

On the sharpness of Green's function estimates for a convection-diffusion problem

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Abstract Linear singularly perturbed convection-diffusion problems with characteristic layers are considered in three dimensions. We demonstrate the sharpness of our recently obtained upper bounds for the associated Green's function and its derivatives in the L_1 norm. For this, in this paper we establish the corresponding lower bounds. Both upper and lower bounds explicitly show any dependence on the singular perturbation parameter.

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1 Introduction

Consider the convection-diffusion problem in the domain $\Omega = (0, 1)^3$:

$$\mathcal{L}_{\mathbf{x}}u(\mathbf{x}) = -\varepsilon\Delta_{\mathbf{x}}u(\mathbf{x}) - 2\alpha\partial_{x_1}u(\mathbf{x}) = f(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega, \quad (1a)$$

$$u(\mathbf{x}) = 0 \quad \text{for } \mathbf{x} \in \partial\Omega. \quad (1b)$$

Here $\varepsilon \in (0, 1]$ is a small positive parameter, while α is a positive constant. Then (1) is a singularly perturbed convection-dominated problem, whose solutions typically exhibit sharp characteristic boundary and interior layers.

This article addresses the sharpness of our recently published obtained upper bounds for the associated Green's function and its derivatives in the L_1 norm. Our interest in considering the Green's function of problem is motivated by the numerical analysis of this computationally challenging problem.

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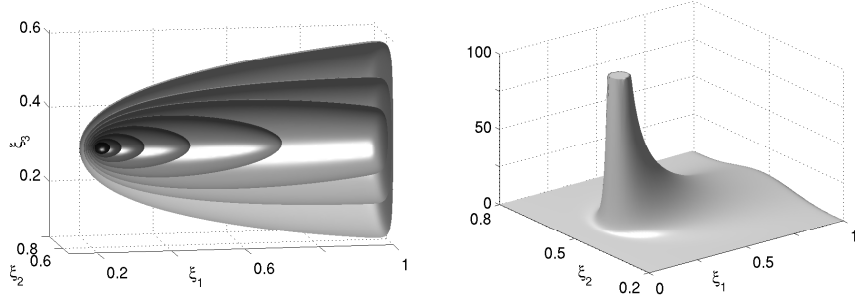


Fig. 1 Anisotropy of the Green's function $G(\mathbf{x}; \boldsymbol{\xi})$ associated with (1) for $\varepsilon = 0.01$ and $\mathbf{x} = (\frac{1}{5}, \frac{1}{2}, \frac{1}{3})$. Left: isosurfaces at values of 1, 4, 8, 16, 32, 64, 128, and 256. Right: a two-dimensional graph for fixed $\xi_3 = x_3$.

More specifically, these estimates will be used in the forthcoming paper [4] to derive robust a posteriori error bounds for computed solutions of this problem using finite-difference methods. (This approach is related to recent articles [10, 2], which address the numerical solution of singularly perturbed equations of reaction-diffusion type.) In a more general numerical-analysis context, we note that sharp estimates for continuous Green's functions (or their generalised versions) frequently play a crucial role in a priori and a posteriori error analyses [3, 8, 11].

For each fixed $\mathbf{x} \in \Omega$, the Green's function G associated with (1) satisfies

$$\mathcal{L}_{\boldsymbol{\xi}}^* G(\mathbf{x}; \boldsymbol{\xi}) := -\varepsilon \Delta_{\boldsymbol{\xi}} G(\mathbf{x}; \boldsymbol{\xi}) + 2\alpha \partial_{\xi_1} G(\mathbf{x}; \boldsymbol{\xi}) = \delta(\mathbf{x} - \boldsymbol{\xi}) \quad \text{for } \boldsymbol{\xi} \in \Omega, \quad (2a)$$

$$G(\mathbf{x}; \boldsymbol{\xi}) = 0 \quad \text{for } \boldsymbol{\xi} \in \partial\Omega. \quad (2b)$$

Here $\mathcal{L}_{\boldsymbol{\xi}}^*$ is the adjoint differential operator to $\mathcal{L}_{\mathbf{x}}$, and $\delta(\cdot)$ is the three-dimensional Dirac δ -distribution.

Note that the Green's function for a singularly perturbed self-adjoint reaction-diffusion operator $-\varepsilon \Delta_{\mathbf{x}} + \alpha$ is almost radially symmetric and exponentially decaying away from the singular point [2]. By contrast, the Green's function for our convection-diffusion problem (1) exhibits a strong anisotropic structure, which is demonstrated by Figure 1.

In [5, 6] we have obtained certain upper bounds for the Green's function associated with a variable-coefficient version of (1), which we now cite.

Theorem 1 ([5, 6]). *Let $\varepsilon \in (0, 1]$. The Green's function G associated with (1) on the unit cube $\Omega = (0, 1)^3$ satisfies, for all $\mathbf{x} \in \Omega$, the following upper bounds:*

$$\|\partial_{\xi_1} G(\mathbf{x}; \cdot)\|_{1;\Omega} \leq C(1 + |\ln \varepsilon|), \quad (3a)$$

$$\|\partial_{\xi_k} G(\mathbf{x}; \cdot)\|_{1;\Omega} \leq C\varepsilon^{-1/2}, \quad k = 2, 3. \quad (3b)$$

Furthermore, for any ball $B(\mathbf{x}', \rho)$ of radius ρ centred at any $\mathbf{x}' \in \Omega$, we have

$$\|G(\mathbf{x}; \cdot)\|_{1,1;\Omega \cap B(\mathbf{x}', \rho)} \leq C\rho/\varepsilon, \quad (3c)$$

while for any ball $B(\mathbf{x}, \rho)$ of radius ρ centred at $\mathbf{x} \in \Omega$, we have

$$\|\partial_{\xi_1}^2 G(\mathbf{x}; \cdot)\|_{1;\Omega \setminus B(\mathbf{x}, \rho)} \leq C\varepsilon^{-1} \ln(2 + \varepsilon/\rho), \quad (3d)$$

$$\|\partial_{\xi_k}^2 G(\mathbf{x}; \cdot)\|_{1;\Omega \setminus B(\mathbf{x}, \rho)} \leq C\varepsilon^{-1} (|\ln \varepsilon| + \ln(2 + \varepsilon/\rho)), \quad k = 2, 3. \quad (3e)$$

Remark 2. Theorem 1 is given in [5, 6] for a more general variable-coefficient operator $-\varepsilon \Delta_{\mathbf{x}} - a(\mathbf{x}) \partial_{x_1} + b(\mathbf{x})$ with sufficiently smooth coefficients that satisfy $a(\mathbf{x}) \geq 2\alpha > 0$ and $b(\mathbf{x}) + \partial_{x_1} a(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \bar{\Omega}$.

