

## GREEN'S FUNCTION ESTIMATES FOR A SINGULARLY PERTURBED CONVECTION-DIFFUSION PROBLEM IN THREE DIMENSIONS

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**Abstract.** A linear singularly perturbed convection-diffusion problem with characteristic layers is considered in three dimensions. Sharp bounds for the associated Green's function and its first- and second-order derivatives are established in the  $L_1$  norm. The dependence of these bounds on the small perturbation parameter is shown explicitly. The obtained estimates will be used in a forthcoming numerical analysis of the considered problem to derive a robust a posteriori error estimator in the maximum norm.

**Key Words.** Green's function, singular perturbations, convection-diffusion, a posteriori error estimates

### 1. Introduction

In many physical transport problems convection is much stronger than diffusion. For example, consider the incompressible Navier-Stokes equations that describe the flow of a fluid through a given domain  $\Omega \subset \mathbb{R}^3$  over the time  $[0, T]$

$$\partial_t \mathbf{v} - \varepsilon \Delta \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = f, \quad \nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega,$$

where  $\mathbf{v} : \Omega \times [0, T] \rightarrow \mathbb{R}^3$  is the velocity of the fluid and  $p : \Omega \times [0, T] \rightarrow \mathbb{R}$  is its pressure. The parameter  $\varepsilon$  is the inverse of the Reynolds number. At large Reynolds number (for turbulent flows), one has  $\varepsilon \ll 1$  so the velocity equation becomes singularly perturbed.

In this paper we focus on a steady-state linear version of this equation posed in the domain  $\Omega = (0, 1)^3$ :

$$(1.1a) \quad L_{\mathbf{x}} u(\mathbf{x}) = -\varepsilon \Delta_{\mathbf{x}} u(\mathbf{x}) - \partial_{x_1}(a(\mathbf{x}) u(\mathbf{x})) + b(\mathbf{x}) u(\mathbf{x}) = f(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega,$$

$$(1.1b) \quad u(\mathbf{x}) = 0 \quad \text{for } \mathbf{x} \in \partial\Omega.$$

Here  $\varepsilon$  is a small positive parameter, and we assume that the coefficients  $a$  and  $b$  are sufficiently smooth (e.g.,  $a, b \in C^\infty(\bar{\Omega})$ ). We also assume, for some positive constant  $\alpha$ , that

$$(1.2) \quad a(\mathbf{x}) \geq \alpha > 0, \quad b(\mathbf{x}) \geq 0, \quad b(\mathbf{x}) - \partial_{x_1} a(\mathbf{x}) \geq 0 \quad \text{for all } \mathbf{x} \in \bar{\Omega}.$$

Under these assumptions, (1.1a) is a singularly perturbed elliptic equation, also referred to as a convection-dominated convection-diffusion equation. Its solutions typically exhibits sharp interior and boundary layers.

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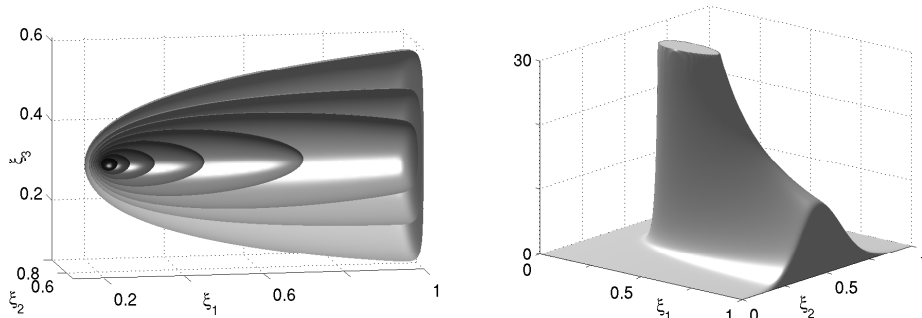


FIGURE 1. Anisotropy of the Green's function  $G$  associated with (1.1) for  $\varepsilon = 0.01$  and  $\mathbf{x} = (\frac{1}{5}, \frac{1}{2}, \frac{1}{3})$ . Left: isosurfaces at values of 1, 4, 8, 16, 32, 64, 128, and 256. Right: a two-dimensional graph for fixed  $\xi_3 = x_3$ .

The Green's function for the convection-diffusion problem (1.1) exhibits a strong anisotropic structure, which is demonstrated by Figure 1. This reflects the complexity of solutions of this problem; it should be noted that problems of this type require an intricate asymptotic analysis [10, Section IV.1], [11]; see also [17, Chapter IV] and [12, 13]. We also refer the reader to Dörfler [3], who, for a similar problem, gives extensive a priori solution estimates.

Our interest in considering the Green's function of problem (1.1) and estimating its derivatives is motivated by the numerical analysis of this computationally challenging problem. More specifically, we shall use the obtained estimates in the forthcoming paper [5] to derive robust a posteriori error bounds for computed solutions of this problem using finite-difference methods. (This approach is related to recent articles [2, 15], which address the numerical solution of singularly perturbed equations of reaction-diffusion type.) More specifically, the basic idea is to rewrite the continuous residual of the computed solution  $u^N$  in the form

$$L_{\mathbf{x}}[u^N - u] = \sum_{i=1}^3 \partial_{x_i} F_i + \bar{f},$$

and then use stability properties of the differential operator, that follow from certain sharp bounds of the Green's function, to obtain a bound on the error  $\|u^N - u\|_{\infty; \Omega}$ . Here the computed solution  $u^N$  is understood as a continuous function (if necessary, an interpolation of the discrete numerical solution has to be used). For more details, we refer the reader to Section 6. In a more general numerical-analysis context, we note that sharp estimates for continuous Green's functions (or their generalised versions) frequently play a crucial role in a priori and a posteriori error analyses [4, 9, 16].

The purpose of the present paper is to establish sharp bounds for the derivatives of the Green's function in the  $L_1$  norm (as they will be used to estimate the error in the computed solution in the dual  $L_{\infty}$  norm [5]). Our estimates will be *uniform in the small perturbation parameter*  $\varepsilon$  in the sense that any dependence on  $\varepsilon$  will be shown explicitly. Note also that our estimates will be *sharp* (in the sense of Theorem 2.2) up to an  $\varepsilon$ -independent constant multiplier. We employ the analysis technique used in [7], which we now extend to a three-dimensional problem. Roughly speaking, we freeze the coefficients and estimate the corresponding explicit

frozen-coefficient Green's function, and then we investigate the difference between the original and the frozen-coefficient Green's functions. To make this paper more readable, we deliberately follow some of the notation and presentation of [7], while some proofs that involve much computation and resemble the ones in [7] have been placed in a separate technical report [6].

The paper is structured as follows. In Section 2, the Green's function associated with problem (1.1) is defined and upper bounds for its derivatives are stated in Theorem 2.1, the main result of the paper. The corresponding lower bounds are then given in Theorem 2.2. In Section 3, we obtain and estimate the fundamental solution for a constant-coefficient version of (1.1) in the domain  $\Omega = \mathbb{R}^3$ . This fundamental solution is then used in Section 4 to construct certain approximations of the frozen-coefficient Green's functions for the domains  $\Omega = (0, 1) \times \mathbb{R}^2$  and  $\Omega = (0, 1)^3$ . The difference between these approximations and the original variable-coefficient Green's function is estimated in Section 5, which completes the proof of Theorem 2.1. Finally, in Section 6 we present certain stability properties of our differential operator that follow from the obtained Green's function estimates, and furthermore, discuss applications of these properties in deriving a robust *a-posteriori* error estimator for a finite difference method.

*Notation.* Throughout the paper,  $C$ , as well as  $c$ , denotes a generic positive constant that may take different values in different formulas, but is *independent of the small diffusion coefficient*  $\varepsilon$ . A subscripted  $C$  (e.g.,  $C_1$ ) denotes a positive constant that takes a fixed value, and is also independent of  $\varepsilon$ . The usual Sobolev spaces  $W^{m,p}(D)$  and  $L_p(D)$  on any measurable domain  $D \subset \mathbb{R}^3$  are used. The  $L_p(D)$  norm is denoted by  $\|\cdot\|_p;D$  while the  $W^{m,p}(D)$  norm is denoted by  $\|\cdot\|_{m,p;D}$ . By  $\mathbf{x} = (x_1, x_2, x_3)$  we denote an element in  $\mathbb{R}^3$ . For an open ball centred at  $\mathbf{x}'$  of radius  $\rho$ , we employ the notation  $B(\mathbf{x}', \rho) = \{\mathbf{x} \in \mathbb{R}^3 : \sum_{k=1,2,3} (x_k - x'_k)^2 < \rho^2\}$ . The notation  $\partial_{x_m} f$ ,  $\partial_{x_m}^2 f$  and  $\Delta_{\mathbf{x}}$  is used for the first- and second-order partial derivatives of a function  $f$  in variable  $x_m$ , and the Laplacian in variable  $\mathbf{x}$ , respectively, while  $\partial_{x_k x_m}^2 f$  will denote a mixed derivative of  $f$ .

## 2. Green's Function: Definition and Main Result

Let  $G = G(\mathbf{x}; \boldsymbol{\xi})$  be the Green's function associated with (1.1). For each fixed  $\mathbf{x} \in \Omega$ , it satisfies

$$(2.1a) \quad L_{\boldsymbol{\xi}}^* G(\mathbf{x}; \boldsymbol{\xi}) := -\varepsilon \Delta_{\boldsymbol{\xi}} G + a(\boldsymbol{\xi}) \partial_{\xi_1} G + b(\boldsymbol{\xi}) G = \delta(\mathbf{x} - \boldsymbol{\xi}) \quad \text{for } \boldsymbol{\xi} \in \Omega,$$

$$(2.1b) \quad G(\mathbf{x}; \boldsymbol{\xi}) = 0 \quad \text{for } \boldsymbol{\xi} \in \partial\Omega.$$

Here  $L_{\boldsymbol{\xi}}^*$  is the adjoint differential operator to  $L_{\mathbf{x}}$ , and  $\delta(\cdot)$  is the three-dimensional Dirac  $\delta$ -distribution. The unique solution  $u$  of (1.1) allows the representation

$$(2.2) \quad u(\mathbf{x}) = \iiint_{\Omega} G(\mathbf{x}; \boldsymbol{\xi}) f(\boldsymbol{\xi}) d\boldsymbol{\xi}.$$

It should be noted that the Green's function  $G$  also satisfies, for each fixed  $\boldsymbol{\xi} \in \Omega$ ,

$$(2.3a) \quad L_{\mathbf{x}} G(\mathbf{x}; \boldsymbol{\xi}) = -\varepsilon \Delta_{\mathbf{x}} G - \partial_{x_1} (a(\mathbf{x}) G) + b(\mathbf{x}) G = \delta(\mathbf{x} - \boldsymbol{\xi}) \quad \text{for } \mathbf{x} \in \Omega,$$

$$(2.3b) \quad G(\mathbf{x}; \boldsymbol{\xi}) = 0 \quad \text{for } \mathbf{x} \in \partial\Omega.$$

Consequently, the unique solution  $v$  of the adjoint problem

$$(2.4a) \quad L_{\mathbf{x}}^* v(\mathbf{x}) = -\varepsilon \Delta_{\mathbf{x}} v + b(\mathbf{x}) \partial_{x_1} v + c(\mathbf{x}) v = f(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega,$$

$$(2.4b) \quad v(\mathbf{x}) = 0 \quad \text{for } \mathbf{x} \in \partial\Omega,$$

is given by

$$(2.5) \quad v(\boldsymbol{\xi}) = \iiint_{\Omega} G(\mathbf{x}; \boldsymbol{\xi}) f(\mathbf{x}) d\mathbf{x}.$$

We now state the main result of the paper.

