MA4402 Computer Mathematics

http://www.staff.ul.ie/natalia/MA4402.html

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Motivation

One MA4402 Learning Outcome:

Employ sequences and series for an efficient representation of mathematical functions in algorithms.

MOTIVATION:

- Computers, Calculators, Phones are all capable of performing basic arithmetic operations cheaply and accurately.
  By basic arithmetic I mean $+$, $-$, $\times$ and $/$.
- Most of these devices can do much more than this. They can calculate $\sin x$ and $\cos x$, $\log x$ and e^x.
  But HOW???
  NOTE: Evaluating these functions is a non-trivial task.
- Since $\sin x$ and $\cos x$... are not basic arithmetic operations, the computer must approximate them somehow.
  Moreover, whatever method is used to approximate them, it must rely entirely on the four basic operations.

  NOTE: There are a number of approaches to approximate these functions, but we shall consider only one of them:

  Power Series Representation

  of functions such as $\sin x$ and $\cos x$

....

To prepare for this, we shall first consider Sequences and Series.

§3.1 Sequences

Definition

A sequence is an finite/infinite list of terms (or numbers) arranged in a definite order, that is, there is a rule by which each term after the first may be found.

EXAMPLES:

1. $\{1, 2, 3, 4, 5, \ldots\}$
   NOTE: the $\ldots$ at the end of a sequence indicates that we have an infinite number of terms.
2. $\{1, 4, 9, 16, 25, \ldots\}$
   NOTE: Denote the nth term by a_n.
   Then $a_1 = 1^1 = 1$, $a_2 = 2^2 = 4$, $a_3 = 3^2 = 9$...
   So this sequence can be described by the formula $a_n = n^2$.

3. Let a sequence be defined by the formula of its nth term:

$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

Then we get $a_1 = \left(1 + \frac{1}{1}\right)^1 = 2,\ a_2 = \left(1 + \frac{1}{2}\right)^2 = \left(\frac{3}{2}\right)^2$,...

So we get $\left\{2, \left(\frac{3}{2}\right)^2, \left(\frac{4}{3}\right)^3, \left(\frac{5}{4}\right)^4, \ldots\right\}$.

4. The Fibonacci sequence is defined as follows

$$a_1 = a_2 = 1, \qquad a_{n+2} = a_n + a_{n+1}.$$

From this definition we get: $\{1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\}$.

NOTATION: A sequence is frequently written as $\{a_1, a_2, a_3, a_4, \ldots\}$.

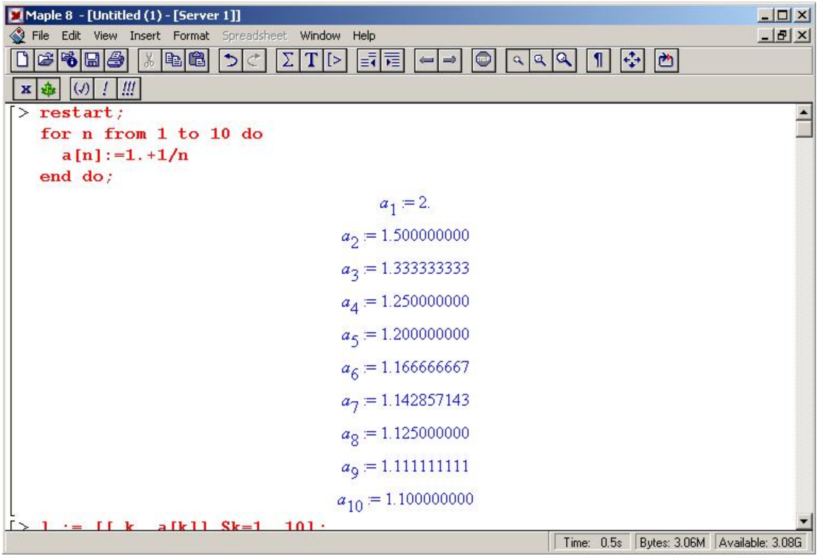
Shorthand for such a sequence is $\{a_n\}_{n=1}^{\infty}$

Another EXAMPLE: List the first 4 terms of the following sequence: $a_n = n^2 + 1$.

