MA4402 — Tutorial Sheet on Linear Algebra

October 23, 2012

1. Consider the following vectors and matrices

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix},$$

$$E = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & 1 \\ 1 & 2 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Calculate the following expressions:

(a) $A\mathbf{u}$
(b) $A\mathbf{u} + B\mathbf{u}$
(c) AB
(d) $AB\mathbf{u}$
(e) BA
(f) $BA\mathbf{u}$
(g) $C\mathbf{v}_1$
(h) $D\mathbf{v}_2$
(i) CD
(j) DC
(k) $C - D$
(l) $D - 2E$
(m) CE
(n) DE
(o) $F\mathbf{w}$
(p) $G\mathbf{w}$
(q) FG
(r) G^2
(s) $\det A$
(t) $\det B$
(u) $\det C$
(v) $\det D$
(w) $\det(C + D)$
(x) $\det(CD)$
(y) $\det E$
(z) $\det F$

Answers:

(a) $$\begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

(b) $$\begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

(c) $$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(d) $$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(e) $$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

(f) $$\begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

(g) $$\begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$$

(h) $$\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

(i) $$\begin{bmatrix} 1 & 1 & 5 \\ -1 & 1 & 1 \\ -2 & 0 & 2 \end{bmatrix}$$

(j) $$\begin{bmatrix} 2 & 1 & -4 \\ -2 & 0 & -2 \\ 2 & 1 & 2 \end{bmatrix}$$

(k)
$$\begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix}$$

(l)
$$\begin{bmatrix} -1 & -4 & 0 \\ 1 & 5 & -1 \\ -1 & -4 & -1 \end{bmatrix}$$

(m)
$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & 1 \\ -2 & -4 & 0 \end{bmatrix}$$

(n)
$$\begin{bmatrix} 3 & 6 & 1 \\ -1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(o)
$$\begin{bmatrix} 2 \\ 0 \\ 3 \\ 1 \end{bmatrix}$$

(p)
$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(q)
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

(r)
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(s) 0

(t) 0

(u) −4

(v) −3

(w) −24

(x) 12

(y) 0

(z) 1

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2. Evaluate the determinants of the following matrices:

$$\begin{vmatrix} 3 & 0 & 8 \\ 6 & 1 & 10 \\ 7 & 3 & -6 \end{vmatrix}; \quad \begin{vmatrix} 7 & 1 & 20 \\ 2 & 0 & 9 \\ 8 & 3 & -5 \end{vmatrix}; \quad \begin{vmatrix} 2 & 0 & 3 \\ 7 & -5 & 4 \\ 9 & 1 & -6 \end{vmatrix}; \quad \begin{vmatrix} 3 & 5 & -1 \\ -3 & 1 & 7 \\ 1 & 0 & -1 \end{vmatrix}; \quad \begin{vmatrix} 2 & -9 & 1 \\ 2 & 3 & 0 \\ -1 & 0 & 2 \end{vmatrix};$$

$$\begin{vmatrix} 1 & -1 & 8 \\ 2 & 5 & 0 \\ 3 & 1 & 3 \end{vmatrix}; \quad \begin{vmatrix} 3 & 1 & -1 \\ 3 & 0 & 4 \\ 6 & -2 & 5 \end{vmatrix}; \quad \begin{vmatrix} 6 & -3 & 2 \\ -1 & 0 & -7 \\ 3 & -2 & 4 \end{vmatrix}; \quad \begin{vmatrix} 1 & 6 & 3 \\ -4 & 5 & 0 \\ 2 & -3 & -1 \end{vmatrix}; \quad \begin{vmatrix} 2 & 7 & 1 \\ 0 & 1 & 5 \\ 4 & 8 & 3 \end{vmatrix};$$

$$\begin{vmatrix} 0 & 1 & -1 \\ -1 & 8 & 0 \\ -3 & 4 & -2 \end{vmatrix}; \quad \begin{vmatrix} -1 & 3 & 0 \\ 1 & 2 & 3 \\ 2 & 1 & -2 \end{vmatrix}; \quad \begin{vmatrix} 1 & 2 & 3 \\ -1 & 3 & 0 \\ 2 & 1 & -2 \end{vmatrix}; \quad \begin{vmatrix} 2 & 1 & 2 \\ 1 & 0 & 3 \\ -1 & 3 & -4 \end{vmatrix};$$

$$\begin{vmatrix} 2 & -1 & 5 \\ 1 & -3 & 0 \\ 3 & 4 & -6 \end{vmatrix} \text{ and } \begin{vmatrix} 2 & 0 & -8 & 7 \\ 0 & 2 & -1 & 5 \\ 0 & 1 & -3 & 0 \\ 0 & 3 & 4 & -6 \end{vmatrix};$$

$$\begin{vmatrix} 5 & -1 & 7 \\ 9 & 2 & 6 \\ 3 & 0 & 1 \end{vmatrix} \text{ and } \begin{vmatrix} 5 & -1 & 7 & 13 \\ 9 & 2 & 6 & -4 \\ 3 & 0 & 1 & 5 \\ 0 & 0 & 0 & 3 \end{vmatrix};$$

$$\begin{vmatrix} -2 & 0 & 3 \\ 3 & 4 & 6 \\ 7 & -1 & -5 \end{vmatrix} \text{ and } \begin{vmatrix} -2 & 0 & 3 & 0 \\ 11 & -2 & 5 & 3 \\ 3 & 4 & 6 & 0 \\ 7 & -1 & -5 & 0 \end{vmatrix} \text{ and then } \begin{vmatrix} 3 & 0 & 0 & 0 & -4 \\ -2 & 0 & 3 & 0 & 0 \\ 11 & -2 & 5 & 3 & 0 \\ 3 & 4 & 6 & 0 & 0 \\ 7 & -1 & -5 & 0 & 0 \end{vmatrix};$$