**Theorem 2.1.** *The Green's function  $G$  associated with (1.1) in the unit cube  $\Omega = (0, 1)^3$  satisfies, for all  $\mathbf{x} \in \Omega$ , the following bounds*

$$(2.6a) \quad \|\partial_{\xi_1} G(\mathbf{x}; \cdot)\|_{1;\Omega} \leq C(1 + |\ln \varepsilon|),$$

$$(2.6b) \quad \|\partial_{\xi_k} G(\mathbf{x}; \cdot)\|_{1;\Omega} \leq C\varepsilon^{-1/2}, \quad k = 2, 3.$$

Furthermore, for any ball  $B(\mathbf{x}', \rho)$  of radius  $\rho$  centred at any  $\mathbf{x}' \in \bar{\Omega}$ , we have

$$(2.6c) \quad \|G(\mathbf{x}; \cdot)\|_{1,1;B(\mathbf{x}',\rho)} \leq C\varepsilon^{-1}\rho,$$

while for the ball  $B(\mathbf{x}, \rho)$  of radius  $\rho$  centred at  $\mathbf{x}$  we have

$$(2.6d) \quad \|\partial_{\xi_1}^2 G(\mathbf{x}; \cdot)\|_{1;\Omega \setminus B(\mathbf{x},\rho)} \leq C\varepsilon^{-1} \ln(2 + \varepsilon/\rho),$$

$$(2.6e) \quad \|\partial_{\xi_k}^2 G(\mathbf{x}; \cdot)\|_{1;\Omega \setminus B(\mathbf{x},\rho)} \leq C\varepsilon^{-1} (|\ln \varepsilon| + \ln(2 + \varepsilon/\rho)), \quad k = 2, 3.$$

Here  $C$  is some positive  $\varepsilon$ -independent constant.

Note that the bound of the streamline derivative (2.6a) is smaller than the bound of the cross-wind derivatives (2.6b).

Furthermore, we note that the upper estimates of Theorem 2.1 are *sharp* in the following sense.

**Theorem 2.2** ([8]). *Let  $\varepsilon \in (0, c_0]$  for some sufficiently small positive  $c_0$ . Set  $a(\mathbf{x}) := \alpha$  and  $b(\mathbf{x}) := 0$  in (1.1). Then the Green's function  $G$  associated with this problem in the unit cube  $\Omega = (0, 1)^3$  satisfies, for all  $\mathbf{x} \in [\frac{1}{4}, \frac{3}{4}]^3$ , the following lower bounds:*

$$(2.7a) \quad \|\partial_{\xi_1} G(\mathbf{x}; \cdot)\|_{1;\Omega} \geq c |\ln \varepsilon|,$$

$$(2.7b) \quad \|\partial_{\xi_k} G(\mathbf{x}; \cdot)\|_{1;\Omega} \geq c\varepsilon^{-1/2}, \quad k = 2, 3.$$

Furthermore, for any ball  $B(\mathbf{x}; \rho)$  of radius  $\rho \leq \frac{1}{8}$ , we have

$$(2.7c) \quad \|G(\mathbf{x}; \cdot)\|_{1,1;\Omega \cap B(\mathbf{x};\rho)} \geq \begin{cases} c\rho/\varepsilon, & \text{for } \rho \leq 2\varepsilon, \\ c(\rho/\varepsilon)^{1/2}, & \text{otherwise,} \end{cases}$$

$$(2.7d) \quad \|\partial_{\xi_1}^2 G(\mathbf{x}; \cdot)\|_{1;\Omega \setminus B(\mathbf{x};\rho)} \geq c\varepsilon^{-1} \ln(2 + \varepsilon/\rho), \quad \text{for } \rho \leq c_1\varepsilon,$$

$$(2.7e) \quad \|\partial_{\xi_k}^2 G(\mathbf{x}; \cdot)\|_{1;\Omega \setminus B(\mathbf{x};\rho)} \geq c\varepsilon^{-1} (\ln(2 + \varepsilon/\rho) + |\ln \varepsilon|) \quad \text{for } \rho \leq \frac{1}{8}, \quad k = 2, 3.$$

Here  $c$  and  $c_1$  are  $\varepsilon$ -independent positive constants.

The rest of the paper is devoted to the proof of Theorem 2.1, and also to a brief discussion of its applications. We have placed some proofs that involve much computation and resemble the ones in [7] in a separate technical report [6].

### 3. Fundamental Solution in the Constant-Coefficient Case

In our analysis, we invoke the observation that constant-coefficient versions of the two problems (2.1) and (2.3) that we have for  $G$ , can be easily solved explicitly when posed in  $\mathbb{R}^3$ . So in this section we shall explicitly solve simplifications of (2.1) and (2.3). To get these simplifications, we freeze the coefficients in these problems by replacing  $a(\boldsymbol{\xi})$  by  $a(\mathbf{x})$  in (2.1), and replacing  $a(\mathbf{x})$  by  $a(\boldsymbol{\xi})$  in (2.3), and also setting  $b := 0$ ; the frozen-coefficient versions of the operators  $L_{\boldsymbol{\xi}}^*$  and  $L_{\mathbf{x}}$

will be denoted by  $\bar{L}_\xi^*$  and  $\tilde{L}_x$ , respectively. Furthermore, we extend the resulting equations to  $\mathbb{R}^3$  and denote their solutions by  $\bar{g}$  and  $\tilde{g}$ . So we get, for each fixed  $\mathbf{x} \in \mathbb{R}^3$ :

$$(3.1) \quad \bar{L}_\xi^* \bar{g}(\mathbf{x}; \boldsymbol{\xi}) = -\varepsilon \Delta_\xi \bar{g}(\mathbf{x}; \boldsymbol{\xi}) + a(\mathbf{x}) \partial_{\xi_1} \bar{g}(\mathbf{x}; \boldsymbol{\xi}) = \delta(\mathbf{x} - \boldsymbol{\xi}) \quad \text{for } \boldsymbol{\xi} \in \mathbb{R}^3,$$

and similarly, for each fixed  $\boldsymbol{\xi} \in \mathbb{R}^3$ :

$$(3.2) \quad \tilde{L}_x \tilde{g}(\mathbf{x}; \boldsymbol{\xi}) = -\varepsilon \Delta_x \tilde{g}(\mathbf{x}; \boldsymbol{\xi}) - a(\boldsymbol{\xi}) \partial_{x_1} \tilde{g}(\mathbf{x}; \boldsymbol{\xi}) = \delta(\mathbf{x} - \boldsymbol{\xi}) \quad \text{for } \mathbf{x} \in \mathbb{R}^3.$$

As  $\mathbf{x}$  appears in (3.1) as a parameter, so the coefficient  $a(\mathbf{x})$  in this equation is considered constant and we can solve the problem explicitly. Setting  $q = \frac{1}{2}a(\mathbf{x})$  for fixed  $\mathbf{x} \in (0, 1)^3$  and  $\bar{g}(\mathbf{x}; \boldsymbol{\xi}) = V(\mathbf{x}; \boldsymbol{\xi}) e^{q\xi_1/\varepsilon}$  (see, e.g., [11]), one gets

$$-\varepsilon^2 \Delta_\xi V + q^2 V = \varepsilon e^{-q\xi_1/\varepsilon} \delta(\mathbf{x} - \boldsymbol{\xi}) = \varepsilon e^{-qx_1/\varepsilon} \delta(\mathbf{x} - \boldsymbol{\xi}).$$

As the fundamental solution for the operator  $-\varepsilon^2 \Delta_\xi + q^2$  is  $\frac{1}{4\pi\varepsilon^2} \frac{e^{-qr/\varepsilon}}{r}$  (see, e.g., [2]), so

$$V(\mathbf{x}; \boldsymbol{\xi}) = \varepsilon e^{-x_1 q/\varepsilon} \frac{1}{4\pi\varepsilon^2} \frac{e^{-rq/\varepsilon}}{r} \quad \text{where } r = \sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2}.$$

Finally, for the fundamental solution of (3.1) we get

$$\bar{g}(\mathbf{x}; \boldsymbol{\xi}) = \frac{1}{4\pi\varepsilon^2} \frac{e^{q(\xi_1 - x_1 - r)/\varepsilon}}{r}, \quad \text{where } q = q(\mathbf{x}) = \frac{1}{2}a(\mathbf{x}).$$

A similar argument yields the fundamental solution of (3.2)

$$\tilde{g}(\mathbf{x}; \boldsymbol{\xi}) = \frac{1}{4\pi\varepsilon^2} \frac{e^{q(\xi_1 - x_1 - r)/\varepsilon}}{r}, \quad \text{where } q = q(\boldsymbol{\xi}) = \frac{1}{2}a(\boldsymbol{\xi}).$$

Let

$$\begin{aligned} \hat{\xi}_{1,[x_1]} &= (\xi_1 - x_1)/\varepsilon, & \hat{\xi}_2 &= (\xi_2 - x_2)/\varepsilon, & \hat{\xi}_3 &= (\xi_3 - x_3)/\varepsilon, \\ \hat{r}_{[x_1]} &= \sqrt{\hat{\xi}_{1,[x_1]}^2 + \hat{\xi}_2^2 + \hat{\xi}_3^2}. \end{aligned}$$

As we shall need bounds for both  $\bar{g}$  and  $\tilde{g}$ , it is convenient to represent them via a more general function

$$(3.3) \quad g = g(\mathbf{x}; \boldsymbol{\xi}; q) := \frac{1}{4\pi\varepsilon^2} \frac{e^{q(\hat{\xi}_{1,[x_1]} - \hat{r}_{[x_1]})}}{\hat{r}_{[x_1]}}$$

as

$$(3.4) \quad \bar{g}(\mathbf{x}; \boldsymbol{\xi}) = g(\mathbf{x}; \boldsymbol{\xi}; q) \Big|_{q=\frac{1}{2}a(\mathbf{x})}, \quad \tilde{g}(\mathbf{x}; \boldsymbol{\xi}) = g(\mathbf{x}; \boldsymbol{\xi}; q) \Big|_{q=\frac{1}{2}a(\boldsymbol{\xi})}$$

We use the subindex  $[x_1]$  in  $\hat{\xi}_{1,[x_1]}$  and  $\hat{r}_{[x_1]}$  to highlight their dependence on  $x_1$  as in many places  $x_1$  will take different values; but when there is no ambiguity, we shall sometimes simply write  $\hat{\xi}_1$  and  $\hat{r}$ .

