# UNIVERSITY *of* LIMERICK

OLLSCOIL LUIMNIGH

Faculty of Science and Engineering
Department of Mathematics and Statistics

MODULE CODE: MS4008

SEMESTER: Autumn 2013

MODULE TITLE: Numerical Solution of PDEs

DURATION OF EXAMINATION: $2\frac{1}{2}$ hours

LECTURER: Dr A. Hegarty

PERCENTAGE OF TOTAL MARKS: 80 %

EXTERNAL EXAMINER: Prof. T. Myers

INSTRUCTIONS TO CANDIDATES:
**Answer questions 1, 2 and 3.**

**To obtain maximum marks you must show all your work clearly and in detail.**

**Standard mathematical tables are provided by the invigilators. Under no circumstances should you use your own tables or be in possession of any written material other than that provided by the invigilators.**

**You must obey the examination rules of the University. Any breaches of these rules (and in particular any attempt at cheating) will result in disciplinary proceedings. For a first offence this can result in a year's suspension from the University.**

**1. Answer part (a) and one of parts (b) and (c).** 33%

(a) Let the space $V = \{\text{all functions } v \in H^1(0,1) \text{ such that } v(1) = 0\}$ with the norm 15%

$$\|v\| = \sqrt{\int_0^1 (v'^2(x) + v^2(x))\,dx}\,.$$

Let the bilinear form $a(\cdot,\cdot)$ be defined by

$$a(v,u) = \int_0^1 v'(x)\,u'(x)\,dx \quad \text{for any } v,u \in V.$$

- Show that this bilinear form is symmetric.
  Then show that, for some positive constants $\alpha$ and $\gamma$, one has

$$\alpha\|v\|^2 \le a(v,v) \le \gamma\|v\|^2 \quad \text{for all } v \in V.$$

  Specify the constants $\alpha$ and $\gamma$.
- Suppose that $V^h$ is a finite-dimensional subspace of $V$ and let $u \in V$ and $u_h \in V^h$ be such that

$$a(u - u_h, w_h) = 0 \quad \text{for all } w_h \in V^h.$$

  Prove that this implies

$$a(u - u_h, u - u_h) \le a(u - v_h, u - v_h) \quad \text{for all } v_h \in V^h.$$

- Then prove that

$$\|u - u_h\| \le C\|u - v_h\| \quad \text{for all } v_h \in V^h$$

  and specify the positive constant $C$ here. (Hint: use $\alpha$ and $\gamma$).

18%

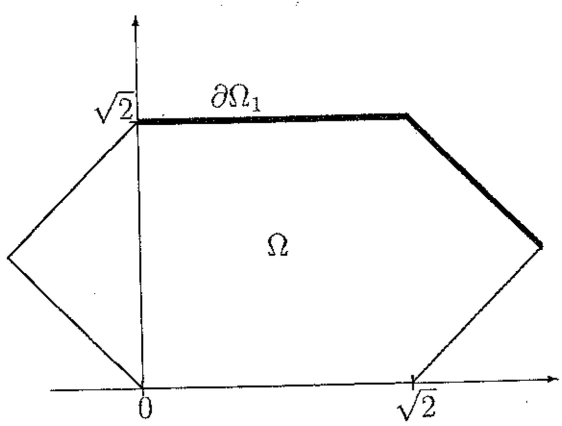
(b) In a two-dimensional domain $\Omega$ consider the problem:

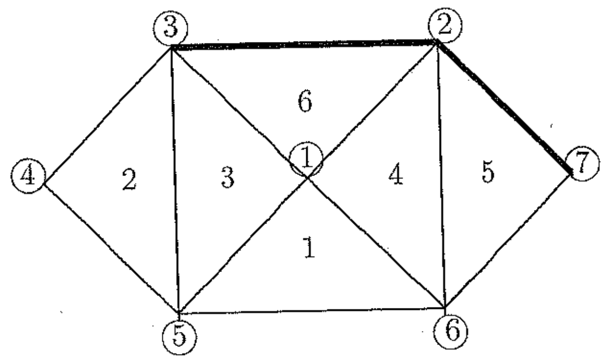
$$-\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=f \quad \text{for } (x,y)\in\Omega,$$

$$u(x,y)=0 \quad \text{for } (x,y)\in\partial\Omega_1, \qquad \frac{\partial u(x,y)}{\partial n}=0 \quad \text{for } (x,y)\in\partial\Omega_2,$$

where $f$ is constant and $\partial\Omega=\partial\Omega_1\cup\partial\Omega_2$ is the boundary of $\Omega$.