The purpose of this paper is to show the sharpness of the bounds of Theorem 1 up to a constant ε -independent multiplier in the following sense.

Theorem 3. *Let $\varepsilon \in (0, c_0]$ for some sufficiently small positive c_0 . The Green's function G associated with the constant-coefficient problem (1) in the unit cube $\Omega = (0, 1)^3$ satisfies, for all $\mathbf{x} \in [\frac{1}{4}, \frac{3}{4}]^3$, the following lower bounds:*

$$\|\partial_{\xi_1} G(\mathbf{x}; \cdot)\|_{1;\Omega} \geq c |\ln \varepsilon|, \quad (4a)$$

$$\|\partial_{\xi_k} G(\mathbf{x}; \cdot)\|_{1;\Omega} \geq c\varepsilon^{-1/2}, \quad k = 2, 3. \quad (4b)$$

Furthermore, for any ball $B(\mathbf{x}; \rho)$ of radius $\rho \leq \frac{1}{8}$, we have

$$\|G(\mathbf{x}; \cdot)\|_{1,1;\Omega \cap B(\mathbf{x}, \rho)} \geq \begin{cases} c\rho/\varepsilon, & \text{if } \rho \leq 2\varepsilon, \\ c(\rho/\varepsilon)^{1/2}, & \text{otherwise,} \end{cases} \quad (4c)$$

$$\|\partial_{\xi_1}^2 G(\mathbf{x}; \cdot)\|_{1;\Omega \setminus B(\mathbf{x}, \rho)} \geq c\varepsilon^{-1} \ln(2 + \varepsilon/\rho), \quad \text{if } \rho \leq c_1\varepsilon, \quad (4d)$$

$$\|\partial_{\xi_k}^2 G(\mathbf{x}; \cdot)\|_{1;\Omega \setminus B(\mathbf{x}, \rho)} \geq c\varepsilon^{-1} (\ln(2 + \varepsilon/\rho) + |\ln \varepsilon|), \quad \text{if } \rho \leq \frac{1}{8}, \quad (4e)$$

where $k = 2, 3$, and c_1 is a sufficiently small positive constant.

Remark 4. The restriction $\mathbf{x} \in [\frac{1}{4}, \frac{3}{4}]^3$ is somewhat arbitrary and can be replaced by $\mathbf{x} \in [\theta, 1 - \theta]^3$ with any ε -independent constant $\theta \in (0, \frac{1}{2})$ (then $\rho \leq \frac{1}{2}\theta$ is imposed instead of $\rho \leq \frac{1}{8}$).

The paper is structured as follows. Sharp lower bounds for the fundamental solution in \mathbb{R}^3 are derived in Section 2. The next Section 3 is devoted to the proof of Theorem 3 in a simpler setting, for the domain $(0, 1) \times \mathbb{R}^2$. Sections 4 and 5 briefly describe an extension of the analysis of Section 2 to the domain $\Omega = (0, 1)^3$ and to convection-reaction-diffusion problems. Finally, in Section 6, we give an outlook for problems in n dimensions.

Notation. Throughout the paper, C denotes a generic positive constant, typically sufficiently large, while c denotes a sufficiently small generic positive constant; they take different values in different formulas, but are independent

of the singular perturbation parameter ε . The usual Sobolev spaces $W^{m,p}(D)$ and $L_p(D)$ on any measurable set $D \subset \mathbb{R}^3$ are used; the $L_p(D)$ norm is denoted by $\|\cdot\|_{p;D}$, while the $W^{m,p}(D)$ norm is denoted by $\|\cdot\|_{m,p;D}$. By $\mathbf{x} = (x_1, x_2, x_3)$ we denote an element of \mathbb{R}^3 . For an open ball in \mathbb{R}^3 , we employ the notation $B(\mathbf{x}', \rho) = \{\mathbf{x} \in \mathbb{R}^3 : \sum_{k=1}^3 (x_k - x'_k)^2 < \rho^2\}$. The notation $\partial_{x_1} f$, $\partial_{x_1}^2 f$ and $\Delta_{\mathbf{x}}$ is used for the first- and second-order partial derivatives of a function f in variable x_1 , and the Laplacian in variable \mathbf{x} , respectively.

2 The fundamental solution

In this section we investigate the fundamental solution g that solves a similar problem to (2) but posed in the domain \mathbb{R}^3 :

$$\mathcal{L}_{\xi}^* g(\mathbf{x}; \xi) = -\varepsilon \Delta_{\xi} g(\mathbf{x}; \xi) + 2\alpha \partial_{\xi_1} g(\mathbf{x}; \xi) = \delta(\mathbf{x} - \xi) \quad \text{for } \xi \in \mathbb{R}^3. \quad (5)$$

Using [2, 9] (see also [5, 6]), a calculation shows that the fundamental solution g is explicitly represented by

$$g = g(\mathbf{x}; \xi) = \frac{1}{4\pi\varepsilon^2} \frac{e^{\alpha(\widehat{\xi}_{1,[x_1]} - \widehat{r}_{[x_1]})}}{\widehat{r}_{[x_1]}} \quad (6)$$

with the scaled variables $\widehat{\xi}_{1,[x_1]} = \frac{\xi_1 - x_1}{\varepsilon}$, $\widehat{\xi}_2 = \frac{\xi_2 - x_2}{\varepsilon}$, $\widehat{\xi}_3 = \frac{\xi_3 - x_3}{\varepsilon}$ and the scaled distance between \mathbf{x} and ξ denoted by $\widehat{r}_{[x_1]} = \sqrt{\widehat{\xi}_{1,[x_1]}^2 + \widehat{\xi}_2^2 + \widehat{\xi}_3^2}$. We use the subindex $[x_1]$ in $\widehat{\xi}_{1,[x_1]}$ and $\widehat{r}_{[x_1]}$ to highlight their dependence on x_1 as in many places x_1 will take different values; but when there is no ambiguity, we shall simply write $\widehat{\xi}_1$ and \widehat{r} .