$$\begin{vmatrix} 3 & -2 & 7 \\ 4 & 0 & -1 \\ 6 & 1 & -5 \end{vmatrix} \text{ and } \begin{vmatrix} 0 & 3 & -2 & 7 \\ -9 & 1 & 8 & -1 \\ 0 & 4 & 0 & -1 \\ 0 & 6 & 1 & -5 \end{vmatrix}; \text{ and then } \begin{vmatrix} 0 & 3 & 2 & -2 & 7 \\ 0 & 0 & 3 & 0 & 0 \\ -9 & 1 & -1 & 8 & -1 \\ 0 & 4 & 11 & 0 & -1 \\ 0 & 6 & -2 & 1 & -5 \end{vmatrix}.$$

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3. For the following pairs of vectors, evaluate the angle θ between them.

(a)
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

(b)
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix},$$

(c)
$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

(d)
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

(e)
$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(f)
$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(g)
$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

(h)
$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(i)
$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \\ -3 \end{bmatrix}$$

Answers:

(a) $\theta = \arccos(0) = \pi/2$
(b) $\theta = \arccos(1) = 0$
(c) $\theta = \arccos(-1/\sqrt{2}) = 3\pi/4$
(d) $\theta = 0$
(e) $\theta = \arccos(\sqrt{2/3}) \simeq 0.615$
(f) $\theta = \arccos(1/3) \simeq 1.23$
(g) $\theta = \arccos(0) = \pi/2$
(h) $\theta = \arccos(-1/\sqrt{2}) = 3\pi/4$
(i) $\theta = \arccos(-1) = \pi$

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4. Calculate the area S of the parallelogram spanned by the following pairs of vectors.

(a)
$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

(b)
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix},$$

(c)
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix},$$

Answers:

(a) $S = \sqrt{2}$

(b) $S = 3$

(c) $S = 6$

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5. (i) If $\mathbf{u} = 3\mathbf{i} - \mathbf{j} - 6\mathbf{k}$, $\mathbf{v} = \mathbf{i} - 7\mathbf{j} + 2\mathbf{k}$, and $\mathbf{w} = 2\mathbf{i} + 3\mathbf{j}$, (a) find the cross product $\mathbf{v} \times \mathbf{w}$; (b) then find the triple vector product $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$.

   (ii) Similarly, for $\mathbf{u} = 4\mathbf{i} - \mathbf{j} + 9\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} + 6\mathbf{k}$, and $\mathbf{w} = 5\mathbf{i} - 2\mathbf{j} + 8\mathbf{k}$, find the triple vector product $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

   Answers:

   (i) $\mathbf{v} \times \mathbf{w} = -6\mathbf{i} + 4\mathbf{j} + 17\mathbf{k}$, $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = 7\mathbf{i} - 15\mathbf{j} + 6\mathbf{k}$.

   (ii) $\mathbf{u} \times \mathbf{v} = -6\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$, $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = 30\mathbf{i} + 63\mathbf{j} - 3\mathbf{k}$.

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6. Find the area of the triangle with vertices:

   (i) $(2, 2, -3)$, $(4, 1, 5)$ and $(4, 6, -1)$.

   (ii) $(2, 4, 3)$, $(12, -1, 8)$ and $(6, 3, 7)$.

   (iii) $(9, -1, 3)$, $(12, 3, 5)$ and $(7, 0, 2)$.

   Answers: (i) $S = 5\sqrt{14} \approx 18.708$; (ii) $S = \frac{5\sqrt{29}}{2} \approx 13.4629$; (iii) $\vec{AB} = (3, 4, 2)$, $\vec{AC} = (-2, 1, -1)$, $\vec{AB} \times \vec{AC} = (-6, -1, 11)$. Area $= \frac{|\vec{AB} \times \vec{AC}|}{2} = \frac{\sqrt{158}}{2} \approx 6.2849$.

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7. Find the area of the triangle with vertices:

   (i) $(2, 2)$, $(4, 1)$ and $(4, 6)$.

   (ii) $(4, 3)$, $(-1, 8)$ and $(3, 7)$.

   (iii) $(-1, 3)$, $(3, 5)$ and $(0, 2)$.

   Answers: (i) $S = 5$; (ii) $S = \frac{15}{2}$; (iii) $S = 3$.

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8. Find the volume of the parallelepiped formed by $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ with vertices

   (i) $A = (2, 2, -3)$, $B = (4, 1, 5)$, $C = (4, 6, -1)$ and $D = (3, 3, -2)$.

   (ii) $A = (2, 4, 3)$, $B = (12, -1, 8)$, $C = (6, 3, 7)$ and $D = (1, 5, 2)$.

   Answers: (i) $S = 12$; (ii) $S = 15$.

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Answers for Tutorial Sheet B:

Exercise 5: 1. Length $= 5$ and mid-point $= (\frac{13}{2}, -9)$.

Exercise 6: 1. The translated segment has endpoints $(8, -8)$ and $(11, -12)$.

Exercise 7: The rotated segment has endpoints $(0, 0)$ and $(0, 3\sqrt{2})$.

Exercise 8: The rotated segment has endpoints $(2, 2)$ and $(1, 3)$.

Exercise 9: The rotated segment has endpoints $(5, 3)$ and $(1, 5)$ and length $2\sqrt{5}$.

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