We now proceed to estimation of the derivatives of the generalised fundamental solution  $g = g(\mathbf{x}; \boldsymbol{\xi}; q)$ . A calculation shows that the first-order derivatives are given

by

$$(3.5a) \quad \partial_{\xi_1} g = \frac{1}{4\pi\varepsilon^3} \widehat{r}^{-2} \left[ q(\widehat{r} - \widehat{\xi}_1) - \frac{\widehat{\xi}_1}{\widehat{r}} \right] e^{q(\widehat{\xi}_1 - \widehat{r})},$$

$$(3.5b) \quad \partial_{\xi_k} g = -\frac{1}{4\pi\varepsilon^3} (q\widehat{r} + 1) \frac{\widehat{\xi}_k}{\widehat{r}^3} e^{q(\widehat{\xi}_1 - \widehat{r})}, \quad k = 2, 3,$$

$$(3.5c) \quad \partial_q g = \frac{1}{4\pi\varepsilon^2} \frac{\widehat{\xi}_1 - \widehat{r}}{\widehat{r}} e^{q(\widehat{\xi}_1 - \widehat{r})}.$$

Here we used  $\partial_{\xi_j} \widehat{r} = \varepsilon^{-1} \widehat{\xi}_j / \widehat{r}$  for  $j = 1, 2, 3$ . In a similar manner, but also using  $\partial_{\xi_i} (\widehat{\xi}_j / \widehat{r}) = -\varepsilon^{-1} \widehat{\xi}_i \widehat{\xi}_j / \widehat{r}^3$  with  $i \neq j$ , one gets second-order derivatives

$$(3.6a) \quad \partial_{\xi_1 \xi_k}^2 g = \frac{1}{4\pi\varepsilon^4} \frac{\widehat{\xi}_k}{\widehat{r}^3} \left[ q^2(\widehat{\xi}_1 - \widehat{r}) + q \frac{3\widehat{\xi}_1 - \widehat{r}}{\widehat{r}} + 3 \frac{\widehat{\xi}_1}{\widehat{r}^2} \right] e^{q(\widehat{\xi}_1 - \widehat{r})}, \quad k = 2, 3$$

$$(3.6b) \quad \partial_{\xi_1 q}^2 g = \frac{1}{4\pi\varepsilon^3} \widehat{r}^{-2} \left[ -q(\widehat{\xi}_1 - \widehat{r})^2 + \frac{\widehat{r}^2 - \widehat{\xi}_1^2}{\widehat{r}} \right] e^{q(\widehat{\xi}_1 - \widehat{r})},$$

$$(3.6c) \quad \partial_{\xi_k}^2 g = \frac{1}{4\pi\varepsilon^4} \widehat{r}^{-3} \left[ q^2 \widehat{\xi}_k^2 + (q\widehat{r} + 1) \frac{3\widehat{\xi}_k^2 - \widehat{r}^2}{\widehat{r}^2} \right] e^{q(\widehat{\xi}_1 - \widehat{r})}, \quad k = 2, 3.$$

Finally, combining  $\partial_{\xi_1}^2 g = -\partial_{\xi_2}^2 g - \partial_{\xi_3}^2 g + \frac{2q}{\varepsilon} \partial_{\xi_1} g$  with (3.5a) and (3.6c) yields

$$(3.6d) \quad \partial_{\xi_1}^2 g = \frac{1}{4\pi\varepsilon^4} \widehat{r}^{-3} \left[ q^2 (\widehat{r} - \widehat{\xi}_1)^2 - q(\widehat{r} - \widehat{\xi}_1) \left( 1 + 3 \frac{\widehat{\xi}_1}{\widehat{r}} \right) + \frac{3\widehat{\xi}_1^2 - \widehat{r}^2}{\widehat{r}^2} \right] e^{q(\widehat{\xi}_1 - \widehat{r})}.$$

Note that partial derivatives of  $g$  in variable  $q$  are needed to deal with the variable-coefficient case; their estimates can be found in the technical report [6].

**Remark 3.1.** *To compare the analysis that follows with the two-dimensional analysis of [7], note that in two dimensions, solutions  $\bar{g}$  and  $\tilde{g}$  of problems (3.1) and (3.2) are also given by (3.4), but, instead of (3.3), the fundamental solution  $g$  is given by*

$$g = g_{\mathbb{R}^2}(\mathbf{x}; \boldsymbol{\xi}; q) := \frac{1}{2\pi\varepsilon} e^{q\widehat{\xi}_1, [x_1]} K_0(q\widehat{r}_{[x_1]}) \quad \text{for } \mathbf{x}, \boldsymbol{\xi} \in \mathbb{R}^2.$$

Here  $K_0(\cdot)$  is the modified Bessel function of the second kind of order zero. Then, for example, one gets  $\partial_{\xi_1} g = \frac{q}{2\pi\varepsilon^2} e^{q\widehat{\xi}_1} \left[ K_0(q\widehat{r}) - \frac{\widehat{\xi}_1}{\widehat{r}} K_1(q\widehat{r}) \right]$  for the first derivative in the streamline direction, where we also used the modified Bessel function  $K_1(\cdot)$  of the second kind of order one. So in two dimensions, a certain difficulty lies in having to deal with the Bessel functions, for which one can simply employ asymptotic expansions [1]. Note also that in three dimensions, there are two cross-wind directions  $\xi_2$  and  $\xi_3$ , but the partial derivatives in these directions are similar and so will be bounded in a similar way.

We now show that the generalised fundamental solution  $g$  satisfies the bounds of Theorem 2.1.

**Lemma 3.2.** *Let  $\mathbf{x} \in [-1, 1] \times \mathbb{R}^2$  and  $0 < \frac{1}{2}\alpha \leq q \leq C$ . Then for the function  $g = g(\mathbf{x}; \boldsymbol{\xi}; q)$  of (3.3) we have the following bounds*

$$(3.7a) \quad \|g(\mathbf{x}; \cdot; q)\|_{1; \Omega} \leq C,$$

$$(3.7b) \quad \|\partial_{\xi_1} g(\mathbf{x}; \cdot; q)\|_{1; \Omega} \leq C(1 + |\ln \varepsilon|),$$

$$(3.7c) \quad \|\partial_{\xi_k} g(\mathbf{x}; \cdot; q)\|_{1; \Omega} \leq C \varepsilon^{-1/2}, \quad k = 2, 3,$$

and for any ball  $B(\mathbf{x}'; \rho)$  of radius  $\rho$  centred at any  $\mathbf{x}' \in [0, 1] \times \mathbb{R}^2$ , we have

$$(3.7d) \quad \|g(\mathbf{x}; \cdot; q)\|_{1,1;\Omega \cap B(\mathbf{x}'; \rho)} \leq C\varepsilon^{-1}\rho,$$

while for the ball  $B(\mathbf{x}; \rho)$  of radius  $\rho$  centred at  $\mathbf{x}$ , we have

$$(3.7e) \quad \|\partial_{\xi_1}^2 g(\mathbf{x}; \cdot; q)\|_{1;\Omega \setminus B(\mathbf{x}; \rho)} \leq C\varepsilon^{-1} \ln(2 + \varepsilon/\rho),$$

$$(3.7f) \quad \|\partial_{\xi_k}^2 g(\mathbf{x}; \cdot; q)\|_{1;\Omega \setminus B(\mathbf{x}; \rho)} \leq C\varepsilon^{-1} (\ln(2 + \varepsilon/\rho) + |\ln \varepsilon|), \quad k = 2, 3.$$

*Proof.* Throughout this proof, whenever  $k$  appears in any relation, it will be understood to be valid for  $k = 2, 3$  (as all the bounds in (3.7) that involve  $k$ , are given for both  $k = 2, 3$ ).

To estimate the derivatives of  $g$ , note that  $d\xi = \varepsilon^3 d\hat{\xi}$ , where  $\hat{\xi} \in \hat{\Omega} := \varepsilon^{-1}(-x_1, 1 - x_1) \times \mathbb{R}^2 \subset (-\infty, 2/\varepsilon) \times \mathbb{R}^2$ . Consider the two sub-domains

$$\hat{\Omega}_1 := \{ \hat{\xi}_1 < 1 + \frac{1}{2}\hat{r} \}, \quad \hat{\Omega}_2 := \{ \max\{1, \frac{1}{2}\hat{r}\} < \hat{\xi}_1 < 2/\varepsilon \}.$$

As  $\hat{\Omega} \subset \hat{\Omega}_1 \cup \hat{\Omega}_2$  for any  $x_1 \in [-1, 1]$ , it is convenient to consider integrals over these two sub-domains separately.

(i) Consider  $\hat{\xi} \in \hat{\Omega}_1$ . Then  $\hat{\xi}_1 \leq 1 + \frac{1}{2}\hat{r}$ , so one gets

$$(3.8) \quad \varepsilon^3 [\varepsilon^{-1}|g| + |\partial_{\xi_1} g| + |\partial_{\xi_k} g|] \leq C\hat{r}^{-2} (1 + \hat{r}) e^{q(\hat{\xi}_1 - \hat{r})} \leq C\hat{r}^{-2} e^{-q\hat{r}/4},$$

where we combined  $e^{q\hat{\xi}_1} \leq e^{q(1 + \hat{r}/2)}$  with  $(1 + \hat{r}) \leq Ce^{q\hat{r}/4}$ . This immediately yields

$$(3.9) \quad \iiint_{\hat{\Omega}_1} [\varepsilon^{-1}|g| + |\partial_{\xi_1} g| + |\partial_{\xi_k} g|] (\varepsilon^3 d\hat{\xi}) \leq C \int_0^\infty e^{-q\hat{r}/4} d\hat{r} \leq C.$$

Similarly,

$$\varepsilon^3 [|\partial_{\xi_1}^2 g| + |\partial_{\xi_k}^2 g|] \leq C\varepsilon^{-1} \hat{r}^{-3} (1 + \hat{r}^2) e^{q(\hat{\xi}_1 - \hat{r})} \leq C\varepsilon^{-1} \hat{r}^{-2} (\hat{r}^{-1} + \hat{r}) e^{-q\hat{r}/2},$$

so

$$(3.10) \quad \iiint_{\hat{\Omega}_1 \setminus B(\mathbf{0}; \hat{\rho})} [|\partial_{\xi_1}^2 g| + |\partial_{\xi_k}^2 g|] (\varepsilon^3 d\hat{\xi}) \leq C\varepsilon^{-1} \int_{\hat{\rho}}^\infty (\hat{r}^{-1} + \hat{r}) e^{-q\hat{r}/2} d\hat{r} \leq C\varepsilon^{-1} \ln(2 + \hat{\rho}^{-1}).$$

Furthermore, for an arbitrary ball  $\hat{B}_{\hat{\rho}}$  of radius  $\hat{\rho}$  in the coordinates  $\hat{\xi}$ , we get

$$(3.11) \quad \iiint_{\hat{\Omega}_1 \cap \hat{B}_{\hat{\rho}}} [|g| + |\partial_{\xi_1} g| + |\partial_{\xi_k} g|] (\varepsilon^3 d\hat{\xi}) \leq C \int_0^{\hat{\rho}} e^{-q\hat{r}/4} d\hat{r} \leq C \min\{\hat{\rho}, 1\}.$$

