MA4402 — Tutorial Sheet on Numerical Methods for $f(x) = 0$

1. Consider the function

$$f(x) = x - \ln x - \sqrt{2}.$$

   i) Evaluate f from $x = 0.5$ to $x = 3$ using a step of 0.5, and hence sketch the graph of $f(x)$ over this interval.

   ii) Use the Bisection Method to estimate the root of $f(x)$ in the interval $[0.5, 3]$. Start with an interval of length 1 and iterate until the size of the interval is less than 0.01.

   iii) Rewrite the equation $f(x) = 0$ as $x = \ln x + \sqrt{2}$. Hence find a root of this equation to 3 decimal places using the *Fixed Point Iteration* method with the initial guess $x_0 = 2$.

   iv) Given that $f'(x) = 1 - \frac{1}{x}$, use the Newton-Raphson method to estimate the root of $f(x)$ in the interval $[0.5, 3]$. Use an integer as an initial estimate for the root. Ans $\cong 2.20489216557483$

   v) Use the Newton-Raphson Method to find a root of $f(x)$ starting with an initial guess of 0.5. Ans $\cong 0.342380252644745$

2. Consider the function

$$f(x) = e^x(1 - x)$$

   Show that this function has a root at $x = 1$. Comment on using Newton's method to estimate this root using the initial guesses $-1, 0, 1, 2$.

3. Use Newton's method to estimate the solution of the following equations $f(x) = 0$ with the suggested initial guess x_0 and precision Δx:

   i) $x^3 - 5x - 2 = 0$, $x_0 = 2$, $\Delta x = 0.001$ Ans. $x_s \simeq 2.414$

   ii) $\sin x - x^2 = 0$, $x_0 = 1$, $\Delta x = 0.001$ Ans. $x_s \simeq 0.877$

   iii) $\cos x - x = 0$, $x_0 = 1$, $\Delta x = 0.001$ Ans. $x_s \simeq 0.739$

   iv) $2x^3 + 5x^2 - 4x + \sin x = 0$, $x_0 = -3.5$, $\Delta x = 0.001$ Ans. $x_s \simeq -3.137$

4. Given $f(x) = \tan^{-1}(x)$ and $f'(x) = \frac{1}{1+x^2}$, use Newton's method to estimate the root of f using the following initial estimates and comment on the results.

   i) $x_0 = 0$ ii) $x_0 = 1$ iii) $x_0 = 2$

5. [Final Exam 2011, Q5] Consider the equation:

$$e^{x+2} - x^2 + 3x = 0$$

   Use the Newton-Raphson method to find the solution, with a precision of 3 decimal places, taking as initial guess $x_0 = 0$.

6. [Final Exam 2010, Q3] Kepler's problem for an orbit with eccentricity $\frac{1}{2}$ and mean anomaly 1 involves finding the root of the function

$$f(x) = x - \frac{1}{2}\sin(x) - 1$$

where the angle x is measured in **radians**.

i) Using intervals of 0.5, evaluate f from $x = 0$ to $x = 3$ and hence sketch the graph of $f(x)$ over this interval.

ii) Given that

$$f'(x) = 1 - \frac{1}{2}\cos(x)$$

use Newton's method with an initial estimate of 2 to estimate the root of $f(x)$. Stop iterating when at the current estimate x_n, $|f(x_n)| < 0.001$.

iii) Newton's method can be used to estimate $x = \sqrt{D}$, by finding the root of an appropriate function $f(x)$.

   i. Find an appropriate polynomial function $g(x)$ whose root is $\sqrt{D}$ and

   ii. Find the derivative $g'(x)$ of this function.