j+1/3}$ and ψ^{j+1}, which appear in (3.9), can be explicitly represented using (3.4); to be more precise, one gets*

$$\psi^{j+1/3} = -\frac{U^{j+1}-U^j}{\tau_{j+1}} - \frac{y^j - U^j}{\tau_{j+1}}, \quad \psi^{j+1} = -\frac{U^{j+1}-U^j}{\tau_{j+1}} + 3\,\frac{y^j - U^j}{\tau_{j+1}}. \tag{3.14}$$

3.2. Application to a general t-independent operator $\mathcal{L}$ and the model problem (1.3). Suppose that the coefficients of the linear elliptic operator $\mathcal{L}(t)$ are independent of the variable t; we shall highlight this case by using the special notation $\mathring{\mathcal{L}} := \mathcal{L}$ for this operator. Note that in this case, the semidiscrete method (3.4), (3.2) can be rewritten as

$$U^{j+1} - U^j + \tfrac{1}{2}\,\tau_{j+1}\,\mathring{\mathcal{L}}\big(y^j + U^{j+1}\big) + \tfrac{1}{4}\,\tau_{j+1}\,\big(3f^{j+1/3} + f^{j+1}\big) = 0, \tag{3.15a}$$

$$y^j - U^j + \tfrac{1}{6}\,\tau_{j+1}\,\mathring{\mathcal{L}}\big(y^j - U^{j+1}\big) + \tfrac{1}{4}\,\tau_{j+1}\,\big(f^{j+1/3} - f^{j+1}\big) = 0, \tag{3.15b}$$

where we use the notation

$$f^{j+1/3} := f(\cdot, t_{j+1/3}, \tfrac{2}{3}\,y^j + \tfrac{1}{3}\,U^{j+1}), \qquad f^{j+1} := f(\cdot, t_j, U^{j+1}). \tag{3.16}$$

Here we also used the observations that

$$\psi^{j+1/3} = \mathring{\mathcal{L}}[\tfrac{2}{3}\,y^j + \tfrac{1}{3}\,U^{j+1}] + f^{j+1/3}, \quad \psi^{j+1} = \mathring{\mathcal{L}}U^{j+1} + f^{j+1}.$$

Now consider the quantities that appear in the estimator given by (3.10). A similar calculation for χ^{j+1} of (3.9) yields

$$\chi^{j+1} = 6\,\mathring{\mathcal{L}}[U^j - y^j] + 3\,[2f^j - 3f^{j+1/3} + f^{j+1}]. \tag{3.17}$$

The estimator (3.10) also involves $\tilde{\psi} - I_{2,t}\tilde{\psi}$. By (3.7), $\tilde{\psi} = \mathring{\mathcal{L}}\,\tilde{U} + f(\cdot,t,\tilde{U})$, where $I_{2,t}[\mathring{\mathcal{L}}\,\tilde{U}] = \mathring{\mathcal{L}}\,[I_{2,t}\tilde{U}] = \mathring{\mathcal{L}}\,\tilde{U}$, so

$$\tilde{\psi} - I_{2,t}\tilde{\psi} = f(\cdot,t,\tilde{U}) - I_{2,t}[f(\cdot,t,\tilde{U})]. \tag{3.18}$$

Note that this quantity does not involve $\mathring{\mathcal{L}}$ and can be estimated using the properties of the function f.

Our findings are summarized in the following result.

COROLLARY 3.8. *Let the elliptic operator $\mathcal{L}(t) = \mathring{\mathcal{L}}$ be independent of the variable t; then the statement of Theorem 3.5 remains valid with the simplifications (3.17), (3.18) and the notation (3.16).*

Finally, recall that in the model problem (1.3) the elliptic operator $\mathcal{L} = -\varepsilon^2\triangle$ is t-independent, so we apply Corollary 3.8 to this problem.

COROLLARY 3.9. *Under assumption (1.2), the a posteriori error estimate (3.10) applies to the model problem (1.3) with the constants κ_0, κ_1, κ_2 from Lemma 2.2, the simplifications (3.17), (3.18) and the notation (3.16).*

4. Elliptic a posteriori error estimators. In this section, we consider a steady-state version of the abstract parabolic problem (1.1):

$$\mathcal{L}v + g(x,v) = 0 \quad \text{for } x \in \Omega, \qquad v = 0 \quad \text{for } x \in \partial\Omega, \tag{4.1}$$

and its discretizations in the form

$$\text{Find } v_h \in \mathring{V}_h : \quad \mathcal{L}_h v_h + \mathcal{P}_h[g(\cdot, v_h)] = 0, \qquad \text{where } \mathring{V}_h := V_h \cap H_0^1(\Omega). \tag{4.2a}$$

Here $V_h \subset C(\bar{\Omega})$ is some finite-element space, and for the related Lagrange interpolation operator I_h onto V_h, we use some operators $\mathcal{L}_h$ and $\mathcal{P}_h$ such that

$$\begin{aligned} &\mathcal{L}_h : \ H_0^1(\Omega) \to \mathring{V}_h - I_h[g(\cdot,0)], \\ &\mathcal{P}_h v \in \mathring{V}_h + I_h v \quad \forall\, v \in C(\bar{\Omega}), \qquad \mathcal{P}_h v_h = v_h \quad \forall\, v_h \in V_h. \end{aligned} \tag{4.2b}$$

Note that as any $v_h \in \mathring{V}_h$ vanishes on $\partial\Omega$, so $\mathring{V}_h - I_h[g(\cdot,0)] = \mathring{V}_h - I_h[g(\cdot, v_h)]$, so the definition (4.2) is consistent.

Assumptions. We assume, for any admissible g, that

(i) there exist unique solutions v and v_h of problems (4.1) and (4.2), respectively;

(ii) an a posteriori error estimate is available for these solutions in the form

$$\|v - v_h\|_{\infty,\Omega} \le \eta\big(V_h, v_h, g(\cdot, v_h)\big). \tag{4.3}$$

Note that the availability of elliptic a posteriori error estimates, such as (4.3), enables one to employ elliptic reconstructions in the a posteriori error estimation of the related parabolic problems. Moreover, $\mathcal{L}_h$ and $\mathcal{P}_h$ are not necessarily needed to be evaluated explicitly to compute the a posteriori estimator either for the elliptic problem or the parabolic problem.

4.1. Elliptic model problem. Many standard finite element discretizations of elliptic equations (including those with quadrature) allow a representation of type (4.2). For example, consider a steady-state elliptic version of our model problem (1.3) posed in a bounded polyhedral domain $\Omega \subset \mathbb{R}^n$:

$$-\varepsilon^2\triangle v + g(x,v) = 0 \ \text{ for } x \in \Omega, \quad v = 0 \ \text{ for } x \in \partial\Omega, \quad \partial_z g(x,z) \ge \gamma^2 > 0. \tag{4.4}$$

With a finite-element space $V_h \subset C(\bar{\Omega})$ and $\mathring{V}_h := V_h \cap H_0^1(\Omega)$, a standard Galerkin finite element method for this problem can be described by

$$\text{Find } v_h \in \mathring{V}_h : \qquad \varepsilon^2 \langle \nabla v_h, \nabla \chi \rangle + \langle g(\cdot, v_h), \chi \rangle_h = 0 \qquad \forall\, \chi \in \mathring{V}_h, \tag{4.5}$$

where $\langle\cdot,\cdot\rangle_h$ is either exactly the inner product $\langle\cdot,\cdot\rangle$ in $L_2(\Omega)$, or some quadrature formula for $\langle\cdot,\cdot\rangle$.