(ii) Next consider  $\hat{\xi} \in \hat{\Omega}_2$ . In this sub-domain, it is convenient to rewrite the integrals in terms of  $(\hat{\xi}_1, t_2, t_3)$ , where

$$(3.12) \quad t_k := \hat{\xi}_1^{-1/2} \hat{\xi}_k, \quad \Rightarrow \quad \hat{\xi}_1^{-1/2} d\hat{\xi}_k = dt_k, \quad \hat{r} - \hat{\xi}_1 = \frac{\hat{\xi}_2^2 + \hat{\xi}_3^2}{\hat{r} + \hat{\xi}_1} \leq t_2^2 + t_3^2 =: t^2.$$

Note that  $\hat{\xi}_1 \leq \hat{r} \leq 2\hat{\xi}_1$  in  $\hat{\Omega}_2$  so  $\hat{r} - \hat{\xi}_1 = (\hat{\xi}_2^2 + \hat{\xi}_3^2)/(\hat{r} + \hat{\xi}_1) \geq c_0 t^2$ , where  $c_0 := \frac{1}{3}$ . Consequently  $e^{-q(\hat{r} - \hat{\xi}_1)} \leq e^{-qc_0 t^2}$  or

$$(3.13) \quad e^{-q(\hat{r} - \hat{\xi}_1)} \leq C\hat{r}Q, \quad \text{where } Q := \hat{\xi}_1^{-1} e^{-qc_0 t^2},$$

and

$$(3.14) \quad \begin{aligned} & \iint_{\mathbb{R}^2} (1 + t + t^2 + t^3 + t^4) Q d\hat{\xi}_2 d\hat{\xi}_3 \\ &= \iint_{\mathbb{R}^2} (1 + t + t^2 + t^3 + t^4) e^{-qc_0 t^2} dt_2 dt_3 \leq C. \end{aligned}$$

Using (3.5),(3.6) and (3.12) it is straightforward to prove the following bounds for  $g$  and its derivatives in  $\widehat{\Omega}_2$

$$(3.15a) \quad \varepsilon^3 |g| \leq C \varepsilon Q,$$

$$(3.15b) \quad \varepsilon^3 |\partial_{\xi_k} g| \leq C \widehat{\xi}_1^{-1/2} t Q,$$

$$(3.15c) \quad \varepsilon^3 |\partial_{\xi_k}^2 g| \leq C \varepsilon^{-1} \widehat{\xi}_1^{-1} (1 + t^2) Q,$$

and also

$$(3.15d) \quad \varepsilon^3 |\partial_{\xi_1} g| \leq C \widehat{\xi}_1^{-1} (1 + t^2) Q,$$

$$(3.15e) \quad \varepsilon^3 |\partial_{\xi_1}^2 g| \leq C \varepsilon^{-1} \widehat{\xi}_1^{-2} (1 + t^2 + t^4) Q.$$

Combining the obtained estimates (3.15) with (3.14) yields

$$(3.16) \quad \iiint_{\widehat{\Omega}_2} [|g| + \varepsilon^{1/2} |\partial_{\xi_k} g| + |\partial_{\xi_1}^2 g|] (\varepsilon^3 d\widehat{\xi}) \leq C \int_1^{2/\varepsilon} [\varepsilon + \varepsilon^{1/2} \widehat{\xi}_1^{-1/2}] d\widehat{\xi}_1 \leq C.$$

Similarly, combining (3.15c) and (3.15d) with (3.14) yields

$$(3.17) \quad \iiint_{\widehat{\Omega}_2} [|\partial_{\xi_1} g| + \varepsilon |\partial_{\xi_k}^2 g|] (\varepsilon^3 d\widehat{\xi}) \leq C \int_1^{2/\varepsilon} \widehat{\xi}_1^{-1} d\widehat{\xi}_1 \leq C(1 + |\ln \varepsilon|).$$

Furthermore, by (3.15b), and (3.15d) for an arbitrary ball  $\widehat{B}_{\widehat{\rho}}$  of radius  $\widehat{\rho}$  in the coordinates  $\widehat{\xi}$ , we get

$$(3.18) \quad \iiint_{\widehat{\Omega}_2 \cap \widehat{B}_{\widehat{\rho}}} (|g| + |\partial_{\xi_1} g| + |\partial_{\xi_k} g|) (\varepsilon^3 d\widehat{\xi}) \leq C \int_1^{1+\widehat{\rho}} [\varepsilon + \widehat{\xi}_1^{-1} + \widehat{\xi}_1^{-1/2}] d\widehat{\xi}_1 \leq C\widehat{\rho}.$$

To complete the proof, we now recall that  $\widehat{\Omega} \subset \widehat{\Omega}_1 \cup \widehat{\Omega}_2$  and combine estimates (3.9) and (3.10) (that involve integration over  $\widehat{\Omega}_1$ ) with (3.16) and (3.17), which yields the desired bounds (3.7a)-(3.7c) and (3.7e), (3.7f). To get the latter two bounds we also used the observation that the ball  $B(\mathbf{x}; \rho)$  of radius  $\rho$  in the coordinates  $\xi$  becomes the ball  $B(\mathbf{0}; \widehat{\rho})$  of radius  $\widehat{\rho} = \varepsilon^{-1}\rho$  in the coordinates  $\widehat{\xi}$ . The remaining assertion (3.7d) is obtained by combining (3.11) with (3.18) and noting that an arbitrary ball  $B(\mathbf{x}'; \rho)$  of radius  $\rho$  in the coordinates  $\xi$  becomes a ball  $\widehat{B}_{\widehat{\rho}}$  of radius  $\widehat{\rho} = \varepsilon^{-1}\rho$  in the coordinates  $\widehat{\xi}$ .  $\square$

Our next result shows that for  $x_1 \geq 1$ , one gets stronger bounds for  $g$  and its derivatives. These bounds involve the weight function

$$(3.19) \quad \lambda := e^{2q(x_1-1)/\varepsilon}.$$

and show that, although  $\lambda$  is exponentially large in  $\varepsilon$ , this is compensated by the smallness of  $g$  and its derivatives.

**Lemma 3.3.** *Let  $\mathbf{x} \in [1, 3] \times \mathbb{R}^2$  and  $0 < \frac{1}{2}\alpha \leq q \leq C$ . Then for the function  $g = g(\mathbf{x}; \xi; q)$  of (3.3) and the weight  $\lambda$  of (3.19), one has the following bounds*

$$(3.20a) \quad \|(\lambda g)(\mathbf{x}; \cdot; q)\|_{1; \Omega} \leq C\varepsilon,$$

$$(3.20b) \quad \|(\lambda \partial_{\xi_1} g)(\mathbf{x}; \cdot; q)\|_{1; \Omega} + \|(\lambda \partial_{\xi_k} g)(\mathbf{x}; \cdot; q)\|_{1; \Omega} \leq C, \quad k = 2, 3,$$

and for any ball  $B(\mathbf{x}'; \rho)$  of radius  $\rho$  centred at any  $\mathbf{x}' \in [0, 1] \times \mathbb{R}^2$ , one has

$$(3.20c) \quad \|(\lambda g)(\mathbf{x}; \cdot; q)\|_{1, 1; \Omega \cap B(\mathbf{x}'; \rho)} \leq C\varepsilon^{-1}\rho,$$



while for the ball  $B(\mathbf{x}; \rho)$  of radius  $\rho$  centred at  $\mathbf{x}$  and  $k = 2, 3$ , one has

(3.20d)

$$\|(\lambda \partial_{\xi_1}^2 g)(\mathbf{x}; \cdot; q)\|_{1; \Omega \setminus B(\mathbf{x}, \rho)} + \|(\lambda \partial_{\xi_k}^2 g)(\mathbf{x}; \cdot; q)\|_{1; \Omega \setminus B(\mathbf{x}, \rho)} \leq C \varepsilon^{-1} \ln(2 + \varepsilon/\rho).$$

Finally, for some positive constant  $c_1$  one has

$$(3.21) \quad \|\lambda g(\mathbf{x}; \cdot)\|_{2,1; [0, \frac{1}{3}] \times \mathbb{R}^2} \leq C e^{-c_1 \alpha / \varepsilon}.$$

*Proof.* Throughout this proof, whenever  $k$  appears in any relation, it will be understood to be valid for  $k = 2, 3$  (as all the bounds in (3.20), that involve  $k$ , are given for both  $k = 2, 3$ ).

We shall use the notation  $A = A(x_1) := (x_1 - 1)/\varepsilon \geq 0$ . Then (3.19) becomes  $\lambda = e^{2qA}$ . We partially imitate the proof of Lemma 3.2. Again  $d\hat{\xi} = \varepsilon^3 d\hat{\xi}$ , but now  $\hat{\xi} \in \hat{\Omega} = \varepsilon^{-1}(-x_1, 1 - x_1) \times \mathbb{R}^2 \subset (-3/\varepsilon, -A) \times \mathbb{R}^2$ . So  $\hat{\xi}_1 < -A \leq 0$  immediately yields

$$(3.22) \quad \lambda e^{q\hat{\xi}_1} = e^{2q(A - |\hat{\xi}_1|)} e^{q|\hat{\xi}_1|} \leq e^{q|\hat{\xi}_1|}.$$

Consider the sub-domains

$$\begin{aligned} \hat{\Omega}'_1 &:= \{ |\hat{\xi}_1| < 1 + \frac{1}{2}\hat{r}, \quad \hat{\xi}_1 < -A \}, \\ \hat{\Omega}'_2 &:= \{ |\hat{\xi}_1| > \max\{1, \frac{1}{2}\hat{r}\}, \quad -3/\varepsilon < \hat{\xi}_1 < -A \}. \end{aligned}$$

As  $\hat{\Omega} \subset \hat{\Omega}'_1 \cup \hat{\Omega}'_2$  for any  $x_1 \in [1, 3]$ , we estimate integrals over these two domains separately.