Next, we evaluate derivatives of g of order one and two:

$$\partial_{\xi_1} g = \frac{e^{\alpha(\widehat{\xi}_1 - \widehat{r})}}{4\pi\varepsilon^3} \widehat{r}^{-2} \left[\alpha(\widehat{r} - \widehat{\xi}_1) - \frac{\widehat{\xi}_1}{\widehat{r}} \right], \quad (7a)$$

$$\partial_{\xi_k} g = -\frac{e^{\alpha(\widehat{\xi}_1 - \widehat{r})}}{4\pi\varepsilon^3} (\alpha\widehat{r} + 1) \frac{\widehat{\xi}_k}{\widehat{r}^3}, \quad k = 2, 3, \quad (7b)$$

$$\partial_{\xi_1}^2 g = \frac{e^{\alpha(\widehat{\xi}_1 - \widehat{r})}}{4\pi\varepsilon^4} \widehat{r}^{-3} \left[\alpha^2(\widehat{r} - \widehat{\xi}_1)^2 - \alpha(\widehat{r} - \widehat{\xi}_1) \left(1 + 3\frac{\widehat{\xi}_1}{\widehat{r}} \right) + \frac{3\widehat{\xi}_1^2 - \widehat{r}^2}{\widehat{r}^2} \right], \quad (7c)$$

$$\partial_{\xi_k}^2 g = \frac{e^{\alpha(\widehat{\xi}_1 - \widehat{r})}}{4\pi\varepsilon^4} \widehat{r}^{-3} \left[\alpha^2 \widehat{\xi}_k^2 + (\alpha\widehat{r} + 1) \frac{3\widehat{\xi}_k^2 - \widehat{r}^2}{\widehat{r}^2} \right], \quad k = 2, 3. \quad (7d)$$

Lemma 5. *Let $\Omega_* := (0, 1) \times \mathbb{R}^2$, $\varepsilon \in (0, c_0]$ for some sufficiently small constant c_0 , and $0 < \alpha \leq C$. Then the function g of (6) satisfies, for any $\mathbf{x} \in [2\varepsilon, \frac{3}{4}] \times \mathbb{R}^2$, the following bounds*

$$\|\partial_{\xi_1} g(\mathbf{x}; \cdot)\|_{1; \Omega_*} \geq c |\ln \varepsilon|, \quad (8a)$$

$$\|\partial_{\xi_k} g(\mathbf{x}; \cdot)\|_{1; \Omega_*} \geq c \varepsilon^{-1/2}, \quad k = 2, 3. \quad (8b)$$

Furthermore, for any ball $B(\mathbf{x}; \rho)$ of radius $\rho \leq \frac{1}{8}$, we have

$$\|g(\mathbf{x}; \cdot)\|_{1,1; \Omega_* \cap B(\mathbf{x}; \rho)} \geq \begin{cases} c \rho / \varepsilon, & \text{if } \rho \leq 2\varepsilon, \\ c(\rho/\varepsilon)^{1/2}, & \text{otherwise,} \end{cases} \quad (8c)$$

$$\|\partial_{\xi_1}^2 g(\mathbf{x}; \cdot)\|_{1; \Omega_* \setminus B(\mathbf{x}; \rho)} \geq c \varepsilon^{-1} \ln(2 + \varepsilon/\rho), \quad \text{if } \rho \leq c_1 \varepsilon, \quad (8d)$$

$$\|\partial_{\xi_k}^2 g(\mathbf{x}; \cdot)\|_{1; \Omega_* \setminus B(\mathbf{x}; \rho)} \geq c \varepsilon^{-1} (\ln(2 + \varepsilon/\rho) + |\ln \varepsilon|), \quad \text{if } \rho \leq \frac{1}{8}, \quad (8e)$$

where $k = 2, 3$, and c_1 is a sufficiently small positive constant.

Remark 6. The statement of the above Lemma 5 is almost identical with the statement of Theorem 3 with G replaced by g and Ω_* replaced by Ω .

Proof. Throughout the proof, x_1 is fixed, so we employ the notation $\widehat{\xi}_1 := \widehat{\xi}_{1, [x_1]}$ and $\widehat{\xi} := (\widehat{\xi}_1, \widehat{\xi}_2, \widehat{\xi}_3)$. First, we rewrite all integrals that appear in (8) in variable $\widehat{\xi} \in \widehat{\Omega}_* := (-\frac{x_1}{\varepsilon}, \frac{1-x_1}{\varepsilon}) \times \mathbb{R}^2$ using $d\mathbf{x} = \varepsilon^3 d\widehat{\xi}$. Now it suffices to prove the desired lower bounds on any sub-domain of $\widehat{\Omega}_*$. In particular, we employ the non-overlapping sub-domains $\widehat{\Omega}_1$ and $\widehat{\Omega}_2$ of $\widehat{\Omega}_*$:

$$\widehat{\Omega}_1 := B(\mathbf{0}, 1) \cap \left\{ \widehat{\xi}_1^2 \geq \widehat{\xi}_2^2 + \widehat{\xi}_3^2 \right\}, \quad \widehat{\Omega}_2 := \left\{ \max\{1, \sqrt{\widehat{\xi}_2^2 + \widehat{\xi}_3^2}\} \leq \widehat{\xi}_1 \leq \frac{1}{4}\varepsilon^{-1} \right\}$$

(but other sub-domains of $B(\mathbf{0}, 1)$ similar to $\widehat{\Omega}_1$ will be considered as well). The notation $[v]^+ := \max\{v, 0\}$ will be used for any function v .