REMARK 4.1. *The discretization (4.5) is of type (4.2) provided that the Gram matrix $\langle\chi_i,\chi_j\rangle_h$ of the basis $\{\chi_i\}$ in $\mathring{V}_h$ is invertible. Then let $\langle\mathcal{L}_h\varphi,\chi\rangle_h=\varepsilon^2\langle\nabla\varphi,\nabla\chi\rangle$ and $\langle\mathcal{P}_h\psi,\chi\rangle_h=\langle\psi,\chi\rangle_h$, subject to (4.2b), for all $\varphi\in H_0^1(\Omega)$, $\psi\in C(\bar\Omega)$ and $\chi\in\mathring{V}_h$*

Suppose, for example, that $\langle\psi,\chi\rangle_h=\langle\psi,\chi\rangle$ for all ψ, $\chi\in V_h$. Then $\langle\mathcal{L}_h\varphi,\chi\rangle=\varepsilon^2\langle\nabla\varphi,\nabla\chi\rangle$ and $\langle\mathcal{P}_h\psi,\chi\rangle=\langle\psi,\chi\rangle_h$, subject to (4.2b), for all $\varphi\in H_0^1(\Omega)$, $\psi\in C(\bar\Omega)$ and $\chi\in\mathring{V}_h$. In particular,

(i) if $\langle\cdot,\cdot\rangle_h:=\langle\cdot,\cdot\rangle$ (i.e. no quadrature is used), then $\mathcal{P}_h$ is the L_2 projection onto V_h;

(ii) if a quadrature of type $\langle\psi,\chi\rangle_h:=\langle I_h\psi,\chi\rangle$ is used for all $\psi\in C(\bar\Omega)$ and $\chi\in\mathring{V}_h$, where I_h is the Lagrange interpolation operator associated with V_h, then $\mathcal{P}_h:=I_h$.

REMARK 4.2. *Suppose that one employs a quadrature of lumped-mass type defined by $\langle\psi,\chi_i\rangle_h=\langle I_h(\psi\chi_i),1\rangle=\psi_i\langle\chi_i,1\rangle$ for all basis functions χ_i of V_h, where $\psi\in C(\bar\Omega)$ and $\sum\psi_i\chi_i=I_h\psi$. Then again $\mathcal{P}_h:=I_h$, but $\mathcal{L}_h\varphi:=\sum a_i\chi_i$ with $a_i:=\varepsilon^2\frac{\langle\nabla\varphi,\nabla\chi_i\rangle}{\langle\chi_i,1\rangle}$ for χ_i associated with the interior mesh nodes, and $a_i:=-[g(\cdot,0)]_i$ for χ_i associated with the boundary mesh nodes. Consequently, $\mathcal{L}_hv_h$ is easily explicitly computable for any $v_h\in\mathring{V}_h$ by applying the normalized stiffness matrix to the column vector of nodal values $\{v_{h,i}\}$.*

We now cite elliptic estimators of type (4.3) for particular cases of (4.4) and (4.5).

4.2. Elliptic model problem: regular regime. We first consider the steady-state version (4.4) of our model problem (1.3) in the regular regime of $\varepsilon:=1$.

Let v solve the problem (4.4) with $\varepsilon=1$, $\gamma\geq 0$, posed in a bounded polyhedral domain $\Omega\subset\mathbb{R}^n$, $n=2,3$, and v_h solve the discrete problem (4.5) with V_h and $\langle\cdot,\cdot\rangle_h$ defined as follows. Given a conforming and shape-regular triangulation $\mathcal{T}_h$ of $\bar\Omega$ made of elements T, we let V_h be the space of continuous piecewise polynomial finite element functions of degree $l\geq 1$, and $\mathring{V}_h:=V_h\cap H_0^1(\Omega)$. We employ $\langle\varphi,\psi\rangle_h:=\sum_{T\in\mathcal{T}_h}Q_T(\varphi\psi)$, where Q_T is a quadrature formula for the integral over T with positive weights, and quadrature points contained in T, such that Q_T is exact for the polynomials of degree q with $q\geq\max\{2l-2,1\}$.

In [14, Theorem 4.2], an a posteriori error estimate of type (4.3) is given with $\eta=\eta_0$ defined by

$$\eta_0\big(V_h,v_h,g(\cdot,v_h)\big):=\Big[c_0\max_{T\in\mathcal{T}_h}\Big\{h_T^2\big\|(\triangle v_h-g(\cdot,v_h))\big\|_{\infty,T}+h_T\big\|[\![\partial_n u_h]\!]\big\|_{\infty,\partial T\setminus\partial\Omega}\Big\}$$
$$+c_1\left\|\nu^q_{n/2,T}\right\|_{l_{n/2}}+c_2\left\|h_T\,\nu^{q-1}_{n,T}\right\|_{l_n}\Big]\times|\ln h_{\min}|^2, \tag{4.6}$$

where $h_{\min}$ is the smallest mesh size, h_T is the diameter of T, $[\![\partial_n u_h]\!]$ is the jump of the normal derivatives across an inter-element side, $\|\cdot\|_{l_p}$ is the l_p norm, and the quantity

$$\nu^{q'}_{n',T}:=|T|^{1/n'}\left\|g(\cdot,v_h)-I_{h,q'}[g(\cdot,v_h)]\right\|_{\infty,T}$$

is defined using the Lagrange interpolation operator $I_{h,q'}$ onto the space of piecewise polynomials of degree $\leq q'$.

4.3. Elliptic model problem: singularly-perturbed regime in one dimension. We now consider the steady-state version (4.4) of our model problem (1.3) in the singularly-perturbed regime of $\varepsilon\ll 1$.

Let v solve the problem (4.4) with $\varepsilon \in (0,1]$ and $\gamma > 0$, posed in the domain $\Omega := (0,1)$, and v_h solve the discrete problem (4.5) using the space V_h of continuous ***piecewise-linear*** finite element functions on an arbitrary nonuniform mesh $\{x_i\}_{i=1}^N$ with $0 = x_0 < x_1 < \cdots < x_N = 1$ and $h_i := x_i - x_{i-1}$. Note that here we make absolutely no mesh regularity assumptions (as solutions of our problem typically exhibit sharp layers so a suitable mesh is expected to be highly-nonuniform; see, e.g., [11]).

Consider two choices of $\langle \cdot, \cdot \rangle_h$, which are defined using the standard piecewise-linear Lagrange polynomial I_h onto V_h:

$$\langle \varphi, \psi \rangle_h := \langle I_h \varphi, \psi \rangle, \qquad \text{(quadrature)} \tag{4.7a}$$

$$\langle \varphi, \psi \rangle_h := \langle I_h[\varphi\psi], 1 \rangle. \qquad \text{(lumped-mass quadrature)} \tag{4.7b}$$