(i) Let  $\hat{\xi} \in \hat{\Omega}'_1$ . Then  $|\hat{\xi}_1| \leq 1 + \frac{1}{2}\hat{r}$  so, by (3.22), one has  $\lambda e^{q\hat{\xi}_1} \leq e^{q(1 + \hat{r}/2)}$ . The first inequality in (3.8) remains valid, but now we combine it with

$$(3.23) \quad \lambda e^{q(\hat{\xi}_1 - \hat{r})} (1 + \hat{r}) \leq C e^{-q\hat{r}/4}$$

(which is obtained similarly to the final line in (3.8)). This leads to a version of (3.9) that involves the weight  $\lambda$ :

$$(3.24) \quad \iiint_{\hat{\Omega}'_1} \lambda [\varepsilon^{-1}|g| + |\partial_{\xi_1} g| + |\partial_{\xi_k} g|] (\varepsilon^3 d\hat{\xi}) \leq C.$$

In a similar manner, we obtain versions of estimates (3.10) and (3.11), that also involve the weight  $\lambda$ :

$$(3.25) \quad \iiint_{\hat{\Omega}'_1 \setminus B(\mathbf{0}; \hat{\rho})} \lambda |\partial_{\xi_k}^2 g| (\varepsilon^3 d\hat{\xi}) \leq C \varepsilon^{-1} \ln(2 + \hat{\rho}^{-1}),$$

$$(3.26) \quad \iiint_{\hat{\Omega}'_1 \cap \hat{B}_{\hat{\rho}}} \lambda [|g| + |\partial_{\xi_1} g| + |\partial_{\xi_k} g|] (\varepsilon^3 d\hat{\xi}) \leq C \min\{\hat{\rho}, 1\},$$

where  $\hat{B}_{\hat{\rho}}$  is an arbitrary ball of radius  $\hat{\rho}$  in the coordinates  $\hat{\xi}$ .

(ii) Now consider  $\hat{\xi} \in \hat{\Omega}'_2$ . In this sub-domain (similarly to  $\hat{\Omega}_2$  in the proof of Lemma 3.2) one has  $|\hat{\xi}_1| \leq \hat{r} \leq 2|\hat{\xi}_1|$  and  $c_0 t^2 \leq \hat{r} - |\hat{\xi}_1| \leq t^2$ , where  $t_k := |\hat{\xi}_1|^{-1/2} \hat{\xi}_k$  and  $t^2 := t_2^2 + t_3^2$ , (compare with (3.12)). We also introduce a new barrier  $Q$

$$(3.27) \quad Q := \lambda^{-1} e^{2q(A - |\hat{\xi}_1|)} \{ |\hat{\xi}_1|^{-1} e^{-qc_0 t^2} \} \Rightarrow e^{-q(\hat{r} - \hat{\xi}_1)} \leq C \hat{r} Q,$$

(compare with (3.13); to get the bound for  $e^{-q(\hat{r} - \hat{\xi}_1)}$  we used (3.22)).

With the new definition (3.27) of  $Q$ , the bounds (3.15a)–(3.15c) remain valid in  $\hat{\Omega}'_2$  only with  $\hat{\xi}_1$  replaced by  $|\hat{\xi}_1|$ . Note that the bound (3.15d) is not valid in  $\hat{\Omega}'_2$ , (as it was obtained using  $\hat{r} - \hat{\xi}_1 \leq t^2$ , which is not the case for  $\hat{\xi}_1 < 0$ ). Instead,

using  $\widehat{r} \geq |\widehat{\xi}_1| \geq 1$  and  $\widehat{r} \leq 2|\widehat{\xi}_1|$ , we prove, directly from (3.5) the following bound in  $\widehat{\Omega}'_2$ :

$$(3.28) \quad \varepsilon^3 |\partial_{\xi_1} g| \leq C Q.$$

Next, note that (3.14) is valid with  $Q$  replaced by the multiplier  $\{|\widehat{\xi}_1|^{-1} e^{-q c_0 t^2}\}$  from the current definition (3.27) of  $Q$ . Combining this observation with the bounds (3.15a)–(3.15c) and (3.28), and also with  $\widehat{r} \leq 2|\widehat{\xi}_1|$ , yields

$$(3.29) \quad \begin{aligned} & \iiint_{\widehat{\Omega}'_2} \lambda [\varepsilon^{-1} |g| + |\partial_{\xi_1} g| + |\partial_{\xi_k} g| + \varepsilon |\partial_{\xi_k}^2 g|] (\varepsilon^3 d\widehat{\xi}) \\ & \leq C \int_{-3/\varepsilon}^{-\max\{A,1\}} [1 + |\widehat{\xi}_1|^{-1/2} + |\widehat{\xi}_1|^{-1}] e^{2q(A-|\widehat{\xi}_1|)} d\widehat{\xi}_1 \leq C. \end{aligned}$$

Furthermore, by (3.15b), and (3.28), for an arbitrary ball  $\widehat{B}_{\widehat{\rho}}$  of radius  $\widehat{\rho}$  in the coordinates  $\widehat{\xi}$ , we get

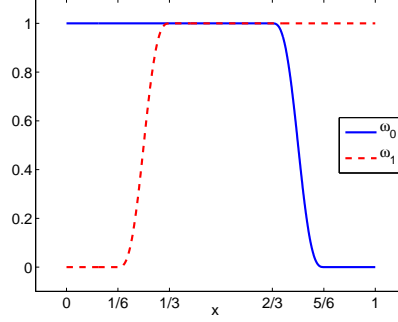
$$(3.30) \quad \begin{aligned} & \iiint_{\widehat{\Omega}'_2 \cap \widehat{B}_{\widehat{\rho}}} \lambda [|g| + |\partial_{\xi_1} g| + |\partial_{\xi_k} g|] (\varepsilon^3 d\widehat{\xi}) \\ & \leq C \int_{-\max\{A,1\}-\widehat{\rho}}^{-\max\{A,1\}} [1 + |\widehat{\xi}_1|^{-1/2}] e^{2q(A-|\widehat{\xi}_1|)} d\widehat{\xi}_1 \leq C \widehat{\rho}. \end{aligned}$$

To complete the proof of (3.20), we now recall that  $\widehat{\Omega} \subset \widehat{\Omega}'_1 \cup \widehat{\Omega}'_2$  and combine estimates (3.24), (3.25) (that involve integration over  $\widehat{\Omega}'_1$ ) with (3.29), which yields the desired bounds (3.20a)–(3.20b) and the bounds for  $\partial_{\xi_2}^2 g$  and  $\partial_{\xi_3}^2 g$  in (3.20d). To get the latter two bounds we also used the observation that the ball  $B(\mathbf{x}; \rho)$  of radius  $\rho$  in the coordinates  $\xi$  becomes the ball  $B(\mathbf{0}; \widehat{\rho})$  of radius  $\widehat{\rho} = \varepsilon^{-1} \rho$  in the coordinates  $\widehat{\xi}$ . The bound for  $\partial_{\xi_1}^2 g$  in (3.20d) follows as  $\partial_{\xi_1}^2 g = -\partial_{\xi_2}^2 g - \partial_{\xi_3}^2 g + \frac{2q}{\varepsilon} \partial_{\xi_1} g$  for  $\xi \neq \mathbf{x}$ . The remaining assertion (3.20c) is obtained by combining (3.26) with (3.30) and noting that an arbitrary ball  $B(\mathbf{x}'; \rho)$  of radius  $\rho$  in the coordinates  $\xi$  becomes a ball  $\widehat{B}_{\widehat{\rho}}$  of radius  $\widehat{\rho} = \varepsilon^{-1} \rho$  in the coordinates  $\widehat{\xi}$ . Thus we have established all the bounds (3.20).

(iii) It now remains to establish (3.21), for which we imitate above proof only now  $\xi_1 < \frac{1}{3}$  or  $\widehat{\xi}_1 < (\frac{1}{3} - x_1)/\varepsilon \leq -\frac{2}{3}/\varepsilon$ . Thus instead of the sub-domains  $\widehat{\Omega}'_1$  and  $\widehat{\Omega}'_2$  we now consider  $\widehat{\Omega}''_1$  and  $\widehat{\Omega}''_2$  defined by  $\widehat{\Omega}''_k := \widehat{\Omega}'_k \cap \{\widehat{\xi}_1 < -(x_1 - \frac{1}{3})/\varepsilon\}$ . So in  $\widehat{\Omega}''_1$  (3.23) remains valid with  $q \geq \frac{1}{2}\alpha$ , but now  $\widehat{r} > \frac{2}{3}/\varepsilon$ . Therefore, when we integrate over  $\widehat{\Omega}''_1$  (instead of  $\widehat{\Omega}'_1$ ), the integrals of type (3.24), (3.25) become bounded by  $C e^{-c_1 \alpha/\varepsilon}$  for any fixed  $c_1 < \frac{1}{8}$ . Next, when considering integrals over  $\widehat{\Omega}''_2$  (instead of  $\widehat{\Omega}'_2$ ), note that  $A - |\widehat{\xi}_1| \leq -\frac{2}{3}/\varepsilon$  so the quantity  $e^{2q(A-|\widehat{\xi}_1|)}$  in the definition (3.27) of  $Q$  is now bounded by  $e^{-\frac{2}{3}\alpha/\varepsilon}$ . Consequently, the integrals of type (3.29) over  $\widehat{\Omega}''_2$  also become bounded by  $C e^{-c_1 \alpha/\varepsilon}$ .  $\square$

#### 4. Approximations for the Green's Function

In Section 3, we have already found bounds for the fundamental solutions of the two frozen-coefficient equations (3.1) and (3.2) posed in  $\mathbb{R}^3$ . But these fundamental solutions do not fulfil the Dirichlet boundary conditions on  $\partial\Omega$ . Our purpose in this section is to introduce and estimate frozen-coefficient approximations  $\bar{G}$  and  $\tilde{G}$  of  $G$  that will vanish on the boundary. We consider a simpler domain  $\Omega = (0, 1) \times \mathbb{R}^2$  in the first part of this section, and the original domain  $\Omega = (0, 1)^3$  in the second part. Note that although  $\bar{G}$  and  $\tilde{G}$  will be constructed as solution approximations for the

FIGURE 2. An example of  $\omega_0$  and corresponding  $\omega_1$ .

frozen-coefficient equations, we shall see that they, in fact, provide approximations to the Green's function  $G$  for our original variable-coefficient problem. We shall employ two related cut-off functions  $\omega_0$  and  $\omega_1$  defined by

$$(4.1) \quad \begin{aligned} \omega_0(t) &\in C^2(0, 1), \quad \omega_0(t) = 1 \text{ for } t \leq \frac{2}{3}, \quad \omega_0(t) = 0 \text{ for } t \geq \frac{5}{6}; \\ \omega_1(t) &:= \omega_0(1 - t), \end{aligned}$$

so  $\omega_k(k) = 1$ ,  $\omega_k(1 - k) = 0$  and  $\omega'_k(x) = \omega''_k(x) = 0$  for  $k = 0, 1$  and  $x = 0, 1$ ; see Figure 2.