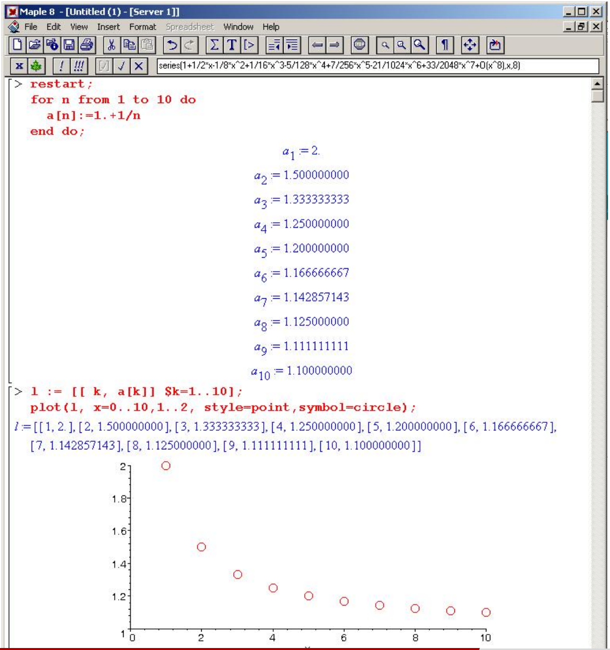
Solution: The first four terms are a_1, a_2, a_3 and a_4.
Therefore using the rule we get

$$a_1 = (1)^2 + 1 = 2; \quad a_2 = (2)^2 + 1 = 5; \quad a_3 = (3)^2 + 1 = 10;$$
$$a_4 = (4)^2 + 1 = 17.$$

NOTE: one can use computer languages/packages to evaluate + plot graphs of sequences. For example, the package Maple for $a_n = 1 + \frac{1}{n}$:



Same example $a_n = 1 + \frac{1}{n}$, but now with a graph:



Bounded Sequences

Definition

The sequence $\{a_n\}$ has a lower bound L if $a_n \geqslant L$ for all n.

Examples:

(i) The Fibonacci sequence has lower bound 1 as $a_n \geqslant 1$ for all $n \geqslant 1$.

(ii)The sequence $\{n^2\}$ has lower bound 1 as $n^2 \geqslant 1$ for all $n \geqslant 1$.

Definition

The sequence $\{a_n\}$ has an upper bound M if $a_n \leqslant M$ for all n.

Example:

The sequence $\{1 - \frac{1}{n}\}$ has upper bound 1 as $1 - \frac{1}{n} \leqslant 1$ for all $n \geqslant 1$.

Definition

The sequence $\{a_n\}$ is bounded
if it has a lower bound L and an upper bound M
so $L \leqslant a_n \leqslant M$ for all n.

Examples:

(i) $a_n = \sin\left(n + \frac{1}{n}\right)$.
The sequence $\{a_n\}$ is bounded as for all $n \geqslant 1$:

$$-1 \leqslant \sin\left(n + \frac{1}{n}\right) \leqslant 1.$$

(ii) The sequence $\{2^{-n}\}_{n=1}^{\infty}$ is the sequence $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$.
So it is bounded: $0 \leqslant 2^{-n} \leqslant 1$.

Another EXAMPLE:

$$a_n = \begin{cases} 1 + \frac{1}{2^n}, & \text{if } n \text{ is odd,} \\ 2^n, & \text{if } n \text{ is even.} \end{cases}$$

From this definition we have:

$$a_1 = 1 + \frac{1}{2} = \frac{3}{2} \quad \text{(here } n = 1\text{, odd);}$$

$$a_2 = 2^2 = 4 \quad \text{(here } n = 2\text{, even);}$$

$$a_3 = 1 + \frac{1}{2^3} = 1 + \frac{1}{8} = \frac{9}{8} \quad \text{(here } n = 3\text{, odd);}$$

$$a_4 = 2^4 = 16 \quad \text{(here } n = 4\text{, even); etc.}$$

So we get the sequence: $\{\frac{3}{2}, 4, \frac{9}{8}, 16, \ldots\}$

This sequence has a lower bound 1 (as $1 + \frac{1}{2^n} \geqslant 1$ and $2^n \geqslant 1$); but no upper bound,

so it is not bounded.

