UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF MATHEMATICS & STATISTICS

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4008 | SEMESTER: Autumn 2009/10

MODULE TITLE: Numerical Partial Differential Equations | DURATION OF EXAMINATION: $2\frac{1}{2}$ hours

LECTURER: Dr. N. Kopteva | PERCENTAGE OF TOTAL MARKS: 75%

INSTRUCTIONS TO CANDIDATES:
Answer questions 1, 2, and 3.

To obtain maximum marks you must show all your work clearly and in detail.
Standard mathematical tables are provided by the invigilators. Under no circumstances should you use your own tables or be in possession of any written material other than that provided by the invigilators.
Non-programmable, non-graphical calculators that have been approved by the lecturer are permitted.
You must obey the examination rules of the University. Any breaches of these rules (and in particular any attempt at cheating) will result in disciplinary proceedings. For a first offence this can result in a year's suspension from the University.

1 Answer part (a) and one of parts (b) and (c). **30%**

. . . .

(a) Let the space V be $H^1(0,1)$ with the norm 15%

$$\|v\| = \sqrt{\int_0^1 (v'^2(x) + v^2(x))\,dx}\,.$$

Let the bilinear form $a(\cdot,\cdot)$ be defined by

$$a(v,u) = \int_0^1 \big(5\,v'(x)\,u'(x) + v(x)\,u(x)\big)\,dx \quad \text{for any } v,u \in H^1(0,1).$$

- Show that this bilinear form is symmetric. Then show that, for some positive constants α and γ, it satisfies

$$\alpha\|v\|^2 \le a(v,v) \le \gamma\|v\|^2 \quad \text{for all } v \in V.$$

Specify the constants α and γ.

- Suppose that V^h is a finite-dimensional subspace of V and for some functions $u \in V$ and $u_h \in V^h$ we have

$$a(u-u_h, u-u_h) \le a(u-v_h, u-v_h) \quad \text{for all } v_h \in V^h.$$

Prove that

$$\|u-u_h\| \le C\|u-v_h\| \quad \text{for all } v_h \in V^h.$$

Specify the constant C here in terms of α and γ.

. . . .

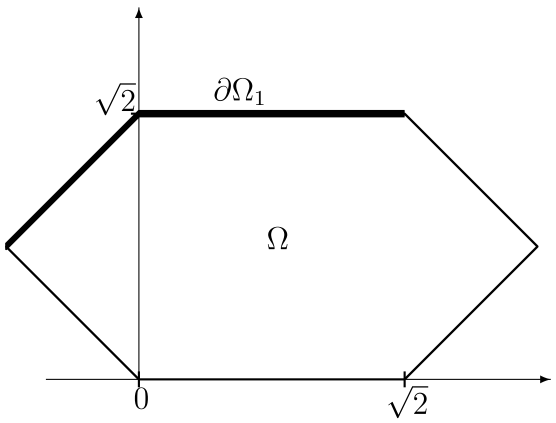
(b) In a two-dimensional domain Ω consider the problem: 15%

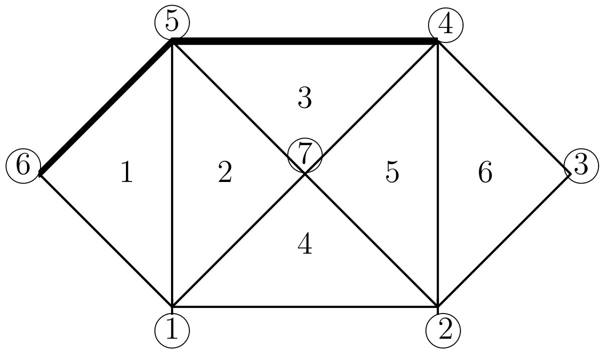
$$-\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f, \quad (x,y) \in \Omega,$$

$$u(x,y) = 0, \quad (x,y) \in \partial\Omega_1; \qquad \frac{\partial u(x,y)}{\partial \mathbf{n}} = 0, \quad (x,y) \in \partial\Omega_2;$$

where $f = \text{const}$ and $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ is the boundary of Ω.

This problem is discretized using *linear finite elements*, where the domain Ω and its triangulation are as follows:

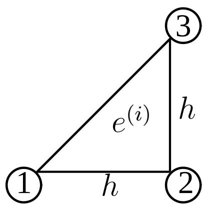




For this discretization:

- Find the global stiffness matrix $K_{(f)}$ and the global load vector $F_{(f)}$ in which the boundary conditions are ignored.
- Find the global stiffness matrix K and the global load vector F which take the boundary conditions into consideration.
- Then write the numerical method as a linear system $KU = F$. For each entry of the unknown vector U specify with which mesh node it is associated.

Note that for the linear element



the local stiffness matrix $K^{(i)}$ and the local load vector $F^{(i)}$ are given by

$$K^{(i)} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \qquad F^{(i)} = \frac{fh^2}{6}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

. . . .