**4.1. Approximations  $\bar{G}$  and  $\tilde{G}$  in the domain  $\Omega = (0, 1) \times \mathbb{R}^2$ .** First, consider a simpler domain  $\Omega := (0, 1) \times \mathbb{R}^2$ . To construct approximations  $\bar{G}$  and  $\tilde{G}$  in this domain, we employ the method of images with an inclusion of the cut-off functions of (4.1). So, using the fundamental solution  $g$  of (3.3), we define

$$(4.2) \quad \bar{G}(\mathbf{x}; \boldsymbol{\xi}) := \bar{\mathcal{G}}|_{q=\frac{1}{2}a(\mathbf{x})}, \quad \tilde{G}(\mathbf{x}; \boldsymbol{\xi}) := \tilde{\mathcal{G}}|_{q=\frac{1}{2}a(\boldsymbol{\xi})},$$

where

$$(4.3a) \quad \bar{\mathcal{G}}(\mathbf{x}; \boldsymbol{\xi}; q) := \frac{e^{q\hat{\xi}_1}}{4\pi\varepsilon^2} \left\{ \left[ \frac{e^{-q\hat{r}_{[x_1]}}}{\hat{r}_{[x_1]}} - \frac{e^{-q\hat{r}_{[-x_1]}}}{\hat{r}_{[-x_1]}} \right] - \left[ \frac{e^{-q\hat{r}_{[2-x_1]}}}{\hat{r}_{[2-x_1]}} - \frac{e^{-q\hat{r}_{[2+x_1]}}}{\hat{r}_{[2+x_1]}} \right] \omega_1(\xi_1) \right\}$$

$$(4.3b) \quad \tilde{\mathcal{G}}(\mathbf{x}; \boldsymbol{\xi}; q) := \frac{e^{q\hat{\xi}_1}}{4\pi\varepsilon^2} \left\{ \left[ \frac{e^{-q\hat{r}_{[x_1]}}}{\hat{r}_{[x_1]}} - \frac{e^{-q\hat{r}_{[2-x_1]}}}{\hat{r}_{[2-x_1]}} \right] - \left[ \frac{e^{-q\hat{r}_{[-x_1]}}}{\hat{r}_{[-x_1]}} - \frac{e^{-q\hat{r}_{[2+x_1]}}}{\hat{r}_{[2+x_1]}} \right] \omega_0(x_1) \right\}$$

Recall that  $\hat{r}_{[x_1]} = \sqrt{\hat{\xi}_{1,[x_1]}^2 + \hat{\xi}_2^2 + \hat{\xi}_3^2}$  where  $\hat{\xi}_{1,[x_1]} = (\xi_1 - x_1)/\varepsilon$ . So for  $\xi_1 = 0$  one has  $\hat{r}_{[x_1]} = \hat{r}_{[-x_1]}$  and  $\omega_1(\xi_1) = 0$ , and therefore  $\bar{G}|_{\xi_1=0} = 0$ . Similarly, for  $\xi_1 = 1$  one has  $\hat{r}_{[x_1]} = \hat{r}_{[2-x_1]}$  and  $\hat{r}_{[-x_1]} = \hat{r}_{[2+x_1]}$ , while  $\omega_1(\xi_1) = 1$ ; consequently  $\bar{G}|_{\xi_1=1} = 0$ . In a similar manner one can check that  $\tilde{G}|_{x_1=0,1} = 0$  so the Dirichlet boundary conditions in streamline direction are satisfied.

Note that  $\bar{G}$  does not satisfy the equation (2.1a), while  $\tilde{G}$  does not satisfy the equation (2.3a) (recall the these equations are satisfied by  $G$ ). But  $\bar{G}$  and  $\tilde{G}$  almost satisfy similar frozen-coefficient equations (3.1) and (3.2), respectively, in the following sense. Let us look at the two residual functions

$$(4.4) \quad \bar{\phi}(\mathbf{x}; \boldsymbol{\xi}) = \bar{L}_{\boldsymbol{\xi}}^* \bar{G} - L_{\boldsymbol{\xi}}^* G, \quad \tilde{\phi}(\mathbf{x}; \boldsymbol{\xi}) := \tilde{L}_{\mathbf{x}} \tilde{G} - L_{\mathbf{x}} G.$$

They are bounded as follows.

**Lemma 4.1.** *Let  $\mathbf{x} \in \Omega = (0, 1) \times \mathbb{R}^2$ . Then for the functions  $\bar{\phi}$  and  $\tilde{\phi}$  of (4.4) some positive constant  $c_1$  and  $k = 2, 3$ , one has*

$$(4.5) \quad \|\bar{\phi}(\mathbf{x}; \cdot)\|_{1,1;\Omega} + \|\partial_{x_k} \bar{\phi}(\mathbf{x}; \cdot)\|_{1;\Omega} + \|\tilde{\phi}(\mathbf{x}; \cdot)\|_{1,1;\Omega} \leq C e^{-c_1 \alpha/\varepsilon} \leq C.$$

We also have

$$(4.6) \quad \bar{\phi}(\mathbf{x}; \boldsymbol{\xi})|_{\boldsymbol{\xi} \in \partial\Omega} = 0.$$

*Proof.* The proof essentially imitates the one for [7, Lemma 5.1] with the only difference of having two cross-wind directions instead of one.  $\square$

The next result shows that  $\bar{G}$  and  $\tilde{G}$  satisfy some of the bounds given by Theorem 2.1 for  $G$ .

**Lemma 4.2.** *The functions  $\bar{G}$  and  $\tilde{G}$  of (4.2) satisfy*

$$(4.7a) \quad \|\bar{G}(\mathbf{x}; \cdot)\|_{1;\Omega} + \|\tilde{G}(\mathbf{x}; \cdot)\|_{1;\Omega} \leq C,$$

$$(4.7b) \quad \|\partial_{\xi_1} \bar{G}(\mathbf{x}; \cdot)\|_{1;\Omega} \leq C(1 + |\ln \varepsilon|),$$

$$(4.7c) \quad \|\partial_{\xi_k} \bar{G}(\mathbf{x}; \cdot)\|_{1;\Omega} \leq C\varepsilon^{-1/2}, \quad k = 2, 3,$$

and for any ball  $B(\mathbf{x}'; \rho)$  of radius  $\rho$  centred at any  $\mathbf{x}' \in [0, 1] \times \mathbb{R}^2$ , one has

$$(4.7d) \quad |\bar{G}(\mathbf{x}; \cdot)|_{1,1;B(\mathbf{x}';\rho) \cap \Omega} \leq C\varepsilon^{-1}\rho,$$

while for the ball  $B(\mathbf{x}; \rho)$  of radius  $\rho$  centred at  $\mathbf{x}$ , we have

$$(4.7e) \quad \|\partial_{\xi_1}^2 \bar{G}(\mathbf{x}; \cdot)\|_{1;\Omega \setminus B(\mathbf{x};\rho)} \leq C\varepsilon^{-1} \ln(2 + \varepsilon/\rho),$$

$$(4.7f) \quad \|\partial_{\xi_k}^2 \bar{G}(\mathbf{x}; \cdot)\|_{1;\Omega \setminus B(\mathbf{x};\rho)} \leq C\varepsilon^{-1} (\ln(2 + \varepsilon/\rho) + |\ln \varepsilon|), \quad k = 2, 3.$$

*Proof.* We imitate the proof of [7, Lemma 5.2] with only minor modifications. In particular, one rewrites the definitions of  $\bar{G}$  and  $\tilde{G}$  using the notation

$$(4.8a) \quad g_{[d]} := g(d, x_2, x_3; \boldsymbol{\xi}; q) = \frac{1}{4\pi\varepsilon^2} \frac{e^{q(\hat{\xi}_{1,[d]} - \hat{r}_{[d]})}}{\hat{r}_{[d]}},$$

$$(4.8b) \quad \lambda^\pm := e^{2q(1 \pm x_1)/\varepsilon}, \quad p := e^{-2qx_1/\varepsilon},$$

and the observation that

$$(4.9) \quad \frac{1}{4\pi\varepsilon^2} \frac{e^{q(\hat{\xi}_{1,[x_1]} - \hat{r}_{[d]})}}{\hat{r}_{[d]}} = e^{q(d-x_1)/\varepsilon} g_{[d]} \quad \text{for } d = \pm x_1, 2 \pm x_1.$$

So

$$(4.10a) \quad \bar{G}(\mathbf{x}; \boldsymbol{\xi}; q) = [g_{[x_1]} - p g_{[-x_1]}] - [\lambda^- g_{[2-x_1]} - p \lambda^+ g_{[2+x_1]}] \omega_1(\xi_1),$$

$$(4.10b) \quad \tilde{G}(\mathbf{x}; \boldsymbol{\xi}; q) = [g_{[x_1]} - \lambda^- g_{[2-x_1]}] - [p g_{[-x_1]} - p \lambda^+ g_{[2+x_1]}] \omega_0(x_1).$$

Note that  $\lambda^\pm$  is obtained by replacing  $x_1$  by  $2 \pm x_1$  in the definition (3.19) of  $\lambda$ . The desired bounds are now obtained from (4.10), where the derivatives of the terms that involve  $g_{[\pm x_1]}$  are estimated using an extended version of Lemma 3.2, which can be found in [6, Lemma 4.1]. Similarly, the derivatives of the terms that involve  $\lambda^\pm g_{[2 \pm x_1]}$  are estimated using an extended version of Lemma 3.3, which can be found in [6, Lemma 4.2]. (Some estimates have been omitted in Lemmas 3.2 and 3.3 to simplify the presentation; their extended versions [6, Lemmas 4.1 and 4.2] also give bounds for the partial derivatives of  $g$  in variable  $q$ , which are needed to deal with  $q = \frac{1}{2} a(\mathbf{x})$  and  $q = \frac{1}{2} a(\boldsymbol{\xi})$ .)  $\square$

**4.2. Approximations for the Green's function in the domain  $\Omega = (0, 1)^3$ .** Now we are prepared to define approximations, denoted by  $\bar{G}_{\square}$  and  $\tilde{G}_{\square}$ , for the Green's function  $G$  in the original domain  $\Omega = (0, 1)^3$ . For this, we employ the previously defined approximations  $\bar{G}$  and  $\tilde{G}$  of (4.2), (4.3) for the domain  $(0, 1) \times \mathbb{R}^2$ . We again use the method of images with an inclusion of the cut-off functions of (4.1) in a two-step process as follows:

$$(4.11a) \quad \begin{aligned} \bar{G}_{\square}(\mathbf{x}; \boldsymbol{\xi}) &:= \bar{G}(\mathbf{x}; \boldsymbol{\xi}) - \omega_0(\xi_2) \bar{G}(\mathbf{x}; \xi_1, -\xi_2, \xi_3) \\ &\quad - \omega_1(\xi_2) \bar{G}(\mathbf{x}; \xi_1, 2 - \xi_2, \xi_3), \\ \tilde{G}_{\square}(\mathbf{x}; \boldsymbol{\xi}) &:= \tilde{G}_{\square}(\mathbf{x}; \boldsymbol{\xi}) - \omega_0(\xi_3) \tilde{G}_{\square}(\mathbf{x}; \xi_1, \xi_2, -\xi_3) \\ &\quad - \omega_1(\xi_3) \tilde{G}_{\square}(\mathbf{x}; \xi_1, \xi_2, 2 - \xi_3), \end{aligned}$$

$$(4.11b) \quad \begin{aligned} \bar{G}_{\square}(\mathbf{x}; \boldsymbol{\xi}) &:= \bar{G}(\mathbf{x}; \boldsymbol{\xi}) - \omega_0(x_2) \bar{G}(x_1, -x_2, x_3; \boldsymbol{\xi}) \\ &\quad - \omega_1(x_2) \bar{G}(x_1, 2 - x_2, x_3; \boldsymbol{\xi}), \\ \tilde{G}_{\square}(\mathbf{x}; \boldsymbol{\xi}) &:= \tilde{G}_{\square}(\mathbf{x}; \boldsymbol{\xi}) - \omega_0(x_3) \tilde{G}_{\square}(x_1, x_2, -x_3; \boldsymbol{\xi}) \\ &\quad - \omega_1(x_3) \tilde{G}_{\square}(x_1, x_2, 2 - x_3; \boldsymbol{\xi}). \end{aligned}$$

The constructed approximations  $\bar{G}_{\square}$  and  $\tilde{G}_{\square}$  vanish on the boundary. Indeed,  $\bar{G}_{\square}|_{\xi_1=0,1} = 0$  and  $\tilde{G}_{\square}|_{x_1=0,1} = 0$  (as this is valid for  $\bar{G}$  and  $\tilde{G}$ , respectively), and furthermore, by (4.1), we have  $\bar{G}_{\square}|_{\xi_k=0,1} = 0$  and  $\tilde{G}_{\square}|_{x_k=0,1} = 0$  for  $k = 2, 3$ .