(i) The bounds (8a), (8b) will be obtained using $\widehat{\Omega}_2$. Note that for $\widehat{\xi} \in \widehat{\Omega}_2$ one has $\widehat{\xi}_1 \leq \widehat{r} \leq \sqrt{2}\widehat{\xi}_1$. Introduce the new variables $\psi_k := \widehat{\xi}_k / \sqrt{2\widehat{\xi}_1}$ for $k = 2, 3$, so

$$d\widehat{\xi}_2 d\widehat{\xi}_3 = 2\widehat{\xi}_1 d\psi_2 d\psi_3, \quad \Omega_\psi = \left\{ \psi_2^2 + \psi_3^2 \leq \frac{1}{2}\widehat{\xi}_1 \right\}, \quad \frac{\widehat{r} - \widehat{\xi}_1}{\psi_2^2 + \psi_3^2} \in [c_2, 1],$$

where $c_2 := 2(1 + \sqrt{2})^{-1}$, and we used $\widehat{r} - \widehat{\xi}_1 = (\widehat{\xi}_2^2 + \widehat{\xi}_3^2) / (\widehat{\xi}_1 + \widehat{r})$ to get the final relation above.

Now a calculation using (7a) yields the first desired bound (8a) as follows:

$$\begin{aligned} \|\partial_{\xi_1} g\|_{1, \Omega_*} &\geq c \iiint_{\widehat{\Omega}_2} e^{\alpha(\widehat{\xi}_1 - \widehat{r})} \widehat{\xi}_1^{-2} \left[\alpha(\widehat{r} - \widehat{\xi}_1) - \frac{\widehat{\xi}_1}{\widehat{r}} \right]^+ d\widehat{\xi} \\ &\geq c \int_1^{\frac{1}{4}\varepsilon^{-1}} \widehat{\xi}_1^{-1} d\widehat{\xi}_1 \iint_{\Omega_\psi} e^{-\alpha(\psi_2^2 + \psi_3^2)} \left[\alpha c_2 (\psi_2^2 + \psi_3^2) - 1 \right]^+ d\psi_2 d\psi_3 \\ &\geq c |\ln \varepsilon|. \end{aligned}$$

A similar calculation using (7b) with $k = 2, 3$ yields (8b); indeed,

$$\begin{aligned} \|\partial_{\xi_k} g\|_{1, \Omega_*} &\geq c \iiint_{\widehat{\Omega}_2} e^{\alpha(\widehat{\xi}_1 - \widehat{r})} (\alpha \widehat{r} + 1) \widehat{\xi}_1^{-3} |\widehat{\xi}_k| d\widehat{\xi} \\ &\geq c \int_1^{\frac{1}{4}\varepsilon^{-1}} d\widehat{\xi}_1 (\alpha \widehat{\xi}_1 + 1) \widehat{\xi}_1^{-3/2} \iint_{\Omega_\psi} e^{-\alpha(\psi_2^2 + \psi_3^2)} |\psi_k| d\psi_2 d\psi_3 \geq c\varepsilon^{-1/2}. \end{aligned}$$

(ii) To show (8c) for $\rho \leq 2\varepsilon$, we note that $\|g\|_{1,1;B(\mathbf{x},\rho)} \geq \|\partial_{\xi_2} g\|_{1;B(\mathbf{x},\rho)}$ so set $\widehat{\rho} := \rho/\varepsilon$ and consider the sub-domain $\widehat{\Omega}_3 := B(\mathbf{0}, \widehat{\rho}) \cap \{\widehat{\xi}_2^2 \geq \widehat{\xi}_1^2 + \widehat{\xi}_3^2\}$. Note that in this sub-domain, $e^{\alpha(\widehat{\xi}_1 - \widehat{r})} \geq c$ and $\widehat{\xi}_2 \geq \widehat{r}/\sqrt{2}$ so (7b) yields

$$\|\partial_{\xi_2} g\|_{1, \Omega_* \cap B(\mathbf{x}, \rho)} \geq c \iiint_{\widehat{\Omega}_3} \widehat{r}^{-2} d\widehat{\xi} \geq c \int_0^{\widehat{\rho}} d\widehat{r} \geq c\widehat{\rho} = c\rho/\varepsilon,$$

which immediately implies (8c) for $\rho \leq 2\varepsilon$.

Next, for $\rho \in [2\varepsilon, \frac{1}{8}]$ consider $\partial_{\xi_2} g$ in the sub-domain $\widehat{\Omega}_2 \cap B(\mathbf{0}, \widehat{\rho})$. Imitating the calculation in part (i), one gets

$$\begin{aligned} \|\partial_{\xi_2} g\|_{1, \Omega_* \cap B(\mathbf{x}, \rho)} &\geq c \iiint_{\widehat{\Omega}_2 \cap B(\mathbf{0}, \widehat{\rho})} e^{\alpha(\widehat{\xi}_1 - \widehat{r})} (\alpha \widehat{r} + 1) \widehat{\xi}_1^{-3} |\widehat{\xi}_2| d\widehat{\xi} \\ &\geq c \int_1^{\widehat{\rho}} d\widehat{\xi}_1 (\alpha \widehat{\xi}_1 + 1) \widehat{\xi}_1^{-3/2} \iint_{\Omega_\psi} e^{-\alpha(\psi_2^2 + \psi_3^2)} |\psi_2| d\psi_2 d\psi_3 \\ &\geq c\widehat{\rho}^{-1/2}, \end{aligned}$$

which completes the proof of (8c) for $\rho \leq \frac{1}{8}$.

(iii) To obtain (8d), we use the sub-domain $\widehat{\Omega}_1 \setminus B(\mathbf{0}, \widehat{\rho})$, where $\widehat{\rho} := \rho/\varepsilon$. Note that for $\widehat{\xi} \in \widehat{\Omega}_1$ one has $e^{\alpha(\widehat{\xi}_1 - \widehat{r})} \geq c$ and $\frac{3\widehat{\xi}_1^2 - \widehat{r}^2}{\widehat{r}^2} \geq 1$. So using (7c), one gets

$$\begin{aligned} \|\partial_{\xi_1}^2 g\|_{1, \Omega_* \setminus B(\mathbf{x}, \rho)} &\geq c\varepsilon^{-1} \iiint_{\widehat{\Omega}_1 \setminus B(\mathbf{0}, \widehat{\rho})} \widehat{r}^{-3} [0 - 4\alpha\widehat{r} + 1]^+ d\widehat{\xi} \\ &\geq c\varepsilon^{-1} \int_{\widehat{\rho}}^{\min\{1, \frac{1}{8\alpha}\}} \widehat{r}^{-1} d\widehat{r} \geq c\varepsilon^{-1} \ln(2 + \rho/\varepsilon). \end{aligned}$$

So we have shown (8d) for $\rho \leq c_1\varepsilon$ with $c_1 := \frac{1}{2} \min\{1, \frac{1}{8\alpha}\}$.

