Chapter 2

Sequences

Exercise 1

Compute the 6 first terms u_0, u_1, u_2, u_3, u_4, u_5 of the following seqences:

(i) $u_n = \frac{2n^2-n}{n+2}$

(ii) $u_n = -n^2 + 100n$

(iii) $u_n = (-1)^n n^2$

(iv) $u_n = 2^n - 3^n$

(v) $u_n = \frac{n}{n+1}$

(vi) $u_n = \frac{n+2}{n+1}$

Exercise 2

Find whether the previous sequences are increasing, decreasing, or neither increasing nor decreasing. Give a proof or a counter example to justify your answer.

Exercise 3

Let $(u_n)_{n\in\mathbb{N}}$ be sequence defined as follows:

$$\left\{ \begin{array}{l} u_0 = 2 \\ u_{n+1} = 6 - u_n \quad \forall n \in \mathbb{N} \end{array} \right\}$$

(i) Compute u_k for $k = 1, 2, ..., 6$.

(ii) Compute u_{100}.

(iii) Prove that $\forall n \in \mathbb{N}$, we have

$$u_{n+2} = u_n$$

(iv) Redefine the sequence $(u_n)_{n\in\mathbb{N}}$ without using a recursive relation.

Exercise 4

Determine the sign of $u_{n+1} - u_n$ for the following sequences and then precise whether the sequence is increasing or decreasing.

(i) $u_n = \frac{3+5n}{6} - 1$

(ii) $u_n = n^2$

(iii) $u_n = n^2 + 4n$

(iv) $u_n = \frac{2n+1}{3n-1}$

(v) $u_n = \left(\frac{5}{4}\right)^n$

(vi) $u_n = -\left(\frac{5}{4}\right)^n$

(vii) $u_n = -\frac{3}{n+1}$

(viii) $u_n = -n^2 + 3$

Chapter 3

Series

Exercise 1

(i) What is the difference between a sequence and a series?

(ii) Is a sequence a series?

(iii) Is a series a sequence?

Exercise 2

Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence.

(i) Is the series defined by $\sum_{n=0}^{\infty} a_n$ necessarlly convergent?

(ii) Can you give a condition on the bound of the sequence $(a_n)_{n\in\mathbb{N}}$ so that the series $\sum_{n=0}^{\infty} a_n$ is convergent?

Exercise 3

(i) Show that the series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$$

defines a convergent series for all $x \in \mathbb{R}$. Note this series defines $\sin(x)$.

(ii) Use the series defined in the previous question to estimate the value of $\sin(\pi/6)$.

Exercise 4

(i) Show that the series defined by the sequence $\left(\frac{x^n}{n!}\right)_{n\in\mathbb{N}}$ is convergent. Note this series defines e^x.

(ii) Use the series defined in the previous question to estimate the value of e^2 correct to 3 decimal places.

Exercise 5

(i) Explain why $\sum_{n=0}^{\infty} x^n$ converges if $|x| < 1$ and diverges if $|x| > 1$. What happens if $|x| = 1$?

(ii) Does the series $\sum_{n=0}^{\infty} n!x^n$ converge? Why?

Exercise 6

Let p a positive number. Define the sequence $(a_n)_{n\in\mathbb{N}}$ by:

$$\begin{cases} a_1 = 1 \\ a_{n+1} = \frac{1}{2}\left(a_n + \frac{p}{a_n^2}\right), \quad \forall n \in \mathbb{N}. \end{cases}$$

(i) Assuming the above recursively defines a convergent sequence to a positive limit, what is its limit?

(ii) Use this series to estimate $\sqrt[3]{5}$ to two decimal places.