**Remark 4.3.** Lemmas 4.1 and 4.2 remain valid if  $\Omega$  is understood as  $(0, 1)^3$ , and  $\bar{G}$  and  $\tilde{G}$  are replaced by  $\bar{G}_{\square}$  and  $\tilde{G}_{\square}$ , respectively, in the definition (4.4) of  $\bar{\phi}$  and  $\tilde{\phi}$  and in the lemma statements.

This is shown by imitating the proofs of these two lemmas. We leave out the details and only note that the application of the method of images in the  $\xi_2$ - and  $\xi_3$ - ( $x_2$ - and  $x_3$ -) directions is relatively straightforward as an inspection of (3.3) shows that in these directions, the fundamental solution  $g$  is symmetric and exponentially decaying away from the singular point.

## 5. Proof of Theorem 2.1 for $\Omega = (0, 1)^3$ (general variable-coefficient case)

We are now ready to establish our main result, Theorem 2.1, for the original variable-coefficient problem (1.1) in the domain  $\Omega = (0, 1)^3$ . In Section 4, we have already obtained various bounds for the approximations  $\bar{G}_{\square}$  and  $\tilde{G}_{\square}$  of  $G$  in  $\Omega = (0, 1)^3$ ; in particular, we have shown that they satisfy the bounds of Theorem 2.1. So now it remains to show that these bounds are also satisfied by the two functions

$$\tilde{v}(\mathbf{x}; \boldsymbol{\xi}) := [G - \tilde{G}_{\square}](\mathbf{x}; \boldsymbol{\xi}), \quad \bar{v}(\mathbf{x}; \boldsymbol{\xi}) := [G - \bar{G}_{\square}](\mathbf{x}; \boldsymbol{\xi}).$$

Note that, by (4.4), we have  $L_{\mathbf{x}}\tilde{v} = L_{\mathbf{x}}[G - \tilde{G}_{\square}] = [\tilde{L}_{\mathbf{x}} - L_{\mathbf{x}}]\tilde{G}_{\square} - \tilde{\phi}$ , and similarly  $L_{\boldsymbol{\xi}}^*\bar{v} = L_{\boldsymbol{\xi}}^*[G - \bar{G}_{\square}] = [\tilde{L}_{\boldsymbol{\xi}}^* - L_{\boldsymbol{\xi}}^*]\bar{G}_{\square} - \bar{\phi}$ . Consequently, the function  $\tilde{v}$  is a solution of the following problem:

$$(5.1a) \quad L_{\mathbf{x}}\tilde{v}(\mathbf{x}; \boldsymbol{\xi}) = \tilde{h}(\mathbf{x}; \boldsymbol{\xi}) \text{ for } \mathbf{x} \in \Omega, \quad \tilde{v}(\mathbf{x}; \boldsymbol{\xi}) = 0 \text{ for } \mathbf{x} \in \partial\Omega,$$

where the right-hand side is given by

$$(5.1b) \quad \tilde{h}(\mathbf{x}; \boldsymbol{\xi}) := \partial_{x_1}\{R\tilde{G}_{\square}\}(\mathbf{x}; \boldsymbol{\xi}) - b(\mathbf{x})\tilde{G}_{\square}(\mathbf{x}; \boldsymbol{\xi}) - \tilde{\phi}(\mathbf{x}; \boldsymbol{\xi}),$$

and

$$(5.2) \quad R(\mathbf{x}; \boldsymbol{\xi}) := a(\mathbf{x}) - a(\boldsymbol{\xi}), \quad \text{so } |R| \leq C \min\{\varepsilon\hat{r}_{|x_1|}, 1\}.$$

Similarly,  $\bar{v}$  is a solution to the problem

$$(5.3a) \quad L_{\xi}^* \bar{v}(\mathbf{x}; \xi) = \bar{h}(\mathbf{x}; \xi) \text{ for } \xi \in \Omega, \quad \bar{v}(\mathbf{x}; \xi) = 0 \text{ for } \xi \in \partial\Omega$$

where the right-hand side is given by

$$(5.3b) \quad \bar{h}(\mathbf{x}; \xi) := \{R \partial_{\xi_1} \bar{G}_{\boxplus}\}(\mathbf{x}; \xi) - b(\xi) \bar{G}_{\boxplus}(\mathbf{x}; \xi) - \bar{\phi}(\mathbf{x}; \xi).$$

Then the solution representation formulae (2.2) and (2.5) applied to problems (5.1a) and (5.3a), respectively, yield

$$(5.4a) \quad \tilde{v}(\mathbf{x}; \xi) = \iiint_{\Omega} G(\mathbf{x}; \mathbf{s}) \tilde{h}(\mathbf{s}; \xi) d\mathbf{s},$$

and

$$(5.4b) \quad \bar{v}(\mathbf{x}; \xi) = \iiint_{\Omega} G(\mathbf{s}; \xi) \bar{h}(\mathbf{x}; \mathbf{s}) d\mathbf{s}.$$

To complete the proof, it remains to show that either  $\tilde{v}$  or  $\bar{v}$ , represented by (5.4), satisfies each of the bounds of Theorem 2.1. For this one imitates, with only mild modifications, the argument given in [7, Section 6] (for further details, we refer the reader to an extended version of this paper [6]). Thus, we have Theorem 2.1.

## 6. Stability Corollaries and Applications

In view of the solution representation (2.2), the bounds of the Green's function given by Theorem 2.1 imply a number of a priori solution estimates for our original problem (1.1). Such estimates can be used in the numerical analysis of this computationally challenging problem. More specifically, they can be used to derive a posteriori error bounds for computed solutions of this problem.

One straightforward stability corollary is as follows.

**Corollary 6.1.** *Let  $f(\mathbf{x}) = \partial_{x_1} \bar{F}_1(\mathbf{x}) + \partial_{x_2} \bar{F}_2(\mathbf{x}) + \partial_{x_3} \bar{F}_3(\mathbf{x}) + \bar{f}(\mathbf{x})$  where  $\bar{F}_1, \bar{F}_2, \bar{F}_3, \bar{f} \in L_{\infty}(\Omega)$ . Then for the solution  $u$  of problem (1.1) we have the bound*

$$\|u\|_{\infty; \Omega} \leq C \left[ (1 + |\ln \varepsilon|) \|\bar{F}_1\|_{\infty; \Omega} + \varepsilon^{-1/2} (\|\bar{F}_2\|_{\infty; \Omega} + \|\bar{F}_3\|_{\infty; \Omega}) + \|\bar{f}\|_{\infty; \Omega} \right].$$

*Proof.* In view of the linearity of the operator  $L$ , we first combine the solution representation (2.2) with the bounds (2.6a), (2.6b) to deal with the terms in  $f$  that involve  $\bar{F}_m$ ,  $m = 1, 2, 3$ . The remaining solution component, that corresponds to the right-hand side  $\bar{f}$ , is easily estimated using the maximum/comparison principle.  $\square$

In fact, our problem enjoys a more subtle stability result. To formulate it, we first introduce a tensor-product mesh  $\{x_{1,i}, x_{2,j}, x_{3,k}\}$  in the domain  $\Omega = (0, 1)^3$ , with  $N_k + 1$  vertices in each coordinate direction, where

$$0 = x_{\ell,0} < x_{\ell,1} < \cdots < x_{\ell,N_k-1} < x_{\ell,N_k} = 1, \quad \ell = 1, 2, 3.$$

The local mesh sizes are  $h_{\ell,i} := x_{\ell,i} - x_{\ell,i-1}$  for  $i = 1, \dots, N_{\ell}$  and  $\ell = 1, 2, 3$ . Our next stability corollary is as follows.

**Corollary 6.2.** *Let*

$$(6.1) \quad f(\mathbf{x}) = \partial_{x_1} [F_1(\mathbf{x}) + \bar{F}_1(\mathbf{x})] + \partial_{x_2} [F_2(\mathbf{x}) + \bar{F}_2(\mathbf{x})] + \partial_{x_3} [F_3(\mathbf{x}) + \bar{F}_3(\mathbf{x})] + \bar{f}(\mathbf{x}),$$

where  $F_1, F_2, F_3, \bar{F}_1, \bar{F}_2, \bar{F}_3, \bar{f} \in L_\infty(\Omega)$ ,  $A_i, B_j, C_k \in L_\infty((0, 1)^2)$  for all  $i, j, k$ , and

$$\begin{aligned} F_1(\mathbf{x}) &= A_i(x_2, x_3)(x_1 - x_{1,i-1/2}), & \mathbf{x} \in (x_{1,i-1}, x_{1,i}) \times (0, 1)^2, & \quad i = 1, \dots, N_1, \\ F_2(\mathbf{x}) &= B_j(x_1, x_3)(x_2 - x_{2,j-1/2}), & \mathbf{x} \in (0, 1) \times (x_{2,j-1}, x_{2,j}) \times (0, 1), & \quad j = 1, \dots, N_2, \\ F_3(\mathbf{x}) &= C_k(x_1, x_2)(x_3 - x_{3,k-1/2}), & \mathbf{x} \in (0, 1)^2 \times (x_{3,k-1}, x_{3,k}), & \quad k = 1, \dots, N_3. \end{aligned}$$

Then for the solution  $u$  of problem (1.1) we have the bound

$$\begin{aligned} \|u\|_{\infty;\Omega} \leq C & \left[ (1 + |\ln \varepsilon|) \|\bar{F}_1\|_{\infty;\Omega} + \varepsilon^{-1/2} (\|\bar{F}_2\|_{\infty;\Omega} + \|\bar{F}_3\|_{\infty;\Omega}) + \|\bar{f}\|_{\infty;\Omega} \right. \\ & + \max_{i=1, \dots, N_1} \left\{ \min \left\{ h_{1,i} (1 + |\ln \varepsilon|), \frac{h_{1,i}^2}{\varepsilon} \ln(2 + \varepsilon/\kappa_1) \right\} \|A_i\|_{\infty;(0,1)^2} \right\} \\ & + \max_{j=1, \dots, N_2} \left\{ \min \left\{ \frac{h_{2,i}}{\varepsilon^{1/2}}, \frac{h_{2,i}^2}{\varepsilon} (\ln(2 + \varepsilon/\kappa_2) + |\ln \varepsilon|) \right\} \|B_j\|_{\infty;(0,1)^2} \right\} \\ & \left. + \max_{k=1, \dots, N_3} \left\{ \min \left\{ \frac{h_{3,i}}{\varepsilon^{1/2}}, \frac{h_{3,i}^2}{\varepsilon} (\ln(2 + \varepsilon/\kappa_3) + |\ln \varepsilon|) \right\} \|C_k\|_{\infty;(0,1)^2} \right\} \right] \end{aligned}$$

with  $\kappa_m = \min_{i=1 \dots N_m} \{h_{m,i}\}$  for  $m = 1, 2, 3$ .