This problem is discretized using *linear finite elements*, where the domain $\Omega$ and its triangulation are as follows (the bold line indicates $\partial\Omega_1$ and the rest of the boundary is $\partial\Omega_2$):

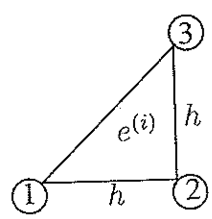



For this discretisation:

- Find the global stiffness matrix $K_{(f)}$ and the global load vector $F_{(f)}$ in which the boundary conditions are ignored.
- Find the global stiffness matrix $K$ and the global load vector $F$ which take the boundary conditions into consideration.
- Then write the numerical method as a linear system $KU=F$. For each entry of the unknown vector $U$ specify with which mesh node it is associated.

Note that for the linear element




the local stiffness matrix $K^{(i)}$ and the local load vector $F^{(i)}$ are given by

$$K^{(i)}=\frac{1}{2}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}, \qquad F^{(i)}=\frac{fh^2}{6}\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

18%

(c) Consider the problem:

$$-u'' + 4u = f \qquad \text{for } x \in (0,1), \qquad u'(0) = u(1) = 0.$$

- Obtain a weak formulation of this problem.
(Note: you should specify the space to which $u$ belongs, from which space test functions $v$ are taken, and which boundary conditions $u$ and $v$ are required to satisfy, if any.)
- Suppose this problem is discretised using piecewise **linear** finite elements with the local shape functions $\phi_1^{(i)}$ and $\phi_2^{(i)}$, defined on each mesh element $e^{(i)} = (x_i, x_{i+1})$, with $h_i = x_{i+1} - x_i$, by

$$\phi_k^{(i)} = \varphi_k\left(\frac{x - x_i}{h_i}\right), \quad k = 1, 2, \qquad \varphi_1(t) = 1 - t, \;\; \varphi_2(t) = t.$$

Find the local stiffness matrix $K^{(i)}$ and the local load vector $F^{(i)}$, assuming that $f$ is constant.
- On the mesh $\{x_i\}_{i=1}^5$, where $x_i = \frac{1}{4}(i-1)$ and $h_1 = h_2 = h_3 = h_3 = \frac{1}{4}$, write the above numerical method as a linear system $KU = F$, where $K$ is the global stiffness matrix, $F$ is the global load vector, and $U$ is the computed-solution vector. For each entry of the unknown vector $U$, specify with which mesh node it is associated.
(Note: do not forget to address the boundary conditions.)

**2. Answer part (a) and any *two* of parts (b), (c), (d).** **24%**

Let $\varepsilon$ be a positive constant. In the square domain $\Omega = (0,1) \times (0,1)$ with the boundary $\partial\Omega$ consider the problem:

$$Lu = -\varepsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + 2\frac{\partial u}{\partial x} + u = f(x,y) \qquad \text{for } (x,y) \in \Omega,$$

$$u(x,y) = 0 \qquad \text{for } (x,y) \in \partial\Omega.$$

This problem is discretised on the uniform mesh $\{(x_i, y_j)\}_{i,j=1,\dots,N+1}$, where $x_i = (i-1)h,\ y_j = (j-1)h,\ h = 1/N$, by the finite difference method:

$$L^h U_{ij} = -\varepsilon\,\frac{U_{i-1,j} + U_{i+1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{ij}}{h^2} + 2\,\frac{U_{i+1,j} - U_{i-1,j}}{2h} + U_{ij} = f(x_i, y_j)$$

for $i, j = 2, \dots, N$, with the boundary conditions:

$$U_{ij} = 0 \quad \text{for } (x_i, y_j) \in \partial\Omega.$$

(a) Estimate the local truncation error $r_{ij} = -L^h u(x_i, y_j) + f(x_i, y_j)$ for this method, where $u(x_i, y_j)$ is the exact solution at the mesh node $(x_i, y_j)$. 8%