Remark 4.3. *To illustrate Remarks 4.1 and 4.2, note that the described two discretizations using either (4.7a) or (4.7b) are of type (4.2). In particular, for (4.7a), we get* $\mathcal{L}_h := -\varepsilon^2 [\partial_x^2]_h$ *and* $\mathcal{P}_h := I_h$. *Here the operator* $[\partial_x^2]_h : H_0^1(\Omega) \to \mathring{V}_h + \varepsilon^{-2} I_h[g(\cdot, 0)]$ *is defined by* $\langle -[\partial_x^2]_h \varphi, \chi \rangle = \langle \varphi', \chi' \rangle$ *for all* $\varphi \in H_0^1(\Omega)$, $\chi \in \mathring{V}_h$. *Consequently, the discrete problem using (4.7a) may be represented as*

$$-\varepsilon^2 \, [\partial_x^2]_h v_h + I_h[g(\cdot, v_h)] = 0. \tag{4.8a}$$

By contrast, (4.7b) can be rewritten as a difference scheme: $-\varepsilon^2 \delta_x^2 v_{h,i} + g(x_i, v_{h,i}) = 0$, *for* $i = 1, \ldots, N-1$, *where* $\delta_x^2 v_{h,i} := \frac{2}{h_i + h_{i+1}} \big[\frac{1}{h_{i+1}} (v_{h,i+1} - v_{h,i}) - \frac{1}{h_i} (v_{h,i} - v_{h,i-1}) \big]$ *is the standard finite-difference operator. Letting* $\delta_x^2 v_{h,i} := \varepsilon^{-2} \, g(x_i, v_{h,i})$ *for* $i = 0, N$ *and applying the linear interpolation* I_h *to* $\{\delta_x^2 v_{h,i}\}_{i=0}^N$, *we can represent the discrete problem using (4.7b) as*

$$-\varepsilon^2 \, I_h[\delta_x^2 v_h] + I_h[g(\cdot, v_h)] = 0, \tag{4.8b}$$

where the values $\delta_x^2 v_{h,i}$ *are easily explicitly computable.*

We cite a posteriori error bounds [7, 10, 11] of type (4.3) with $\eta := \eta_\varepsilon\big(V_h, g(\cdot, v_h)\big)$ for (4.7a) and $\eta := \eta_{\varepsilon;\,\mathrm{l.m.}}\big(V_h, g(\cdot, v_h)\big)$ for (4.7b), respectively, defined by

$$\eta_\varepsilon\big(V_h, g_*\big) := \max_{i=1,\ldots,N} \Big\{ \tfrac{h_i^2}{4\varepsilon^2} \, \|I_h g_*\|_{\infty,(x_{i-1},x_i)} \Big\} + \gamma^{-2} \|g_* - I_h g_*\|_{\infty,(0,1)}, \tag{4.9a}$$

$$\eta_{\varepsilon;\,\mathrm{l.m.}}\big(V_h, g_*\big) := \eta_\varepsilon + \max_{i=1,\ldots,N} \Big\{ \tfrac{h_i^2}{6\gamma\,\varepsilon} \, \|\partial_x (I_h g_*)\|_{\infty,(x_{i-1},x_i)} \Big\}, \tag{4.9b}$$

where $\qquad g_* := g(\cdot, v_h).$

Remark 4.4. *The error estimators (4.9a) and (4.9b) are* <u>*robust*</u> *although they involve negative powers of the small parameter* ε. *Indeed, an inspection of representations (4.8a) and (4.8b) for the two considered numerical methods shows that* $\varepsilon^{-2} h_i^2 |I_h g_*| = \varepsilon^{-2} h_i^2 |I_h[g(\cdot, v_h)]|$ *becomes* $h_i^2 |[\partial_x^2]_h v_h|$ *or* $h_i^2 |\delta_x^2 v_h|$, *so it approximates* $h_i^2 |\partial_x^2 v|$, *where* v *is the exact solution of our equation* $-\varepsilon^2 \partial_x^2 v + g(\cdot, v) = 0$. *Similarly, the term* $\varepsilon^{-1} h_i^2 |\partial_x (I_h g_*)|$ *approximates* $\varepsilon \, |\partial_x^3 v|$, *which has similar magnitude to* $h_i^2 |\partial_x^2 v|$ *in the layer regions.*

5. Fully discrete Discontinuous Galerkin method dG(1). We now describe a full discretization of dG(1) type for the abstract parabolic problem (1.1). To this end, we apply a spatial discretization of type (4.2) to the semidiscrete problem (3.2), (3.4) as follows. A finite-element space $V_h^{j+1} \subset C(\bar{\Omega})$ and a computed solution

$u_h^{j+1} \in \mathring{V}_h^{j+1} := V_h^{j+1} \cap H_0^1(\Omega)$ are associated with the time level t_{j+1}, while auxiliary computed solutions $\hat{u}_h^j$, $\hat{y}_h^j \in \mathring{V}_h^{j+1}$ are associated with the time level t_j^+ (this is indicated by the hat notation).

Imitating (3.2), let

$$u_h := \hat{y}_h^j\, \phi_j + u_h^{j+1}\phi_{j+1}, \quad \Psi := \mathcal{L}_h(t)\, u_h + \mathcal{P}_h^{j+1}\big[f(\cdot, t, u_h)\big] \quad \text{for } t \in (t_j, t_{j+1}]. \tag{5.1a}$$

Here, in agreement with (4.2b), for the Lagrange interpolation operator I_h^{j+1} onto V_h^{j+1}, we assume that

$$\begin{aligned} &\mathcal{L}_h(t) : H_0^1(\Omega) \to \mathring{V}_h^{j+1} - I_h^{j+1}[f(\cdot, t, 0)] \quad \text{for } t \in (t_j, t_{j+1}], \\ &\mathcal{P}_h^{j+1} v \in \mathring{V}_h^{j+1} + I_h^{j+1} v \quad \forall\, v \in C(\bar{\Omega}), \qquad \mathcal{P}_h^{j+1} v_h = v_h \quad \forall\, v_h \in V_h^{j+1}. \end{aligned} \tag{5.1b}$$

Note that u_h vanishes on $\partial\Omega$, so $I_h[f(\cdot, t_k, 0)] = I_h[f(\cdot, t_k, u_h)]$ on $\partial\Omega$, so $\Psi \in \mathring{V}_h^{j+1}$.

Now, imitating (3.4), we require, for $j = 0, \ldots, M-1$, that

$$u_h^{j+1} - \hat{u}_h^j + \tfrac{1}{4}\, \tau_{j+1} \left(3\Psi^{j+1/3} + \Psi^{j+1}\right) = 0, \tag{5.2a}$$

$$\hat{y}_h^j - \hat{u}_h^j + \tfrac{1}{4}\, \tau_{j+1} \left(\Psi^{j+1/3} - \Psi^{j+1}\right) = 0, \tag{5.2b}$$

where

$$\Psi^{j+1/3} := \Psi(\cdot, t_{j+1/3}), \qquad \Psi^{j+1} := \Psi(\cdot, t_{j+1}), \tag{5.2c}$$

are computed using $u_h(\cdot, t_{j+1/3}) = \frac{2}{3}\hat{y}_h^j + \frac{1}{3}u_h^{j+1}$ and $u_h(\cdot, t_j) = u_h^{j+1}$, respectively, as well as $\mathcal{L}_h(t_{j+1/3})$ and $\mathcal{L}_h(t_{j+1})$, by virtue of (5.1a).

The auxiliary computed solution $\hat{u}_h^j$, which appears in (5.2a), (5.2b), is computed by applying some linear interpolation operator I_*^{j+1} to u_h^j:

$$\hat{u}_h^j := I_*^{j+1}\, u_h^j, \qquad I_*^{j+1} : \mathring{V}_h^j \to \mathring{V}_h^{j+1}, \qquad I_*^{j+1}\, u_h^j = u_h^j \;\text{ if }\; u_h^j \in \mathring{V}_h^{j+1}, \tag{5.2d}$$

with the convention $\hat{u}_h^0 = u_h^0 \in \mathring{V}_h^1$. Note that if $V_h^j \subset V_h^{j+1}$ (which includes the case of $V_h^j = V_h$ being fixed for all j), then $\hat{u}_h^j = u_h^j$. Also, to define I_*^{j+1}, one may employ, e.g., the standard Lagrange interpolation or the L_2 projection.