Increasing/Decreasing Sequences, etc

Definition

The sequence $\{a_n\}$ is

- positive if $a_n \geqslant 0$ for all n.
- negative if $a_n \leqslant 0$ for all n.

- increasing if $a_{n+1} > a_n$ for all n.
- decreasing if $a_{n+1} < a_n$ for all n.
- monotonic if it is either increasing or decreasing.

- alternating if $a_n a_{n+1} < 0$ for all n. i.e., consecutive terms have opposite signs.

NOTE: For a positive sequence, one can
—replace the criterion $a_{n+1} > a_n$ by the equivalent $\frac{a_{n+1}}{a_n} > 1$;
—and the criterion $a_{n+1} < a_n$ by the equivalent $\frac{a_{n+1}}{a_n} < 1$.

EXAMPLES (Increasing/Decreasing Sequences)

1. The sequence $\{a_n = n\}_{n=1}^{\infty}$ is increasing
   as $a_{n+1} - a_n = (n+1) - n = 1 > 0$.
2. The sequence $\{a_n = 1 - n\}_{n=1}^{\infty}$ is decreasing
   as $a_{n+1} - a_n = (1 - (n+1)) - (1 - n) = -1 < 0$.
3. The positive sequence $\{a_n = 2^n\}_{n=1}^{\infty}$ is increasing
   as $\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{2^n} = 2 > 1$.
4. The sequence $\{a_n\}_{n=1}^{\infty}$ with $a_n = 1 + \frac{(-1)^n}{n}$ is neither increasing nor decreasing
   as we have $\{0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \ldots\}$.
   (This sequence keeps bouncing backwards and forwards around 1 and so neither increases nor decreases.)

Limits of Sequences

Definition

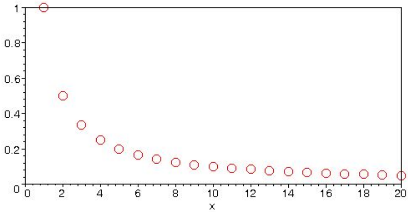
As $n \to \infty$, the terms a_n may approach some number L.

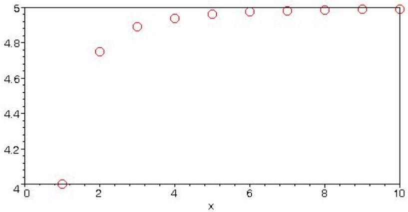
Then we say that the sequence $\{a_n\}$ converges to the limit L and write

$$\lim_{n\to\infty} a_n = L.$$

The series is said to be convergent.

Example: $\lim_{n\to\infty} \frac{1}{n} = 0;$ $\lim_{n\to\infty} (5 - \frac{1}{n^2}) = 5.$

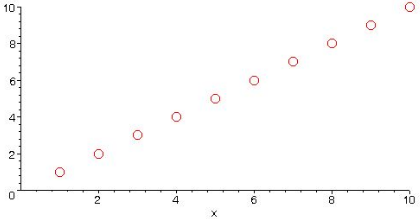




Definition

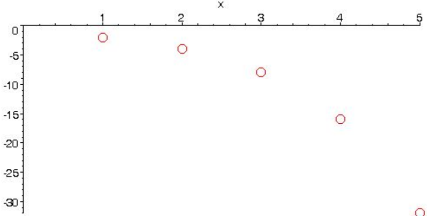
If the sequence $\{a_n\}$ does not converge to a finite limit, it is said to be divergent.

Example: If $a_n = n$, then $\{a_n\}$ diverges (as it goes to $+\infty$.)