(iv) In a similar manner as in part (iii), using (7d), one can show that $\|\partial_{\xi_k}^2 g\|_{1, \Omega \setminus B(\mathbf{x}, \rho)} \geq c\varepsilon^{-1} \ln(2 + \rho/\varepsilon)$ for $k = 2, 3$ and $\rho \leq \frac{1}{2}\varepsilon$. Note that now we use the sub-domain $B(\mathbf{0}, 1) \cap \{\widehat{\xi}_k^2 \geq \widehat{\xi}_1^2 + \widehat{\xi}_j^2\}$ instead of $\widehat{\Omega}_1$, with $j = 3$ for $k = 2$ and $j = 2$ for $k = 3$.

Consequently, to obtain (8e) for any $\rho \leq \frac{1}{8}$, it remains to show that $\|\partial_{\xi_k}^2 g\|_{1, \Omega_* \setminus B(\mathbf{x}, \rho)} \geq c\varepsilon^{-1} |\ln \varepsilon|$. For this, consider (7d) in $\widehat{\Omega}_2 \setminus B(\mathbf{0}, \widehat{\rho})$. Com-

binning the observations made in part (i) with $\alpha\hat{r} + 1 \leq (\alpha\sqrt{2} + 1)\hat{\xi}_1$ and $\frac{3\hat{\xi}_k^2 - \hat{r}^2}{\hat{r}^2} \geq -1$, yields

$$\begin{aligned} & \|\partial_{\xi_k}^2 g\|_{1, \Omega_* \setminus B(\mathbf{x}, \rho)} \\ & \geq c\varepsilon^{-1} \iiint_{\hat{\Omega}_2 \setminus B(\mathbf{0}, \hat{\rho})} e^{\alpha(\hat{\xi}_1 - \hat{r})} \hat{\xi}_1^{-3} \left[2\alpha^2 \hat{\xi}_1 \psi_k^2 - (\alpha\sqrt{2} + 1)\hat{\xi}_1 \right]^+ d\hat{\xi} \\ & \geq c \int_{\max\{1, \hat{\rho}\}}^{\frac{1}{4}\varepsilon^{-1}} \hat{\xi}_1^{-1} d\hat{\xi}_1 \iint_{\Omega_\psi} e^{-\alpha(\psi_2^2 + \psi_3^2)} \left[2\alpha^2 \psi_k^2 - (\alpha\sqrt{2} + 1) \right]^+ d\psi_2 d\psi_3 \\ & \geq c\varepsilon^{-1} |\ln \varepsilon|. \end{aligned}$$

Here we also used $\hat{\rho} \leq \frac{1}{8}\varepsilon^{-1}$. So we have proved the final desired assertion (8e). \square

3 Approximation of the Green's function and proof of Theorem 3 for the domain $\Omega_* = (0, 1) \times \mathbb{R}^2$

To approximate the Green's function, we use the fundamental solution g of Section 2 and the cut-off function ω , defined by

$$\omega(t) \in C^2(0, 1), \quad \omega(t) = 0 \text{ for } t \leq \frac{1}{6}, \quad \omega(t) = 1 \text{ for } t \geq \frac{1}{3}, \quad (9)$$

so $\omega'(x) = 0$ for $x = 0, 1$. Then we set

$$\bar{G}(\mathbf{x}; \boldsymbol{\xi}) = \frac{e^{\alpha\hat{\xi}_1[x_1]}}{4\pi\varepsilon^2} \left\{ \left[\frac{e^{-\alpha\hat{r}[x_1]}}{\hat{r}[x_1]} - \frac{e^{-\alpha\hat{r}[-x_1]}}{\hat{r}[-x_1]} \right] - \left[\frac{e^{-\alpha\hat{r}[2-x_1]}}{\hat{r}[2-x_1]} - \frac{e^{-\alpha\hat{r}[2+x_1]}}{\hat{r}[2+x_1]} \right] \omega(\xi_1) \right\},$$

which approximates the Green's function G associated with the domain $\Omega_* = (0, 1) \times \mathbb{R}^2$; in particular, it satisfies the boundary condition $\bar{G}|_{\partial\Omega_*} = \bar{G}|_{\xi_1=0,1} = 0$. Using the notations

$$g_{[d]} := g(d, x_2, x_3; \boldsymbol{\xi}), \quad \lambda^\pm := e^{2\alpha(1 \pm x_1)/\varepsilon}, \quad p := e^{-2\alpha x_1/\varepsilon},$$

we can rewrite the definition of \bar{G} as

$$\bar{G}(\mathbf{x}; \boldsymbol{\xi}) = [g_{[x_1]} - p g_{[-x_1]}] - [\lambda^- g_{[2-x_1]} - p \lambda^+ g_{[2+x_1]}] \omega(\xi_1). \quad (10)$$

We now present a version of [5, Lemma 4.2], which gives certain upper bounds for g that involve a weight function λ of type λ^\pm .