5.1. A posteriori error estimate using piecewise-quadratic elliptic reconstructions. To estimate the error of the fully discrete dG(1) method (5.2), we partially imitate the arguments of Section 3.1 for the related semidiscrete method. First, we introduce some notation. Similarly to (3.5), define a piecewise-quadratic function in time by $\tilde{u}_h(\cdot, 0) := \hat{u}_h^0 = u_h^0$ and

$$\tilde{u}_h := \hat{u}_h^j \phi_j + u_h^{j+1}\phi_{j+1} + \nu_h, \qquad \nu_h := 3\, \phi_j\, \phi_{j+1}\, \{\hat{y}_h^j - \hat{u}_h^j\} \qquad \text{for } t \in (t_j, t_{j+1}]. \tag{5.3}$$

Note that, similarly to (3.6),

$$\tilde{u}_h(\cdot, t_{j+1/3}) = u_h(\cdot, t_{j+1/3}) = \tfrac{2}{3}\, \hat{y}_h^j + \tfrac{1}{3}\, u_h^{j+1}, \;\; \tilde{u}_h(\cdot, t_{j+1}) = u_h(\cdot, t_{j+1}) = u_h^{j+1}. \tag{5.4}$$

Next, similarly to (3.7), we introduce $\tilde{\Psi}(\cdot, 0) := \tilde{\Psi}(\cdot, 0^+)$,

$$\tilde{\Psi} := \mathcal{L}_h(t)\, \tilde{u}_h + \mathcal{P}_h^{j+1}\big[f(\cdot, t, \tilde{u}_h)\big] \qquad \text{for } t \in (t_j, t_{j+1}], \tag{5.5}$$

for which (5.4) implies that

$$\tilde{\Psi}(\cdot, t_{j+1/3}) = \Psi^{j+1/3}, \qquad \tilde{\Psi}(\cdot, t_{j+1}) = \Psi^{j+1}. \tag{5.6}$$

In our analysis in this section, we shall employ the ***elliptic reconstruction***, which was introduced in the recent papers [12, 9, 2] as a counterpart of the Ritz-projection in the a posteriori error estimation for parabolic problems.

Using $\tilde{\Psi}$ of (5.5), we define an elliptic reconstruction R as the unique solution $R(\cdot, t) \in H_0^1(\Omega) \cap C(\bar{\Omega})$ of the elliptic problem

$$\mathcal{L}(t)\, R + g_{[t]}(\cdot, R) = 0, \quad g_{[t]}(\cdot, v) := f(\cdot, t, v) - \tilde{\Psi}(t, \cdot), \quad \text{for } t \in \{t_j^+, t_{j+1/3}, t_{j+1}\}, \tag{5.7}$$

in which t appears as a parameter.

Note that (5.7) describes a version of the elliptic problem (4.1) with $\mathcal{L} := \mathcal{L}(t)$, $g := g_{[t]}$, and the exact solution $R(\cdot, t)$. Furthermore, the numerical method (4.2), using the finite element space V_h^{j+1}, applied to this problem yields $R_h(\cdot, t)$

$$\mathcal{L}_h(t)\, R_h + \mathcal{P}_h^{j+1}\, [g_{[t]}(x, R_h)] = 0. \tag{5.8}$$

We have assumed that a solution of this discrete problem is unique. Thus, $R_h = \tilde{u}_h$. This is easily checked by combining (5.8) with the definition of $g_{[t]}$ in (5.7), in which the terms $\mathcal{P}_h^{j+1}\tilde{\Psi} = \tilde{\Psi}$, and the definition of $\tilde{\Psi}$ in (5.5). Consequently, applying the elliptic a posteriori error estimate (4.3) to the exact solutions R and the corresponding computed solutions u_h, one gets, for $j = 1, \ldots, M$,

$$\begin{aligned} \eta^{j+1} &:= \max_{t \in \{t_j^+,\, t_{j+1/3},\, t_{j+1}\}} \Big\{ \|R(\cdot, t) - \tilde{u}_h(\cdot, t)\|_{\infty,\Omega} \Big\}, \\ \text{where} \quad & \|R(\cdot, t) - \tilde{u}_h(\cdot, t)\|_{\infty,\Omega} \le \eta\big(V_h^j, \tilde{u}_h, f(\cdot, t, \tilde{u}_h) - \tilde{\Psi}(t, \cdot)\big). \end{aligned} \tag{5.9}$$

Finally, to formulate our a posteriori error estimate for $u_h - u$, we generalize the ***piecewise-quadratic interpolation*** $I_{2,t}$ of (3.8) to any *left-continuous* function $w = w(t)$ by using the interpolation nodes t_j^+, $t_{j+1/3}$ and t_j, so $I_{2,t}^* w(0) := w(0)$ and

$$\begin{aligned} I_{2,t}^* w(t) &:= w(t_j^+)\, \phi_j + w(t_{j+1})\, \phi_{j+1} - \tfrac{1}{2}\, \phi_j\, \phi_{j+1}\, W^{j+1}, \\ \text{where} \quad W^{j+1} &:= 3\, [2w(t_j^+) - 3w(t_{j+1/3}) + w(t_{j+1})], \end{aligned}$$

for $t \in (t_j, t_{j+1}]$, $j = 0, \ldots, M-1$. By applying $I_{2,t}^*$ to the elliptic reconstruction R of (5.7), we extend it to all $t \in [0, T]$ as

$$R(\cdot, 0) := R(\cdot, 0^+), \qquad R(\cdot, t) := I_{2,t}^* R(\cdot, t) \quad \text{for } t \in (0, T]. \tag{5.10}$$

Note also that (5.7) implies $\tilde{\Psi} = \mathcal{L}(t)\, R + f(\cdot, t, R)$ for $t \in \{t_j^+,\, t_{j+1/3},\, t_{j+1}\}$ so

$$I_{2,t}^* \tilde{\Psi} = I_{2,t}^* [\mathcal{L}(t)\, R + f(\cdot, t, R)] \qquad \text{for } t \in [0, T]. \tag{5.11}$$

Here $I_{2,t}^* \tilde{\Psi}$, by virtue of (5.6)), allows a representation for $t \in (t_j, t_{j+1}]$:

$$I_{2,t}^* \tilde{\Psi} = \Psi^{j+1} - \tfrac{3}{2}\{\Psi^{j+1} - \Psi^{j+1/3}\}\, \phi_j - \tfrac{1}{2}\, \tau_{j+1}^{-1}(t - t_{j+1/3})\, \phi_{j+1}\, \chi_h^{j+1}, \tag{5.12}$$

where $\chi_h^{j+1} = \partial_t^2(I_{2,t}^*\tilde{\Psi})$ is given by

$$\chi_h^{j+1} := 3\,[2\tilde{\Psi}(\cdot,t_j^+) - 3\Psi^{j+1/3} + \Psi^{j+1}], \tag{5.13}$$

(compare with (3.8), (3.9)).

We are now prepared to formulate our main result.