*Proof.* In view of the linearity of the operator  $L$  and our earlier Corollary 6.1, it suffices to establish the desired result for the case  $\bar{F}_1 = \bar{F}_2 = \bar{F}_3 = \bar{f} := 0$ . Furthermore, we can deal with the solution components induced by each of  $F_1, F_2, F_3$  separately.

(i) First we deal with  $F_1$  so let  $\bar{F}_1 = \bar{F}_2 = \bar{F}_3 = \bar{f} = F_2 = F_3 := 0$ . Here we imitate the argument used in [2, 15] to deal with a simpler reaction-diffusion operator. Fix  $\mathbf{x} \in \Omega$  and use the notation  $W(\boldsymbol{\xi}) = G(\mathbf{x}; \boldsymbol{\xi})$  for the Green's function. Now the solution representation (2.2) yields

$$\begin{aligned} u(\mathbf{x}) &= - \iiint_{\Omega} F_1(\boldsymbol{\xi}) \partial_{\xi_1} W(\boldsymbol{\xi}) d\boldsymbol{\xi} \\ &= - \sum_{i=1}^{N_1} \iiint_{\Omega_i} A_i(\xi_2, \xi_3) (\xi_1 - \xi_{1,i-1/2}) \partial_{\xi_1} W(\boldsymbol{\xi}) d\boldsymbol{\xi}, \end{aligned}$$

where  $\Omega_i := (x_{1,i-1}, x_{1,i}) \times (0, 1)^2$ .

We shall consider the integrals over a certain neighbourhood  $\Omega'$  of the singular point  $\boldsymbol{\xi} = \mathbf{x}$  of  $G$  separately. Suppose  $x_1 \in [x_{1,n-1/2}, x_{1,n+1/2}]$  for some  $0 < n < N_1$  (the cases  $x_1 \in [0, x_{1/2}]$  and  $x_1 \in (x_{N_1-1/2}, 1]$  are similar). Now let

$$\Omega' := (x_{1,n-1}, x_{1,n+1}) \times (x_2 - \frac{1}{2}\kappa_1, x_2 + \frac{1}{2}\kappa_1) \times (x_3 - \frac{1}{2}\kappa_1, x_3 + \frac{1}{2}\kappa_1),$$

and define a regular function  $\widetilde{W}$  by

$$\widetilde{W}(\boldsymbol{\xi}) = \begin{cases} W(\boldsymbol{\xi}) & \text{for } \boldsymbol{\xi} \in \Omega \setminus \Omega', \\ 0 & \text{for } \boldsymbol{\xi} \in \Omega'. \end{cases}$$

With this definitions, rewrite  $u$  as

$$u(\mathbf{x}) = - \underbrace{\sum_{i=1}^{N_1} \iiint_{\Omega_i} A_i(\xi_2, \xi_3) (\xi_1 - \xi_{1,i-1/2}) \partial_{\xi_1} \widetilde{W}(\boldsymbol{\xi}) d\boldsymbol{\xi}}_{=:S_1} - \underbrace{\sum_{i=n}^{n+1} \iiint_{\Omega_i \cap \Omega'} A_i(\xi_2, \xi_3) (\xi_1 - \xi_{1,i-1/2}) \partial_{\xi_1} W(\boldsymbol{\xi}) d\boldsymbol{\xi}}_{=:S_2}.$$

For the first sum  $S_1$  we have

$$|S_1| \leq \sum_{i=1}^{N_1} \|A_i\|_{\infty; (0,1)^2} \iint_{(0,1)^2} \left| \int_{x_{1,i-1}}^{x_{1,i}} (\xi_1 - \xi_{1,i-1/2}) \partial_{\xi_1} \widetilde{W}(\boldsymbol{\xi}) d\xi_1 \right| d\xi_2 d\xi_3.$$

Here the interior-integral terms can be estimated using either

$$\left| \int_{x_{1,i-1}}^{x_{1,i}} (\xi_1 - x_{1,i-1/2}) \partial_{\xi_1} \widetilde{W}(\boldsymbol{\xi}) d\xi_1 \right| \leq \frac{1}{2} h_{1,i} \int_{x_{1,i-1}}^{x_{1,i}} |\partial_{\xi_1} \widetilde{W}(\boldsymbol{\xi})| d\xi_1$$

or

$$\begin{aligned} \left| \int_{x_{1,i-1}}^{x_{1,i}} (\xi_1 - x_{1,i-1/2}) \partial_{\xi_1} \widetilde{W}(\boldsymbol{\xi}) d\xi_1 \right| &= \left| \int_{x_{1,i-1}}^{x_{1,i}} (\xi_1 - x_{1,i-1/2}) \int_{x_{1,i-1}}^{\xi_1} \partial_{\xi_1}^2 \widetilde{W}(s, \xi_2, \xi_3) ds d\xi_1 \right| \\ &\leq \frac{1}{4} h_{1,i}^2 \int_{x_{1,i-1}}^{x_{1,i}} |\partial_{\xi_1}^2 \widetilde{W}(\boldsymbol{\xi})| d\xi_1. \end{aligned}$$

where  $\partial_{\xi_1}^2 \widetilde{W}(\boldsymbol{\xi})$  is well defined and integrable. Consequently, for any splitting  $\|A_i\|_{\infty; (0,1)^2} = \mathcal{A}_i^1 + \mathcal{A}_i^2$ , one gets

$$|S_1| \leq \max_{i=1, \dots, N_1} \left\{ \frac{1}{2} h_{1,i} \mathcal{A}_i^1 \right\} \cdot \|\partial_{\xi_1} W\|_{1; \Omega \setminus \Omega'} + \max_{i=1, \dots, N_1} \left\{ \frac{1}{4} h_{1,i}^2 \mathcal{A}_i^2 \right\} \cdot \|\partial_{\xi_1}^2 W\|_{1; \Omega \setminus \Omega'}.$$

As  $W(\boldsymbol{\xi}) = G(\boldsymbol{\xi}; \mathbf{x})$ , the bounds (2.6a) and (2.6d) respectively imply  $\|\partial_{\xi_1} W\|_{1; \Omega \setminus \Omega'} \leq C(1 + |\ln \varepsilon|)$  and  $\|\partial_{\xi_1}^2 W\|_{1; \Omega \setminus \Omega'} \leq C\varepsilon^{-1} \ln(2 + \varepsilon/\kappa_1)$ , where to get the second estimate we also used  $B(\mathbf{x}; \frac{1}{2}\kappa_1) \subset \Omega'$ . Now, let  $\mathcal{A}_i^1 := \|A_i\|_{\infty; (0,1)^2}$  and  $\mathcal{A}_i^2 := 0$  if  $\frac{1}{2} h_{1,i}(1 + |\ln \varepsilon|) \leq \frac{1}{4} h_{1,i}^2 \varepsilon^{-1} \ln(2 + \varepsilon/\kappa_1)$ , and  $\mathcal{A}_i^1 := 0$  and  $\mathcal{A}_i^2 := \|A_i\|_{\infty; (0,1)^2}$  otherwise. This definition yields

$$(6.2) \quad |S_1| \leq \max_{i=1, \dots, N_1} \left\{ \min \left\{ h_{1,i}(1 + |\ln \varepsilon|), \frac{h_{1,i}^2}{\varepsilon} \ln(2 + \varepsilon/\kappa_1) \right\} \|A_i\|_{\infty; (0,1)^2} \right\}.$$

In a similar manner, for  $S_2$  we get

$$(6.3) \quad |S_2| \leq \sum_{i=n, n+1} \left\{ \frac{1}{2} h_{1,i} \|A_i\|_{\infty; (0,1)^2} \|\partial_{\xi_1} W\|_{1; \Omega_i \cap \Omega'} \right\}.$$

Here  $(\Omega_i \cap \Omega') \subset B((x_{1,i-1/2}, x_2, x_3); h_{1,i})$ , so, combining (2.6a) and (2.6c) yields

$$(6.4) \quad \|\partial_{\xi_1} W\|_{1; \Omega_i \cap \Omega'} \leq C \min \{ (1 + |\ln \varepsilon|), \varepsilon^{-1} h_{1,i} \}.$$

As the sum in (6.3) involves only two terms, combining (6.3) and (6.4), we conclude that the bound (6.2) for  $S_1$  holds true for  $S_2$  as well. Comparing the right-hand side in (6.2) with the assertion of the corollary, we are done.

(ii) In a similar manner, to deal with  $F_2$  one lets  $\bar{F}_1 = \bar{F}_2 = \bar{F}_3 = \bar{f} = F_1 = F_3 := 0$ , and to deal with  $F_3$  one lets  $\bar{F}_1 = \bar{F}_2 = \bar{F}_3 = \bar{f} = F_1 = F_2 := 0$ . Then the argument used in part (i) is imitated with the only modification that the



bounds of Theorem 2.1 for the crosswind derivatives of the Green's function are now employed.  $\square$

**Applications.** Stability properties of our convection-reaction-diffusion problem such as given by Corollaries 6.1 and 6.2 can be used in the numerical analysis of this problem. In particular, Corollary 6.2 is a key ingredient in deriving a robust **a posteriori error estimator** for a finite difference method in a forthcoming paper [5]. The error is estimated in the maximum norm, which is a sufficiently strong norm to capture layers. Any dependence on the small singular perturbation parameter will be shown explicitly. One particular feature of our estimator is that it holds true on an arbitrary tensor-product mesh under no mesh aspect ratio restrictions. This is crucial as highly-anisotropic meshes are needed for efficient numerical solution of singularly perturbed problems such as our problem (1.1).

To establish robust a posteriori error estimates for problem (1.1), we combine and extend the arguments used in [2, 15] to deal with a singularly perturbed reaction-diffusion operator in two and three dimensions, and the a posteriori error analysis of a one-dimensional convection-diffusion equation [14]. More specifically, the residual of the continuous (or interpolated) computed solution  $u^N$ , which is defined by

$$L_{\mathbf{x}}[u^N - u] = L_{\mathbf{x}}u^N - f,$$

is represented in the form (6.1). Then Corollary 6.2 is applied to bound the error  $\|u^N - u\|_{\infty; \Omega}$  in terms of the given arbitrary mesh and the computed solution obtained on this mesh. This yields a robust a posteriori error estimator [5].

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