Lemma 7. *Let $\mathbf{x} \in [1 + \varepsilon, 3] \times \mathbb{R}^2$. Then for the function g of (6) and the weight $\lambda := e^{2\alpha(x_1 - 1)/\varepsilon}$ one has the following bounds*

$$\|(\lambda g)(\mathbf{x}; \cdot)\|_{1; \Omega_*} \leq C\varepsilon, \quad (11a)$$

$$\|(\lambda \partial_{\xi_k} g)(\mathbf{x}; \cdot)\|_{1; \Omega_*} \leq C, \quad k = 1, 2, 3, \quad (11b)$$

$$\|(\lambda \partial_{\xi_k}^2 g)(\mathbf{x}; \cdot)\|_{1; \Omega_*} \leq C\varepsilon^{-1}, \quad k = 1, 2, 3, \quad (11c)$$

while for any ball $B(\mathbf{x}'; \rho)$ of radius $\rho \leq \frac{1}{8}$ centred at $\mathbf{x}' \in [0, \frac{3}{4}] \times \mathbb{R}^2$, one has

$$\|(\lambda g)(\mathbf{x}; \cdot)\|_{1,1; \Omega_* \cap B(\mathbf{x}'; \rho)} \leq C \min\{\rho/\varepsilon, (\rho/\varepsilon)^{1/2}\} e^{-\frac{1}{8}\alpha/\varepsilon}. \quad (11d)$$

Proof. The bounds (11a), (11b) appear in [5, Lemma 4.2]. The estimate (11c) is slightly sharper compared to the similar bound in [5, Lemma 4.2]; the latter is for the domain $\Omega_* \setminus B(\mathbf{x}, \rho)$ and involves the logarithmic term $\ln(2 + \varepsilon/\rho)$ as it is valid for $x_1 \in [1, 3]$. In the above Lemma 7 we make a stronger assumption that $x_1 \in [1 + \varepsilon, 3]$, under which $\Omega_* \setminus B(\mathbf{x}, \varepsilon) = \Omega_*$ and the logarithmic term becomes $\ln 3$ so can be dropped.

To obtain the final desired bound (11d), note that $x_1 \geq 1 + \varepsilon$ implies that $\hat{r} > 1$ in Ω_* so (6), (7a) and (7b) yield $\lambda(|g| + |\partial_{\xi_k} g|) \leq C\varepsilon^{-3} \lambda e^{\alpha(\hat{\xi}_1 - \hat{r})}$ for $k = 1, 2, 3$. Here $\hat{\xi}_1 - \hat{r} = -(|\hat{\xi}_1| + \hat{r}) \geq -2|\hat{\xi}_1|$ so $\lambda e^{\alpha(\hat{\xi}_1 - \hat{r})} \leq e^{2\alpha(|x_1 - 1|/\varepsilon - |\hat{\xi}_1|)}$. Note also that in $B(\mathbf{x}'; \rho)$ we have $\xi_1 \leq 1 - \frac{1}{8}$ so $|\hat{\xi}_1| = \frac{x_1 - \xi_1}{\varepsilon} \geq \frac{x_1 - 1}{\varepsilon} + \frac{1}{8}\varepsilon^{-1}$. Consequently, $\lambda(|g| + |\partial_{\xi_k} g|) \leq C\varepsilon^{-3} \lambda e^{-\frac{1}{4}\alpha/\varepsilon}$, so $\|(\lambda g)(\mathbf{x}; \cdot)\|_{1,1; \Omega_* \cap B(\mathbf{x}'; \rho)} \leq C(\rho/\varepsilon)^3 e^{-\frac{1}{4}\alpha/\varepsilon}$. Combining this with $(\rho/\varepsilon)^3 \leq C(\rho/\varepsilon)^p e^{\frac{1}{8}\alpha/\varepsilon}$ for $p = 1, \frac{1}{2}$ yields (11d). \square

Lemma 8. *Let $\Omega_* := (0, 1) \times \mathbb{R}^2$, $\varepsilon \in (0, c_0]$ for some sufficiently small constant c_0 , and $0 < \alpha \leq C$. Then the function \tilde{G} of (10) satisfies, for any $\mathbf{x} \in [1/4, 3/4] \times \mathbb{R}^2$, all the bounds (8) of Lemma 5 with g replaced by \tilde{G} .*

Proof. Let D be any of the first- or second-order differential operators that appear in (8). Now for any $\Omega'_* \subset \Omega_*$, the representation (10) yields

$$\begin{aligned} \|D\tilde{G}(\mathbf{x}; \cdot)\|_{1, \Omega'_*} &= \|Dg_{[x_1]} - p Dg_{[-x_1]} - \lambda^- Dg_{[2-x_1]} + p \lambda^+ Dg_{[2+x_1]}\|_{1, \Omega'_*} \\ &\geq \|Dg_{[x_1]}\|_{1, \Omega'_*} - p \|Dg_{[-x_1]}\|_{1, \Omega'_*} - \|\lambda^- Dg_{[2-x_1]}\|_{1, \Omega'_*} \\ &\quad - p \|\lambda^+ Dg_{[2+x_1]}\|_{1, \Omega'_*}. \end{aligned} \quad (12)$$

For the first term that involves $g_{[x_1]}$, we use the corresponding lower bound from Lemma 5 so it remains to show that this bound will dominate the remaining three terms. For the second term we note that $g_{[-x_1]}$ satisfies the upper bounds of type (3) with Ω replaced by Ω_* [5]. Now, as $x_1 \geq \frac{1}{4}$ implies that $p \leq e^{-\frac{1}{2}\alpha/\varepsilon}$, the second term will be dominated by the first term if ε is sufficiently small (i.e. if the constant c_0 is sufficiently small).

To estimate the terms that involve $\lambda^\pm g_{[2\pm x_1]}$ in (12), we use the bounds (11) of Lemma 7. In particular, by (8d), (11c),

$$\begin{aligned} \frac{1}{3} \|\partial_{\xi_1}^2 g(\mathbf{x}; \cdot)\|_{1; \Omega_* \setminus B(\mathbf{x}; \rho)} - \|\lambda^\pm \partial_{\xi_1}^2 g_{[2 \pm x_1]}\|_{1, \Omega_* \setminus B(\mathbf{x}; \rho)} \\ \geq \varepsilon^{-1} [c \ln(2 + \varepsilon/\rho) - C] \\ \geq c \varepsilon^{-1} \ln(2 + \varepsilon/\rho) \end{aligned}$$

for $\rho \leq c_1 \varepsilon$ if c_1 is sufficiently small, so we get a version of (8d) for \bar{G} . Similarly, by (8e), (11c), for $k = 2, 3$ one gets