THEOREM 5.1. *Let u solve (1.1), (1.2) with a parabolic operator $\mathcal{M}$ satisfying Condition 2.1, u_h^j solve the discrete problem (5.1), (5.2), and R be the elliptic reconstruction defined by (5.7), (5.10) and satisfying (5.9). Then, for $m = 1, \dots, M$, one has*

$$\begin{aligned}\big\|u_h^m - u(\cdot,t_m)\big\|_{\infty,\Omega} \le \kappa_0\, \mathrm{e}^{-\gamma^2 t_m}\big\|u_h^0 - \varphi\big\|_{\infty,\Omega} &\\ + (\kappa_1\,\ell_m + \kappa_2) \max_{j=0,\dots,m-2}\Big\{\tfrac{2}{81}\,\tau_{j+1}\big\|\chi_h^{j+1}\big\|_{\infty,\Omega} + \tfrac{5}{3}\eta^{j+1}\Big\}&\\ + \tfrac{1}{18}\kappa_0\,\tau_m\big\|\chi_h^m\big\|_{\infty,\Omega} + (10\kappa_0+1)\,\eta^m&\\ + \kappa_0\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\mathrm{e}^{-\gamma^2(t_m-s)}\,\big\|\vartheta_R(\cdot,s)\big\|_{\infty,\Omega}\,\mathrm{d}s&\\ + \kappa_0\sum_{j=1}^{m-1}\mathrm{e}^{-\gamma^2(t_m-t_j)}\big\|\hat{u}_h^j - u_h^j\big\|_{\infty,\Omega}\,,&\end{aligned} \tag{5.14}$$

with the notation (5.3), (5.5), (5.13), and

$$\vartheta_R := [\mathcal{L}(t)\,R + f(\cdot,t,R)] - I_{2,t}^*[\mathcal{L}(t)\,R + f(\cdot,t,R)]. \tag{5.15}$$

Proof. In view of (5.9), to get the desired bound (5.14) for $u_h^m - u(\cdot,t_m)$, it suffices to obtain a bound of type (5.14) for $[R-u](\cdot,t_m)$, with $(10\kappa_0+1)$ replaced by $10\kappa_0$, and then apply the triangle inequality. So we consider $[R-u](\cdot,t_m)$ only.

We partially imitate the proof of Theorem 3.5. Let $t \in (t_j, t_{j+1}]$. It is convenient to treat the left-continuous function $\tilde{u}_h$ of (5.3) as being discontinuous at t_j^+ rather than at t_j. Now, combining (5.2a) with $\partial_t(\hat{u}_h^j\phi_j + u_h^{j+1}\phi_{j+1}) = (u_h^{j+1} - \hat{u}_h^j)/\tau_{j+1}$, for $\tilde{u}_h$ one gets

$$\partial_t\tilde{u}_h = \partial_t\nu_h - \tfrac{1}{4}\big(3\Psi^{j+1/3} + \Psi^{j+1}\big) + \vartheta_* \qquad \text{for } t \in (t_j, t_{j+1}]. \tag{5.16}$$

Here the discontinuity of $\tilde{u}_h$ at t_j^+ yielded the term

$$\vartheta_*(\cdot,t) := [\hat{u}_h^j - u_h^j]\;\delta(t - t_j^+) \qquad \text{for } t \in (t_j, t_{j+1}], \tag{5.17}$$

with $\delta(\cdot)$ denoting the one-dimensional Dirac δ-distribution. (Note that $\hat{u}_h^0 = u_h^0$ so $\vartheta_* = 0$ on $[0, t_1]$.) Note also that

$$\int_0^{t_m}\mathrm{e}^{-\gamma^2(t_m-s)}\,\big\|\vartheta_*(\cdot,s)\big\|_{\infty,\Omega}\,\mathrm{d}s \le \sum_{j=1}^{m-1}\mathrm{e}^{-\gamma^2(t_m-t_j)}\big\|\hat{u}_h^j - u_h^j\big\|_{\infty,\Omega}\,. \tag{5.18}$$

Next, imitating the derivation of (3.12), from (5.12) for $t \in (t_j, t_{j+1}]$ one gets

$$I_{2,t}^*\tilde{\Psi} - \tfrac{1}{4}\big(3\Psi^{j+1/3} + \Psi^{j+1}\big) = \partial_t[\mu_h - \nu_h], \qquad \mu_h := \tfrac{1}{6}\,\tau_{j+1}\,\phi_j^2\phi_{j+1}\,\chi_h^{j+1}. \tag{5.19}$$

For this function μ_h, similarly to (3.13), a calculation shows that

$$|\mu_h| \le \tfrac{2}{81}\tau_{j+1}|\chi_h^{j+1}|, \quad \tau_{j+1}\,|\partial_t\mu_h| \le \tfrac{1}{18}\tau_{j+1}|\chi_h^{j+1}|, \quad \text{for } t\in(t_j,t_{j+1}]. \tag{5.20}$$

Now, combining (5.16) with (5.19), we arrive at

$$\partial_t\tilde u_h = \partial_t\mu_h - I^*_{2,t}\tilde\Psi + \vartheta_* \qquad \text{for } t\in(t_j,t_{j+1}].$$

As $\mathcal{M}R = \partial_t R + \mathcal{L}(t)\,R + f(\cdot,t,R)$ and $\mathcal{M}u = 0$, so

$$\begin{aligned}\mathcal{M}R - \mathcal{M}u &= \partial_t[R-\tilde u_h+\mu_h] + \big[\mathcal{L}(t)\,R + f(\cdot,t,R) - I^*_{2,t}\tilde\Psi\big] + \vartheta_*\\ &= \partial_t[R-\tilde u_h+\mu_h] + \vartheta_R + \vartheta_* \qquad \text{for } t\in(t_j,t_{j+1}].\end{aligned}$$

Here the second relation is obtained using (5.11) and (5.15). So the desired bound of type (5.14) for $[R-u](\cdot,t_m)$, only with $(10\kappa_0+1)$ replaced by $10\kappa_0$, is then obtained by an application of Lemma 2.3 with $\mu := (R-\tilde u_h)+\mu_h$ and $\vartheta := \vartheta_R+\vartheta_*$, for which we make a few observations. First, note that $[R-u-\mu](\cdot,0) = R^0-\varphi-(R^0-u_h^0) = u_h^0-\varphi$. For μ_h, we recall (5.20) and also note that $\mu_h(\cdot,t_{m-1}^+)=0$. For the piecewise-quadratic function $R-\tilde u_h$, by virtue of (5.9), $|[R-\tilde u_h](\cdot,t_{m-1}^+)| \le \eta^m$, and a calculation yields

$$|R-\tilde u_h| \le \tfrac{5}{3}\eta^{j+1}, \qquad \tau_{j+1}|\partial_t(R-\tilde u_h)| \le 9\eta^{j+1} \qquad \text{for } t\in(t_j,t_{j+1}]. \tag{5.21}$$

Finally, for ϑ_*, we invoke (5.17). Combining these observations in the application of Lemma 2.3 completes the proof. ∎

REMARK 5.2. *The final term in the error estimate (5.14) vanishes when one has* $V_h^{j-1}\subset V_h^j$ *for all* $j=1,\dots,M$.

