MA4402 — Tutorial Sheet on Sequences

October 30, 2012

1. Write the elements a_1, a_2 and a_4 of the following sequences

   (a) $a_n = 4$

   (b) $a_n = \sqrt{1+(-1)^n} + \sqrt{1+(-1)^{n+1}}$

   (c) $a_n = \frac{(n+1)!}{(n+2)!}$

   (d) $a_n = \frac{n^2+3n-2}{2n+1}$

   (e) $a_n = \frac{n^2+1}{2}$

   (f) $a_n = \frac{n^n}{n!}$

   Answers:

   (a) $a_1 = a_2 = a_4 = 4$

   (b) $a_1 = a_2 = a_4 = \sqrt{2}$

   (c) $a_1 = \frac{1}{3}; a_2 = \frac{1}{4}; a_4 = \frac{1}{6}$

   (d) $a_1 = \frac{2}{3}; a_2 = \frac{8}{5}; a_4 = \frac{26}{9}$

   (e) $a_1 = 1; a_2 = \frac{5}{2}; a_4 = \frac{17}{2}$

   (f) $a_1 = 1; a_2 = 2; a_4 = \frac{32}{3}$

2. Calculate the limit for $n \to +\infty$ of the following sequences:

   (a) $a_n = \frac{n-1}{n+1}$

   (b) $a_n = \frac{2n^2+2n-3}{n^2+1}$

   (c) $a_n = \frac{2n^2+2n-3}{n^3+1}$

   (d) $a_n = \sin\frac{\pi}{n}$

   (e) $a_n = e^{-n} + e^{-n^2}$

   (f) $a_n = \frac{\sqrt{2n+2}}{\sqrt{2n+1}-\sqrt{n+2}}$

   (g) $a_n = \exp\left(\frac{n}{n^2+\sqrt{n}}\right)$

   Answers:

   (a) $\lim_{n\to+\infty} a_n = 1$

   (b) $\lim_{n\to+\infty} a_n = 2$

   (c) $\lim_{n\to+\infty} a_n = 0$

   (d) $\lim_{n\to+\infty} a_n = 0$

   (e) $\lim_{n\to+\infty} a_n = 0$

   (f) $\lim_{n\to+\infty} a_n = \frac{\sqrt{2}}{\sqrt{2}-1}$

   (g) $\lim_{n\to+\infty} a_n = 1$

3. Write the elements a_1, a_2 and a_4 of the following sequences

   (a)
   $$\begin{cases} a_1 = 2 \\ a_n = (a_{n-1})^n \end{cases}$$

   (b)
   $$\begin{cases} a_0 = a_1 = 1 \\ a_{n+1} = a_n - a_{n-1} \end{cases}$$

   (c)
   $$\begin{cases} a_0 = 2 \\ a_{n+1} = 6 - a_n \end{cases}$$

   Answers:

(a) $a_n = 2^{n!}; a_1 = 2; a_2 = 4; a_4 = 2^{24}$

(b) $a_1 = 1; a_2 = 0; a_4 = -1$

(c) $a_n = 2$, if n even, $a_n = 4$, if n odd; $a_1 = 4; a_2 = a_4 = 2$

4. For the following strictly positive sequences, evaluate the ratio $\frac{a_{n+1}}{a_n}$ and hence say whether the sequence is increasing, decreasing or neither. In addition, say whether the sequence is bounded or unbounded.

(a) $a_n = n$

(b) $a_n = \frac{n+4}{n+5}$

(c) $a_n = \frac{n+5}{n+4}$

(d) $a_n = n!$

(e) $a_n = \frac{n!}{n^n},\ n \neq 0$

(f) $a_n = \frac{2^n+1}{2^n-1}$

Answers:

(a) increasing, unbounded

(b) increasing, bounded

(c) decreasing, bounded

(d) increasing, unbounded

(e) decreasing, bounded

(f) decreasing, bounded

5. Evaluate the limits of the following sequences as $n \to \infty$.

(a) $a_n = \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n!}$

(b) $a_n = \frac{2n+2}{3n+7}$

(c) $a_n = \frac{4n-3}{2n^2+2n+1}$

(d) $a_n = \frac{3n+2n^2+1}{5n^2+6}$

(e) $a_n = \frac{7n^2}{20000n+n^2}$

(f) $a_n = \frac{10^{10^{10}}n+n^2}{n^2}$

(g) $a_n = \frac{3n}{n!}$

(h) $a_n = \frac{5n^4+6n^2+4}{2^n+1}$

(i) $a_n = \left(\frac{1}{4}\right)^n$

Answers:

(a) 0

(b) $\frac{2}{3}$

(c) 0

(d) $\frac{2}{5}$

(e) 7

(f) 1

(g) 0

(h) 0

(i) 0

6. Assuming that the following recursive sequence has a limit $\lim_{n\to\infty} a_n = L$, find this limit in terms of p.

$$\begin{cases} a_0 &= 1 \\ a_{n+1} &= \frac{2}{3}a_n + \frac{p}{3a_n^2} \end{cases}$$

What is the limit of the sequence defined by

$$\begin{cases} a_0 &= 1 \\ a_{n+1} &= \frac{k-1}{k}a_n + \frac{p}{ka_n^{k-1}} \end{cases}$$

Answers:

(a)

$$L = \frac{2}{3}L + \frac{p}{3L^2} \quad \Rightarrow \quad L = \sqrt[3]{p}$$

(b)

$$L = \frac{k-1}{k}L + \frac{p}{kL^{k-1}} \quad \Rightarrow \quad L = \sqrt[k]{p}$$

7. Consider the sequence:

$$\begin{cases} a_1 = 6 \\ a_{n+1} = \frac{1}{2}\left(a_n + \frac{p}{a_n}\right) \end{cases}$$

show that if the limit exists, it is $\lim_{n\to+\infty} a_n = \sqrt{p}$, and use this result to calculate $\sqrt{33}$ with initial guess 6 and with an error smaller than 0.01.

Answer:

$$\begin{aligned} &p = 33 \\ &a_1 = 6 \\ &a_2 = 5.75 \\ &a_3 = 5.744565 \\ &a_4 = 5.744563 \\ &a_5 = 5.744563 \\ &\sqrt{33} \simeq 5.74 \end{aligned}$$