(c) Consider the problem: 15%

$$-u'' = f, \quad x \in (0,1), \qquad u'(0) = u(1) = 0.$$

- Obtain a weak formulation of this problem.
  (Note: you are expected to specify the space in which u is found, from which space arbitrary functions v are taken, and which boundary conditions u and v are required to satisfy if any.)
- Suppose this problem is discretized using piecewise **linear** finite elements with the local shape functions $\phi_1^{(i)}$ and $\phi_2^{(i)}$ defined on each element $e^{(i)} = [x_i, x_{i+1}]$, with $h_i = x_{i+1} - x_i$, by

  $$\phi_k^{(i)} = \varphi_k\left(\frac{x - x_i}{h_i}\right), \quad k = 1, 2, \qquad \varphi_1(t) = 1 - t, \quad \varphi_2(t) = t.$$

  Find the local stiffness matrix $K^{(i)}$ and the local load vector $F^{(i)}$, assuming that $f = \mathrm{const}$.
- On the mesh $\{x_i\}_{i=1}^4$ with $h_1 = \frac{1}{2}$, $h_2 = \frac{1}{3}$ and $h_3 = \frac{1}{6}$, write the above numerical method as a linear system $KU = F$, where K is the global stiffness matrix, F is the global load vector, and U is the computed-solution vector. For each entry of the unknown vector U specify with which mesh node it is associated.
  (Note: do not forget to address the boundary conditions.)

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2 Answer part (a) and any *two* of parts (b), (c), (d). 20%

In the two-dimensional domain $\Omega = (0,1) \times (0,1)$ with the boundary $\partial\Omega$ consider the problem:

$$Lu = -a\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - 10\frac{\partial u}{\partial y} + 3u = f(x,y), \quad (x,y) \in \Omega,$$

$$u(x,y) = 0, \quad (x,y) \in \partial\Omega.$$

This problem is discretized on the uniform mesh $\{(x_i, y_j)\}_{i,j=1,\dots,N+1}$, where $x_i = (i-1)h,\ y_j = (j-1)h,\ h = 1/N$, by the finite difference method:

$$L^h U_{ij} = -a\,\frac{U_{i-1,j} + U_{i+1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{ij}}{h^2} - 10\,\frac{U_{i,j} - U_{i,j-1}}{h} + 3\,U_{ij} = f(x_i, y_j),$$

for $i, j = 2, \dots, N$, with the boundary conditions:

$$U_{ij} = 0, \qquad (x_i, y_j) \in \partial\Omega.$$

. . . .

(a) Estimate the local truncation error $r_{ij} = -L^h u(x_i, y_j) + f(x_i, y_j)$ of this method, where $u(x_i, y_j)$ is the exact solution at the mesh node (x_i, y_j). 6%

. . . .

(b) Show that the finite difference operator L^h, possibly under a certain condition that involves h and a, satisfies the discrete maximum principle in the form: 7%

$$\left.\begin{array}{r} L^h V_{ij} \le 0 \quad \forall\, i, j = 2, \ldots, N \\ V_{ij} \le 0 \quad \forall\, (x_i, y_j) \in \partial\Omega \end{array}\right\} \quad \Rightarrow \quad V_{ij} \le 0 \quad \forall\, i, j = 1, \ldots, N{+}1.$$

Specify the discrete maximum principle condition on h.

. . . .

(c) Using the maximum principle described in part (b), show that 7%

$$V_{ij} = 0 \;\; \forall\, (x_i, y_j) \in \partial\Omega \quad \Rightarrow \quad \max_{i,j=1,\ldots,N+1} |V_{ij}| \le C \max_{i,j=2,\ldots,N} |L^h V_{ij}|$$

for some constant C. Specify this constant.

. . . .

(d) Using the result of part (a) and the property described in part (c), estimate the error of the finite difference method 7%

$$\max_{i,j=1,\ldots,N+1} |U_{ij} - u(x_i, y_j)|,$$

where U_{ij} is the computed solution, while $u(x_i, y_j)$ is the exact solution at the mesh node (x_i, y_j).

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3 Answer part (a) and one of parts (b) and (c). 25%

Consider the problem

$$u_t = u_{xx}, \quad x \in (0,1),\ t > 0; \quad u(0,t) = u(1,t) = 0; \quad u(x,0) = g(x).$$

This problem is discretized on the uniform mesh $\{(x_j, t_m),\ j = 1, ..., N+1,$ $m = 1, 2, \ldots\}$, where $x_j = (j-1)h,\ h = 1/N$ and $t_m = (m-1)k$. Furthermore, let U_j^m be the computed solution at the mesh node (x_j, t_m).

. . . .

(a) Using Von Neumann's method, prove that the Du Fort-Frankel method 13%

$$\frac{U_j^{m+1} - U_j^{m-1}}{2k} = \frac{U_{j-1}^m - U_j^{m+1} - U_j^{m-1} + U_{j+1}^m}{h^2},$$

is *unconditionally stable*.

. . . .

(b) Find the local truncation error of the method in part (a). 12%

. . . .

(c) Find the local truncation error of the following method: 12%

$$\frac{U_j^{m+1} - U_j^m}{k} = \frac{1}{3}\,\frac{U_{j-1}^{m+1} - 2U_j^{m+1} + U_{j+1}^{m+1}}{h^2} + \frac{2}{3}\,\frac{U_{j-1}^m - 2U_j^m + U_{j+1}^m}{h^2}.$$

Furthermore, using Von Neumann's method, find out whether this method is unconditionally stable, unconditionally unstable, or conditionally stable. If it is conditionally stable, find the stability condition.