REMARK 5.3 (Computability). *The computation of the right-hand side in the estimate (5.14) involves computing* χ_h^{j+1} *of (5.13) and* η^{j+1} *of (5.9) for* $j<m$. *Note that the terms* $\Psi^{j+1/3}$ *and* Ψ^{j+1}, *which appear in* χ_h^{j+1}, *can be explicitly represented using (5.2); to be more precise, one gets a discrete version of (3.14):*

$$\Psi^{j+1/3} = -\frac{u_h^{j+1}-\hat u_h^j}{\tau_{j+1}} - \frac{\hat y_h^j-\hat u_h^j}{\tau_{j+1}}, \qquad \Psi^{j+1} = -\frac{u_h^{j+1}-\hat u_h^j}{\tau_{j+1}} + 3\,\frac{\hat y_h^j-\hat u_h^j}{\tau_{j+1}}. \tag{5.22}$$

These two terms are also needed to compute η^{j+1} *since (5.9) can be rewritten as*

$$\begin{aligned}\eta^{j+1} \le \max\Big\{ &\eta\big(V_h^{j+1}, u_h^{j+1}, f_h^{j+1}-\Psi^{j+1}\big),\\ &\eta\big(V_h^{j+1}, \tfrac{2}{3}\hat y_h^j+\tfrac{1}{3}u_h^{j+1}, f_h^{j+1/3}-\Psi^{j+1/3}\big),\\ &\eta\big(V_h^{j+1}, \hat u_h^j, f(\cdot,t_j,\hat u_h^j)-\tilde\Psi(\cdot,t_j^+)\big)\Big\},\end{aligned} \tag{5.23}$$

with the notation

$$f_h^{j+1/3} := f(\cdot,t_{j+1/3},\tfrac{2}{3}\,\hat y_h^j+\tfrac{1}{3}\,u_h^{j+1}), \qquad f_h^{j+1} := f(\cdot,t_j,u_h^{j+1}). \tag{5.24}$$

However, the computation of both χ_h^{j+1} *and* η^{j+1} *also requires* $\tilde\Psi(\cdot,t_j^+)$. *Note that, by virtue of (5.3), (5.5),*

$$\text{if } V_h^j = V_h^{j+1} \quad\Rightarrow\quad \hat u_h^j = u_h^j, \quad f(\cdot,t_j,\hat u_h^j) = f_h^j, \quad \tilde\Psi(\cdot,t_j^+) = \Psi^j, \tag{5.25}$$

so in this case $\tilde\Psi(\cdot,t_j^+)$ *can be explicitly represented using (5.22). We further discuss the computability of* $\tilde\Psi(\cdot,t_j^+)$ *in the general case of* $V_h^j\neq V_h^{j+1}$ *in Section 5.4.*

REMARK 5.4. *The term ϑ_R in (5.14), defined by (5.15), involves the elliptic reconstruction R. In view of $R - \tilde u_h$ being piecewise-quadratic, the discrepancy of ϑ_R from $\vartheta_{\tilde u_h} := \mathcal{L}(t)\,\tilde u_h + f(\cdot,t,\tilde u_h) - I^*_{2,t}[\mathcal{L}(t)\,\tilde u_h + f(\cdot,t,\tilde u_h)]$, can be estimated using the bound (5.9). For example, let $\mathcal{L}$ be t-independent. Then $\mathcal{L}\,R - I^*_{2,t}[\mathcal{L}\,R] = 0$, so $\vartheta_R = f(\cdot,t,R) - I^*_{2,t}[f(\cdot,t,R)]$. Consequently, by virtue of (5.21), one easily gets a very crude bound for $t \in (t_j, t_{j+1}]$:*

$$\begin{aligned}\|[\vartheta_R - \vartheta_{\tilde u_h}](\cdot,t)\|_{\infty,\Omega} &= \|[f(\cdot,t,\tilde u_h) - f(\cdot,t,R)] - I^*_{2,t}[f(\cdot,t,\tilde u_h) - f(\cdot,t,R)]\|_{\infty,\Omega}\\ &\le \tfrac{10}{3}\eta^{j+1} \sup_{(t_j,t_{j+1}]\times\mathbb{R}} \big\|\partial_z f(\cdot,t,z)\big\|_{\infty,\Omega}.\end{aligned}$$

Furthermore, in some special cases (e.g., if f is linear in the third argument) one can, in fact, get a sharper bound of type $\|[\vartheta_R - \vartheta_{\tilde u_h}](\cdot,t)\|_{\infty,\Omega} \le C_f\,\tau_{j+1}\,\eta^{j+1}$ for $t \in (t_j, t_{j+1}]$, with some constant C_f. Then the discrepancy $\|[\vartheta_R - \vartheta_{\tilde u_h}](\cdot,t)\|_{\infty,\Omega}$ between $\vartheta_{f,R}$ and ϑ_{f,u_h} becomes negligible compared with the terms η^{j+1} that explicitly appear in (5.14).

5.2. Application to a general t-independent operator $\mathcal{L}$. Let the coefficients of the elliptic operator $\mathcal{L}$ be independent of the variable t; as in Section 3.2, we highlight this case by using the special notation $\mathring{\mathcal{L}} := \mathcal{L}$ for this operator. Note that its discrete counterpart $\mathcal{L}_h^{j+1}(t) : H^1_0(\Omega) \to \mathring{V}_h^{j+1} - I_h^{j+1}[f(\cdot,t,0)]$ of (5.1b) remains dependent on t (and it is not linear), so it is convenient to also use its linear t-independent version $\mathring{\mathcal{L}}_h^{j+1}$ and the related version $\mathring{\mathcal{P}}^{j+1}$ of $\mathcal{P}^{j+1}$ defined by

$$\begin{aligned}&\mathring{\mathcal{L}}_h^{j+1} : H^1_0(\Omega) \to \mathring{V}_h^{j+1}, \qquad \mathring{\mathcal{P}}_h^{j+1}\mathcal{L}_h(t) = \mathring{\mathcal{L}}_h^{j+1} \;\text{ for }\; t \in (t_j, t_{j+1}],\\ &\mathring{\mathcal{P}}_h^{j+1} : C(\bar\Omega) \to \mathring{V}_h^{j+1}, \qquad \mathring{\mathcal{P}}_h^{j+1} v = \mathcal{P}_h^{j+1} v \quad \forall\, v \in H^1_0(\Omega) \cup C(\bar\Omega).\end{aligned} \tag{5.26}$$

Note that the function Ψ of (5.1a) is in $\mathring{V}_h^{j+1}$, so it can now be represented for $t \in (t_j, t_{j+1}]$ as $\Psi = \mathring{\mathcal{L}}_h^{j+1} u_h + \mathring{\mathcal{P}}_h^{j+1}\big[f(\cdot,t,u_h)\big]$. Consequently, the semidiscrete method (5.1), (5.2) can be rewritten (similarly to (3.15)) as

$$u_h^{j+1} - \hat u_h^j + \tfrac12\,\tau_{j+1}\,\mathring{\mathcal{L}}_h^{j+1}\big(\hat y_h^j + u_h^{j+1}\big) + \tfrac14\,\tau_{j+1}\,\mathring{\mathcal{P}}_h^{j+1}\big(3f_h^{j+1/3} + f_h^{j+1}\big) = 0, \tag{5.27a}$$

$$\hat y_h^j - \hat u_h^j + \tfrac16\,\tau_{j+1}\,\mathring{\mathcal{L}}_h^{j+1}\big(\hat y_h^j - u_h^{j+1}\big) + \tfrac14\,\tau_{j+1}\,\mathring{\mathcal{P}}_h^{j+1}\big(f_h^{j+1/3} - f_h^{j+1}\big) = 0, \tag{5.27b}$$

with the notation (5.24). Furthermore, we have a version of (3.17) for χ_h^{j+1} of (5.13):

$$\chi_h^{j+1} = 6\,\mathring{\mathcal{L}}_h^{j+1}[\hat u_h^j - \hat y_h^j] + 3\,\mathring{\mathcal{P}}_h^{j+1}[2f(\cdot,t_j,\hat u_h^j) - 3f_h^{j+1/3} + f_h^{j+1}]. \tag{5.28}$$