$$\begin{aligned} \frac{1}{3} \|\partial_{\xi_k}^2 g(\mathbf{x}; \cdot)\|_{1; \Omega_* \setminus B(\mathbf{x}; \rho)} - \|\lambda^\pm \partial_{\xi_k}^2 g_{[2 \pm x_k]}\|_{1, \Omega_* \setminus B(\mathbf{x}; \rho)} \\ \geq \varepsilon^{-1} [c \ln(2 + \varepsilon/\rho) + c |\ln \varepsilon| - C] \\ \geq c \varepsilon^{-1} (\ln(2 + \varepsilon/\rho) + |\ln \varepsilon|) \end{aligned}$$

provided that ε is sufficiently small. This yields a version of (8e) for \bar{G} . Finally, (11d) implies that

$$\begin{aligned} \|(\lambda^\pm g_{[2 \pm x_k]})(\mathbf{x}; \cdot)\|_{1, 1; \Omega_* \cap B(\mathbf{x}; \rho)} \leq C \min\{\rho/\varepsilon, (\rho/\varepsilon)^{1/2}\} e^{-\frac{1}{8}\alpha/\varepsilon} \\ \leq \frac{1}{3} c \min\{\rho/\varepsilon, (\rho/\varepsilon)^{1/2}\} \end{aligned}$$

for any arbitrarily small c provided that ε is sufficiently small. This observation yields a version of (8c) for \bar{G} . \square

Proof of Theorem 3 for the domain $\Omega_ = (0, 1) \times \mathbb{R}^2$.* As, by Lemma 8, the approximation \bar{G} satisfies the bounds that we need to prove for G , it suffices to estimate the function $v = \bar{G} - G$, which satisfies the differential equation

$$\mathcal{L}_\xi^* v(\mathbf{x}; \xi) = -\varepsilon \Delta_\xi v(\mathbf{x}; \xi) + 2\alpha \partial_{\xi_1} v(\mathbf{x}; \xi) = \phi(\mathbf{x}; \xi) \quad \text{for } \xi \in \Omega_*, \quad (13)$$

and the boundary condition $v(\mathbf{x}; \xi)|_{\xi \in \partial\Omega_*} = 0$. Here for the right-hand side ϕ , it was shown in [5, Lemma 5.1] that

$$\|\phi(\mathbf{x}; \cdot)\|_{1; \Omega_*} \leq C e^{-c_3 \alpha/\varepsilon} \quad (14)$$

for some constant c_3 . In view of (13), the function v can be represented using the Green's function G as

$$v(\mathbf{x}; \xi) = \iiint_{\Omega_*} G(\mathbf{s}; \xi) \phi(\mathbf{x}; \mathbf{s}) d\mathbf{s}.$$

So applying $\partial_{\xi_k}^p$ to this representation with $p = 0, 1, 2$, $k = 1, 2, 3$, for any sub-domain $\Omega'_* \subset \Omega_*$, one gets

$$\begin{aligned} \|\partial_{\xi_k}^p v(\mathbf{x}; \cdot)\|_{1; \Omega'_*} &\leq \left(\sup_{\mathbf{s} \in \Omega_*} \|\partial_{\xi_k}^p G(\mathbf{s}; \cdot)\|_{1, \Omega'_*} \right) \cdot \|\phi(\mathbf{x}; \cdot)\|_{1, \Omega_*} \\ &\leq C e^{-c_3 \alpha/\varepsilon} \cdot \sup_{\mathbf{s} \in \Omega_*} \|\partial_{\xi_k}^p G(\mathbf{s}; \cdot)\|_{1, \Omega_*}, \end{aligned}$$

where we also used (14). We shall use the above estimate with $\Omega'_* := \Omega_*$ for $p = 0, 1$ and $\Omega'_* := \Omega_* \setminus B(\mathbf{x}, \rho)$ for $p = 2$. As the bounds (3) of Theorem 1 remain valid for the domain Ω_* and also $\|G(\mathbf{x}; \cdot)\|_{1, \Omega_*} \leq C$ [5, 6], one now concludes that

$$\begin{aligned} \|\partial_{\xi_k}^2 v(\mathbf{x}; \cdot)\|_{1; \Omega_* \setminus B(\mathbf{x}, \rho)} &\leq C e^{-c_3 \alpha / \varepsilon} \cdot \varepsilon^{-1} (\ln(2 + \varepsilon / \rho) + |\ln \varepsilon|) \\ &\leq c' \varepsilon^{-1} \ln(2 + \varepsilon / \rho) \quad \text{for } k = 1, 2, 3, \end{aligned} \quad (15a)$$

where c' is arbitrarily small provided that ε is sufficiently small. Similarly

$$\|v(\mathbf{x}; \cdot)\|_{1, 1; \Omega_*} \leq C e^{-c_3 \alpha / \varepsilon} \cdot \varepsilon^{-1/2} \leq c' \varepsilon, \quad (15b)$$

which also implies

$$\|v(\mathbf{x}; \cdot)\|_{1, 1; \Omega_* \cap B(\mathbf{x}, \rho)} \leq c' \min\{\rho / \varepsilon, (\rho / \varepsilon)^{1/2}\}. \quad (15c)$$

As c' in the bounds (15) can be made arbitrarily small, combining them with Lemma 8 yields the desired bounds of Theorem 3 for the domain $\Omega_* = (0, 1) \times \mathbb{R}^2$. \square

4 Proof of Theorem 3 for the domain $\Omega = (0, 1)^3$

The proof of Theorem 3 for the domain $\Omega = (0, 1)^3$ is very similar to the above proof for the domain $\Omega_* = (0, 1) \times \mathbb{R}^2$ presented in Section 3. The only difference is that instead of the approximation \bar{G} for the domain $(0, 1) \times \mathbb{R}^2$ we now use the approximation \bar{G}_{\square} defined by

$$\begin{aligned} \bar{G}_{\square}(\mathbf{x}; \boldsymbol{\xi}) &:= \bar{G}(\mathbf{x}; \boldsymbol{\xi}) - \omega_0(\xi_2) \bar{G}(\mathbf{x}; \xi_1, -\xi_2, \xi_3) - \omega_1(\xi_2) \bar{G}(\mathbf{x}; \xi_1, 2 - \xi_2, \xi_3), \\ \bar{G}_{\square}(\mathbf{x}; \boldsymbol{\xi}) &:= \bar{G}_{\square}(\mathbf{x}; \boldsymbol{\xi}) - \omega_0(\xi_3) \bar{G}_{\square}(\mathbf{x}; \xi_1, \xi_2, -\xi_3) - \omega_1(\xi_3) \bar{G}_{\square}(\mathbf{x}; \xi_1, \xi_2, 2 - \xi_3). \end{aligned}$$

Here $\omega_0(t) := \omega(1 - t)$ and $\omega_1(t) := \omega(t)$ with ω defined in (9), so that $\omega_k(k) = 1$ and $\omega_k(1 - k) = 0$ for $k = 0, 1$. This approximation was constructed employing the method of images; an inclusion of cut-off functions ensures that it vanishes on $\partial\Omega$.