(b) Show that the finite difference operator $L^h$, possibly under a certain condition that involves $h$ and $\varepsilon$, satisfies the discrete maximum principle in the form: 8%

$$\left.\begin{array}{r} L^h V_{ij} \le 0 \quad \forall\, i, j = 2, \dots, N \\ V_{ij} \le 0 \quad \forall\, (x_i, y_j) \in \partial\Omega \end{array}\right\} \Rightarrow V_{ij} \le 0 \;\forall\, i, j = 1, \dots, N+1.$$

Specify the discrete-maximum-principle condition on $h$, if any.

(c) Using the maximum principle described in part (b), show that 8%

$$V_{ij} = 0 \ \ \forall\, (x_i, y_j) \in \partial\Omega \ \Rightarrow \max_{i,j=1,\dots,N+1} |V_{ij}| \le C \max_{i,j=2,\dots,N} |L^h V_{ij}|$$

for some positive constant $C$. Specify this constant.

(d) Using the result of part (a) and the property described in part (c), estimate the error of the finite difference method 8%

$$\max_{i,j=1,\dots,N+1} |U_{ij} - u(x_i, y_j)|,$$

where $U_{ij}$ is the computed solution, and $u(x_i, y_j)$ is the exact solution at the mesh node $(x_i, y_j)$.

**3. Answer one of parts (a) or (b) and answer part (c).** 23%

Consider the problem

$$\begin{aligned} u_t &= u_{xx} && \text{for } x \in (0,1),\ t > 0, \\ u(0,t) = u(1,t) &= 0 && \text{for } t > 0, \\ u(x,0) &= g(x) && \text{for } x \in [0,1]. \end{aligned}$$

This problem is discretised on the uniform mesh

$$\{(x_j, t_m),\ j = 1, ..., N+1,\ m = 1, 2, \ldots\},$$

where $x_j = (j-1)h$ with $h = 1/N$, and $t_m = (m-1)k$. Let $U_j^m$ be the computed solution associated with the point $(x_j, t_m)$.

(a) Using Von Neumann's method, prove that the *Leap Frog* method 11%

$$\frac{U_j^{m+1} - U_j^{m-1}}{2k} = \frac{U_{j-1}^m - 2U_j^m + U_{j+1}^m}{h^2},$$

is *unconditionally unstable.*

(b) Using Von Neumann's method, find out whether the *Du Fort-Frankel* method 11%

$$\frac{U_j^{m+1} - U_j^{m-1}}{2k} = \frac{U_{j-1}^m - U_j^{m+1} - U_j^{m-1} + U_{j+1}^m}{h^2}$$

is unconditionally stable, unconditionally unstable, or conditionally stable. If it is conditionally stable, find the stability condition.

(c) The linearised shallow water equations,

$$\begin{aligned} H_t + v_0 H_x + H_0 v_x &= 0 \\ v_t + v_0 v_x + g H_x &= 0 \end{aligned}$$

where $H_0$ and $v_0$ are positive constants, may be written as

$$U_t + AU_x = 0$$

where $U = \begin{pmatrix} H \\ v \end{pmatrix}$ and $A$ is the constant matrix $\begin{pmatrix} v_0 & H_0 \\ g & v_0 \end{pmatrix}$.

Let $U_j^m$ be the computed solution vector associated with the point $(x_j, t_m)$. Assuming that $v_0 > 0$, show that the explicit upwinded method

$$\frac{U_j^{m+1} - U_j^m}{k} + A\frac{U_j^m - U_{j-1}^m}{h} = 0$$

is stable if a CFL condition is satisfied. 12