5.3. Application to the model problem (1.3). Consider a fully discrete Discontinuous Galerkin method dG(1) for (1.3), obtained by applying the spatial discretization (4.5) to the semidiscrete problem (3.4), (3.2): Find $\hat y_h^j$, $u_h^{j+1} \in \mathring{V}_h^{j+1}$ such that

$$\begin{aligned}&\Big\langle \frac{u_h^{j+1} - \hat u_h^j}{\tau_{j+1}}, \chi\Big\rangle_h + \varepsilon^2\big\langle \tfrac12\nabla(\hat y_h^j + u_h^{j+1}), \nabla\chi\big\rangle + \tfrac14\big\langle 3f_h^{j+1/3} + f_h^{j+1}, \chi\big\rangle_h = 0,\\ &\Big\langle \frac{\hat y_h^j - \hat u_h^j}{\tau_{j+1}}, \chi\Big\rangle_h + \varepsilon^2\big\langle \tfrac16\nabla(\hat y_h^j - u_h^{j+1}), \nabla\chi\big\rangle + \tfrac14\big\langle f_h^{j+1/3} - f_h^{j+1}, \chi\big\rangle_h = 0,\end{aligned} \tag{5.29}$$

$\forall\, \chi \in \mathring{V}_h^{j+1}$, subject to (5.2d) and (5.24) (compare with (5.27)). Here $\langle\cdot,\cdot\rangle_h$ is either exactly the inner product $\langle\cdot,\cdot\rangle$ in $L_2(\Omega)$.

Note that the full discretization (5.29) is of type (5.1), (5.2). For some particular cases of $\langle\cdot,\cdot\rangle_h$, the operators $\mathcal{L}_h(t)$ and $\mathcal{P}_h^{j+1}$ are defined as in Remarks 4.1 and 4.2 only using V_h^{j+1} instead of V_h.

Note also that the elliptic operator $\mathcal{L} = -\varepsilon^2\triangle$ in (1.3) is t-independent, so all the results of Section 5.2 apply to this problem.

5.3.1. Model problem (1.3): regular regime. Let u solve problem (1.3) with $\varepsilon = 1$, $\gamma \geq 0$, posed in a bounded polyhedral spatial domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, and u_h^j solve the discrete problem (5.29) with V_h^j and $\langle\cdot,\cdot\rangle_h$ defined, for each time level t_j, as in §4.2. To be more specific, we let $\mathcal{T}_h^j$ be a conforming and shape-regular triangulation of $\bar{\Omega}$ made of elements T, V_h^j be the space of continuous piecewise polynomial finite element functions of degree $l \geq 1$, and $\mathring{V}_h^j := V_h^j \cap H_0^1(\Omega)$. We then employ a quadrature formula $\langle\varphi,\psi\rangle_h := \sum_{T\in\mathcal{T}_h^j} Q_T(\varphi\psi)$, as described in §4.2.

Corollary 5.5. *Let the above numerical method be applied to problem (1.3) with $\varepsilon = 1$, $\gamma \geq 0$. Then the a posteriori error estimate of Theorem 5.1 is valid with the bound (5.23) for the quantities η^{j+1}, in which $\eta := \eta_0$ is defined in (4.6).*

Remark 5.6. *A discontinuous Galerkin method dG(1) for a linear version of (1.3) with $\varepsilon = 1$ was considered in [3]. In this particular case, $f = f(x,t)$ implies that (5.28) can be rewritten as $\chi_h^{j+1} = 6\,\mathring{\mathcal{L}}_h^{j+1}[\hat{u}_h^j - \hat{y}_h^j] + \tau_{j+1}^2\partial_t^2 f(x,t')$ for some intermediate $t' \in [t_j, t_{j+1}]$. With this simplification, our a posteriori error estimate (5.14) resembles (but is not identical with) the one of [3, (1.14)] in that it involves terms of type $\tau_{j+1}\|\mathring{\mathcal{L}}_h^{j+1}[\hat{u}_h^j - \hat{y}_h^j]\|_{\infty,\Omega}$ and $\tau_{j+1}^3\|\partial_t^2 f\|_{\infty,\Omega}$. (Note also that [3, (1.14)] is given without proof, and does not appear to be proved elsewhere).*

5.3.2. Model problem (1.3): singularly perturbed regime in one dimension. Now consider the regime of $\varepsilon \ll 1$. Let u solve the problem (1.3) with $\varepsilon \in (0,1]$, $\gamma > 0$, posed in the domain $\Omega := (0,1)$, and u_h solve the discrete problem (5.29) with V_h^j and $\langle\cdot,\cdot\rangle_h$ defined, for each time level t_j, as in §4.3. Thus V_h^j is the space of continuous ***piecewise-linear*** finite element functions on an arbitrary nonuniform mesh $\{x_i^j\}_{i=1}^{N_j}$ with $0 = x_0^j < x_1^j < \cdots < x_{N_j}^j = 1$ under absolutely no mesh regularity assumptions. We consider the two choices (4.7a) and (4.7b) of $\langle\cdot,\cdot\rangle_h$, using the piecewise-linear Lagrange polynomial $I_h := I_h^j$ onto V_h^j.

Corollary 5.7. *Let the above numerical method be applied to problem (1.3) with $\varepsilon \in (0,1]$, $\gamma > 0$, $\Omega := (0,1)$. Then the a posteriori error estimate of Theorem 5.1 is valid with the bound (5.23) for the quantities η^{j+1}, in which $\eta := \eta_\varepsilon$ for (4.7a), and $\eta := \eta_{\varepsilon;\mathrm{l.m.}}$ for (4.7b). Here η_ε and $\eta_{\varepsilon;\mathrm{l.m.}}$ are defined in (4.9a) and (4.9b), respectively, with I_h now replaced by I_h^{j+1}.*

Remark 5.8. *The a posteriori error estimators of Corollary 5.7 are <u>robust</u> as the argument of Remark 4.4 applies to η^{j+1}. Indeed, recall the earlier version (5.9) of (5.23), in which, e.g., set $\eta = \eta_\varepsilon$ from (4.9a). Now $g_* = f(\cdot,t,\tilde{u}_h) - \tilde{\Psi}$, while $\tilde{\Psi} = \mathcal{L}_h(t)\,\tilde{u}_h + I_h^{j+1}f(\cdot,t,\tilde{u}_h)$, with $I_h^{j+1}\tilde{\Psi} = \tilde{\Psi}$. Consequently, in (4.9a) we use $I_h^{j+1}g_* = -\mathcal{L}_h(t)\,\tilde{u}_h$ and $g_* - I_h^{j+1}g_* = f(\cdot,t,\tilde{u}_h) - I_h^{j+1}f(\cdot,t,\tilde{u}_h)$. As $\mathcal{L} = -\varepsilon^2\partial_x^2$, so $I_h^{j+1}g_*$ approximates $\varepsilon^2\partial_x^2 u$. This implies that the term $\varepsilon^{-2}h_i^2|I_h^{j+1}g_*|$, although it involves the negative power of ε, approximates $h_i^2|\partial_x^2 u|$.*

5.4. Estimator computability. Consider the estimator given by (5.14). In view of Remarks 5.3 and 5.4, all the terms involved in this estimator can be computed/bounded, in particular, using explicit formulas (5.13), (5.22) and (5.23), provided that one can compute

$$\tilde{\Psi}(\cdot, t_j^+) = \mathcal{L}_h(t_j^+)\, \hat{u}_h^j + \mathcal{P}_h^{j+1}\big[f(\cdot, t_j, \hat{u}_h^j)\big]. \tag{5.30}$$

(Note that this relation follows from (5.3), (5.5)).

