MA4402 — Tutorial Sheet on Series

1. Use sigma notation to represent the series:

$$\text{(i)} \qquad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots$$

$$\text{(ii)} \qquad \frac{1}{1\cdot 2} - \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} - \frac{1}{4\cdot 5} + \cdots$$

$$\text{(iii)} \qquad 1 + \frac{3}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \cdots$$

2. For each of the following sequences a_n, evaluate the corresponding series $s_n = \sum_{k=0}^{n} a_k$, up to the term s_5. In addition, find s_{100}.

   i) $a_n = \frac{9}{10^n}$
   ii) $a_n = \frac{1}{2^n}$
   iii) $a_n = 2^n$
   iv) $a_n = \frac{2^n}{3^n}$
   v) $a_n = 3 + 4n$
   vi) $a_n = \frac{1}{4} + \frac{5n}{4}$
   vii) $a_n = 1 + 2n + 2^n$
   viii) $a_n = 30 + 40n + 8\frac{2^n}{3^n}$

3. Apply the Divergence Test to the following series and establish for each of them whether it is divergent or the test is inconclusive.
(Hint: use the answers for one of the questions from the Tutorial Sheet on Sequences.)

   i) $\sum_{n=1}^{\infty} \frac{n-1}{n+1}$
   ii) $\sum_{n=1}^{\infty} \frac{2n^2+2n-3}{n^2+1}$
   iii) $\sum_{n=1}^{\infty} \frac{2n^2+2n-3}{n^3+1}$
   iv) $\sum_{n=1}^{\infty} \sin\frac{\pi}{n}$
   v) $\sum_{n=1}^{\infty} e^{-n} + e^{-n^2}$
   vi) $\sum_{n=1}^{\infty} \frac{\sqrt{2n+2}}{\sqrt{2n+1}-\sqrt{n+2}}$
   vii) $\sum_{n=1}^{\infty} \exp\left(\frac{n}{n^2+\sqrt{n}}\right)$

4. Use the ratio test to determine whether the following infinite series are convergent or divergent, or say whether the test is inconclusive. If possible, say if the series is divergent in the inconclusive cases.

   i) $\sum_{n=0}^{\infty} \frac{1}{3^n}$
   ii) $\sum_{n=0}^{\infty} \frac{n^3}{5^n}$
   iii) $\sum_{n=0}^{\infty} \frac{5^n}{n^3}$
   iv) $\sum_{n=1}^{\infty} \frac{1}{n}$
   v) $\sum_{n=0}^{\infty} \frac{n}{n+1}$
   vi) $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$
   vii) $\sum_{n=0}^{\infty} x^n$
   viii) $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$
   ix) $\sum_{n=0}^{\infty} \frac{x^{2n}}{(n+1)!}$
   x) $\sum_{n=1}^{\infty} \frac{n!\,(n+1)!\,(n+2)!\,5^n}{(3n)!}$
   xi) $\sum_{n=1}^{\infty} \frac{n!\,(n+1)!\,(n+2)!\,10^n}{(3n)!}$

5. The exponential function $e^x = \exp(x)$ can be defined using an infinite series as

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Given that this infinite series is convergent for all $x \in \mathbb{R}$, use the series and the following table to estimate the value of $e^{0.6}$ to 3 decimal places.

n		a_n		s_n
0		1 . 0 0 0 0 0 0 0 0 0		1 . 0 0 0 0 0 0 0 0 0
1		.		.
2		.		.
3		.		.
4		.		.
5		.		.
6		.		.
7		.		.
8		.		.

6. Using the power series representations of the functions $\cos x$, $\sinh x$, $\tan^{-1} x$ and $\ln(1+x)$ (see §3.5 of the Lecture Notes), estimate the following values to 3 decimal places.

   (Hint: use a table similar to the one used in the solution of the previous question.)

   i) $\cos 0.3$

   ii) $\sinh 0.5$

   iii) $\tan^{-1} 0.4$

   iv) $\ln 1.2 = \ln(1 + 0.2)$

Solutions

1.

2. i)

n	a_n	s_n
0	9.00000	9
1	0.90000	$10 - \frac{1}{10} = 9.90000$
2	0.09000	$10 - \frac{1}{100} = 9.99000$
3	0.00900	$10 - \frac{1}{1000} = 9.99900$
4	0.00090	$10 - \frac{1}{10000} = 9.99990$
5	0.00009	$10 - \frac{1}{100000} = 9.99999$

$s_n = 9\frac{1-\frac{1}{10^{n+1}}}{1-\frac{1}{10}} = 10 - \frac{1}{10^n}$

$s_{100} = 10\left(1 - 10^{-101}\right) \simeq 10$

v)

n	a_n	s_n
0	3	3
1	7	10
2	11	21
3	15	36
4	19	55
5	23	78

$s_n = (n+1)(3+2n)$

$s_{100} = 20503$

viii) $s_n = 30(n+1) + 20n(n+1) + 24\left[1 - \left(\frac{2}{3}\right)^{n+1}\right]$

$s_{100} = 205030 + 24\left[1 - \left(\frac{2}{3}\right)^{101}\right] \simeq 205030 + 24 = 205054$ (as $(2/3)^{101} \simeq 1.6 \cdot 10^{-18}$).

3.

4. i) $R = \frac{1}{3} < 1$ convergent.

ii) $R = \frac{1}{5} < 1$ convergent.

iii) $R = 5 > 1$ divergent.

iv) $R = 1$ inconclusive[1].

v) $R = 1$ inconclusive. However, series divergent because $a_k \to 1$.

vi) $R = 0 < 1$ convergent for all x.

vii) $R = |x|$ convergent if $|x| < 1$. If $|x| \geq 1$, not convergent because $a_k \nrightarrow 0$.

viii) $R = |x|$ convergent if $|x| < 1$. Not convergent if $|x| > 1$.

ix) $R = 0 < 1$ convergent for all x.

[1]However, this series is the harmonic series, which is in fact divergent.

5.

n		a_n												s_n										
0		1	·	0	0	0	0	0	0	0	0	0		1	·	0	0	0	0	0	0	0	0	0
1		0	·	6	0	0	0	0	0	0	0	0		1	·	6	0	0	0	0	0	0	0	0
2		0	·	1	8	0	0	0	0	0	0	0		1	·	7	8	0	0	0	0	0	0	0
3		0	·	0	3	6	0	0	0	0	0	0		1	·	8	1	6	0	0	0	0	0	0
4		0	·	0	0	5	4	0	0	0	0	0		1	·	8	2	1	4	0	0	0	0	0
5		0	·	0	0	0	6	4	8	0	0	0		1	·	8	2	2	0	4	8	0	0	0
6		0	·	0	0	0	0	6	4	8	0	0		1	·	8	2	2	1	1	2	8	0	0
7		0	·	0	0	0	0	0	5	5	5	4		1	·	8	2	2	1	1	8	3	5	4
8			·												·									

$\Rightarrow e^{0.6} \cong 1.822$