All the properties of \bar{G} given in Section 3 remain valid for this new approximation \bar{G}_{\square} with Ω_* replaced by Ω provided that $\mathbf{x} \in [\frac{1}{4}, \frac{3}{4}]^3$. We leave out the details and only note that the application of the method of images in the ξ_2 - and ξ_3 -directions is relatively straightforward as an inspection of (6) shows that in these directions, the fundamental solution g is symmetric and exponentially decaying away from the singular point. \square

5 The Convection-Reaction-Diffusion Case

We now slightly generalize (1) by including a reaction term with a constant coefficient $\beta \geq 0$:

$$\tilde{\mathcal{L}}_{\mathbf{x}} u(\mathbf{x}) = -\varepsilon \Delta_{\mathbf{x}} u(\mathbf{x}) - 2\alpha \partial_{x_1} u(\mathbf{x}) + \beta u(\mathbf{x}) = f(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega \quad (16a)$$

$$u(\mathbf{x}) = 0 \quad \text{for } \mathbf{x} \in \partial\Omega. \quad (16b)$$

Now the fundamental solution g in \mathbb{R}^3 satisfies, for each fixed $\mathbf{x} \in \mathbb{R}^3$ the following version of (5) with the adjoint operator $\tilde{\mathcal{L}}_{\boldsymbol{\xi}}^*$:

$$\tilde{\mathcal{L}}_{\boldsymbol{\xi}}^* g(\mathbf{x}; \boldsymbol{\xi}) = -\varepsilon \Delta_{\boldsymbol{\xi}} g(\mathbf{x}; \boldsymbol{\xi}) + 2\alpha \partial_{\xi_1} g(\mathbf{x}; \boldsymbol{\xi}) + \beta g(\mathbf{x}; \boldsymbol{\xi}) = \delta(\mathbf{x} - \boldsymbol{\xi}) \quad \text{for } \boldsymbol{\xi} \in \mathbb{R}^3.$$

Again imitating a calculation of [2, 9], one gets a version of (6):

$$g(\mathbf{x}; \boldsymbol{\xi}) = \frac{1}{4\pi\varepsilon^2} \frac{e^{\alpha\hat{\xi}_1, [x_1] - \gamma\hat{r}_{[x_1]}}}{\hat{r}_{[x_1]}}, \quad \text{where } \gamma := \sqrt{\alpha^2 + \varepsilon\beta}.$$

In view of $\gamma = \alpha + \frac{\beta}{2\alpha}\varepsilon + \mathcal{O}(\varepsilon^2)$, an inspection of the proof of Theorem 3 shows that the lower bounds (4) remain valid for the convection-reaction-diffusion problem (16).

6 Outlook for problems in n dimensions

It was shown in [7] that the upper bounds of Theorem 1 remain valid for a two-dimensional variable-coefficient version of (1). Note that the fundamental solution $g_{\mathbb{R}^2}$ that solves (5) in \mathbb{R}^2 , and also its derivatives involve the modified Bessel functions of second kind of order zero $K_0(\cdot)$ and order one $K_1(\cdot)$. This fundamental solution is given by

$$g_{\mathbb{R}^2}(\mathbf{x}; \boldsymbol{\xi}) = \frac{1}{2\pi\varepsilon} e^{\alpha\hat{\xi}_1} K_0(\alpha\hat{r})$$

with the notations \mathbf{x} , $\boldsymbol{\xi}$, and \hat{r} appropriately adapted. Using this explicit representation, one can imitate the proof of Theorem 3 and get similar lower bounds for the two-dimensional case. A certain difficulty lies in having to deal with the Bessel functions, for which one can simply employ asymptotic expansions [1, 12]:

$$\begin{aligned} K_0(z) &= \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \left(1 + \mathcal{O}(z^{-1})\right) && \text{for } |z| \gg 1, \\ K_1(z) &= K_0(z) \left(1 + \frac{1}{2z} + \mathcal{O}(z^{-2})\right) && \text{for } |z| \gg 1, \\ K_0(z) &= -\ln z + \mathcal{O}(1), \quad K_1(z) = \frac{1}{z} + \mathcal{O}(1) && \text{for } |z| \ll 1. \end{aligned}$$

In this manner one gets the following result.

Theorem 9. *The Green's function associated with problem (1) in the unit-square domain $\Omega := (0, 1)^2$, satisfies a two-dimensional version of Theorem 3.*

Finally, let us take a look at the problem (1) in the n -dimensional domain $\Omega := (0, 1)^n$ of an arbitrary dimension $n \geq 2$. The corresponding fundamental solution $g_{\mathbb{R}^n}$ is given by

$$g_{\mathbb{R}^n}(\mathbf{x}; \boldsymbol{\xi}) = \frac{1}{(2\pi)^{n/2} \varepsilon^{n-1}} \left(\frac{\alpha}{\hat{r}} \right)^{n/2-1} e^{\alpha \hat{\xi}_1} K_{n/2-1}(\alpha \hat{r})$$

with the modified Bessel functions $K_{n/2-1}(\cdot)$ of second kind of (half-)integer order $n/2 - 1$, and the notations \mathbf{x} , $\boldsymbol{\xi}$, and \hat{r} appropriately adapted. Using asymptotic expansions of these Bessel functions [1, 12], one can again get a version of Theorem 3.

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