We now briefly discuss possible approaches to the computation of $\tilde{\Psi}(\cdot, t_j^+)$, when applied to the model problem (1.3). (It may help the reader to recall Remarks 4.1 and 4.2; see also Remark 4.3.)

(i) Suppose $V_h^j = V_h^{j+1}$. Then, by (5.25), one enjoys $\tilde{\Psi}(\cdot, t_j^+) = \Psi^j$, so in this case $\tilde{\Psi}(\cdot, t_j^+)$ can be explicitly represented using (5.22).

(ii) Suppose that in (5.29), one employs a *lumped-mass* quadrature $\langle \psi, \chi_i \rangle_h$. Then, as described in Remark 4.2, $\mathcal{P}_h^{j+1} := I_h^{j+1}$ is the Lagrange interpolation operator onto V_h^{j+1}, while $\mathcal{L}_h(t_j^+)\, \hat{u}_h^j$ is easily computable for any $\hat{u}_h^j \in \mathring{V}_h^{j+1}$ by applying the normalized stiffness matrix to the column vector of nodal values $\{\hat{u}_{h,i}^j\}$. Consequently, the computation of $\tilde{\Psi}(\cdot, t_j^+)$ using (5.30) involves only explicit computations.

(iii) Recall that $\hat{u}_h^j$ is obtained from u_h^j by means of some linear interpolation (5.2d). One possible choice is $\hat{u}_h^j \in \mathring{V}_h^{j+1}$ such that

$$\varepsilon^2 \big\langle \nabla \hat{u}_h^j, \nabla \chi \big\rangle + \big\langle f(\cdot, t_j, \hat{u}_h^j), \chi \big\rangle_h = \big\langle \Psi^j, \chi \big\rangle_h \qquad \forall \chi \in \mathring{V}_h^{j+1}.$$

Note that then the computation of $\hat{u}_h^j$ is more expensive than when it is obtained by an explicit computation (also, Remark 5.2 is no longer valid). But this choice implies that $\tilde{\Psi}(\cdot, t_j^+) = \mathcal{P}_h^{j+1} \Psi^j$, where Ψ^j can be explicitly represented using (5.22). If, furthermore, a quadrature of type $\langle \psi, \chi \rangle_h := \langle I_h^{j+1} \psi, \chi \rangle$ is used for all $\psi \in C(\bar{\Omega})$ and $\chi \in \mathring{V}_h^{j+1}$, then $\mathcal{P}_h^{j+1}$ is identical with the Lagrange interpolation I_h^{j+1} (see Remark 4.1(ii)), so the computation of $\tilde{\Psi}(\cdot, t_j^+)$ becomes explicit.

(iv) In the general case, the computation of $\tilde{\Psi}(\cdot, t_j^+)$ by means of (5.30) involves an application of $\mathcal{L}_h(t_j^+)$ and $\mathcal{P}_h^{j+1}$. Note that Remark 4.1 implies that, roughly speaking, $\mathcal{L}_h(t_j^+) v_h$ for any $v_h \in \mathring{V}_h^{j+1}$ can be obtained by an application of $M_{j+1}^{-1} K_{j+1}$ to the column vector of nodal values $\{v_{h,i}\}$, where M_{j+1} is the mass matrix and K_{j+1} is the stiffness matrix associated with the time level t_{j+1}. Such computations may be expensive.

Note also that, in some cases, an inversion of the mass matrix may be entirely avoided as follows. Suppose $\tilde{\Psi}(\cdot, t_j^+) - \lambda$ is involved in the estimator with some function λ, and an inversion of $M := M_{j+1}$ is required to compute $\tilde{\Psi}(\cdot, t_j^+)$. Then one can instead use the bound $\big\|\tilde{\Psi}(\cdot, t_j^+) - \lambda\big\|_{\infty,\Omega} \le \|M^{-1}\|_\infty \cdot \big\|M(\tilde{\Psi}(\cdot, t_j^+) - \lambda)\big\|_{\infty,\Omega}$, where $\|M^{-1}\|_\infty$ denotes the associated matrix norm (which may be bounded a priori). As $M\tilde{\Psi}(\cdot, t_j^+)$ is explicitly computable (using an application of the normalized stiffness matrix to the column vector of nodal values associated with $\hat{u}_h^j+$), all the computations become explicit.

REFERENCES

[1] G. Akrivis, C. Makridakis, and R. H. Nochetto, *A posteriori error estimates for the Crank-Nicolson method for parabolic equations*, Math. Comput., 75 (2006), pp. 511–531.

[2] A. Demlow, O. Lakkis, and C. Makridakis, *A posteriori error estimates in the maximum norm for parabolic problems*, SIAM J. Numer. Anal., 47 (2009), pp. 2157–2176.

[3] K. Eriksson and C. Johnson, *Adaptive finite element methods for parabolic problems II: Optimal error estimates in $L_\infty L_2$ and $L_\infty L_\infty$*, SIAM J. Numer. Anal., 32 (1995), pp. 706–740.

[4] K. Eriksson, C. Johnson, and V. Thomée, *Time discretization of parabolic problems by the discontinuous Galerkin method*, RAIRO Modél. Math. Anal. Numér., 19 (1985), pp. 611-643.

[5] A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall, Englewood Cliffs, N.J., 1964.

[6] D. H. Griffel, *Applied functional analysis*, Dover Publications, Mineola, NY, 2002.

[7] N. Kopteva, *Maximum norm a posteriori error estimates for a 1D singularly perturbed semilinear reaction-diffusion problem.*, IMA J. Numer. Anal., 27 (2007), pp. 576–592.

[8] N. Kopteva and T. Linss, *A posteriori error estimation for parabolic problems using elliptic reconstructions. I: Backward-Euler and Crank-Nicolson methods*, (2011), submitted for publication, www.staff.ul.ie/natalia/pdf/Kopt_Linss2011_1.pdf.

[9] O. Lakkis and C. Makridakis, *Elliptic reconstruction and a posteriori error estimates for fully discrete linear parabolic problems*, Math. Comp., 75 (2006), pp. 1627–1658.

[10] T. Linss, *Maximum-norm error analysis of a non-monotone fem for a singularly perturbed reaction-diffusion problem*, BIT Numer. Math., 47 (2007), pp. 379–391.

[11] T. Linss, *Layer-adapted meshes for reaction-convection-diffusion problems*, vol. 1985 of Lecture Notes in Mathematics, Springer, Berlin, 2010.

[12] C. Makridakis and R. H. Nochetto, *Elliptic reconstruction and a posteriori error estimates for parabolic problems*, SIAM J. Numer. Anal., 41 (2003), pp. 1585–1594.

[13] C. Makridakis and R. H. Nochetto, *A posteriori error analysis for higher order dissipative methods for evolution problems*, Numer. Math., 104 (2006), pp. 489–514.

[14] R. H. Nochetto, A. Schmidt, K. G. Siebert, and A. Veeser, *Pointwise a posteriori error estimates for monotone semi-linear equations*, Numer. Math., 104 (2006), pp. 515–538.

[15] H.-G. Roos, M. Stynes and L. Tobiska, *Robust Numerical Methods for Singularly Perturbed Differential Equations*, Springer, Berlin, 2008.

[16] V. Thomée, *Galerkin finite element methods for parabolic problems*, Springer, Berlin, 2006.