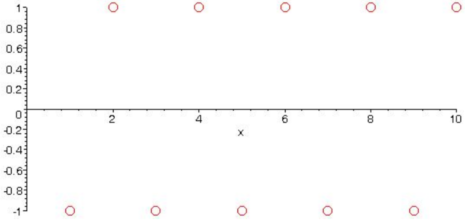


Further EXAMPLES:

1. If $a_n = -2^n$, then $\{a_n\}$ diverges (as it goes to $-\infty$).



2. If $a_n = (-1)^n$, then $\{a_n\} = \{-1, +1, -1, +1 \ldots\}$ just diverges.



Elementary Properties of Convergent Sequences

Suppose $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$.

(i) If $c_n = a_n + b_n$, then $\lim_{n\to\infty} c_n = a + b$.

(ii) If $c_n = a_n \times b_n$, then $\lim_{n\to\infty} c_n = a \times b$.

(iii) If $c_n = \frac{a_n}{b_n}$ and $b \neq 0$, then $\lim_{n\to\infty} c_n = \frac{a}{b}$.

Example:

Set $a_n = 1 + \frac{1}{n}$ and $b_n = 5 - \frac{2}{n}$ and check the above properties (i)–(iii).

NOTE: these properties will be used in all remaining Examples of this section §3.1.

NOTE: sequences such as $\frac{1}{n}$, $\frac{1}{n^2}$, $\frac{1}{n^3}$, $\frac{1}{\sqrt{n}}$ converge to 0 as $n \to \infty$.

We shall use this observations in the next few examples.

Further EXAMPLES:

3. $$\lim_{n\to\infty} \frac{7n-5}{2+3n} = \lim_{n\to\infty} \frac{7-\frac{5}{n}}{\frac{2}{n}+3} = \frac{7-0}{0+3} = \frac{7}{3}.$$

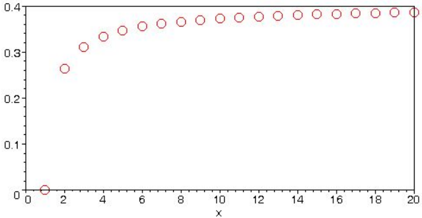
   NOTE: we divided both the numerator and denominator by the strongest term n

4. $$\lim_{n\to\infty} \frac{3\sqrt{n}-1}{8-5\sqrt{n}} =$$

Further EXAMPLES:

5. $$\lim_{n\to\infty} \frac{2n^2 - n - 1}{5n^2 + n - 3} = \lim_{n\to\infty} \frac{2 - \frac{1}{n} - \frac{1}{n^2}}{5 + \frac{1}{n} - \frac{3}{n^2}} = \frac{2 - 0 - 0}{5 + 0 + 0} = \frac{2}{5}.$$

NOTE: we divided both the numerator and denominator by the strongest term n^2



Further EXAMPLES:

7. $$\lim_{n\to\infty} \frac{n^2 - 5n^4 + 10}{3n^2 + n - 7n^4} = \ldots$$

8. $$\lim_{n\to\infty} \frac{n^2 - 5n^3 + 10}{3n^2 + n - 7n^4} = \ldots$$

9. $$\lim_{n\to\infty} \frac{n^2 - 5n^4 + 10}{3n^2 + n - 7n^3} = \ldots$$

§3.2 Sequences in Root Algorithms

How are square roots computed on a computer??
One possible way: define a sequence by a recursive rule.

Square Root $\sqrt{p}$

$$a_1 = 1, \qquad a_{n+1} = \frac{1}{2}\left(a_n + \frac{p}{a_n}\right)$$

NOTE: this is in fact the Newton-Raphson Method applied to the equation $x^2 = p$ (whose positive root is $\sqrt{p}$); see §4 for further details.

Particular Case $\sqrt{2}$

$$a_1 = 1, \qquad a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$$

EXAMPLE: compute the first 3 terms in the $\sqrt{2}$ algorithm.

$$a_1 = 1,$$

$$a_2 = \frac{1}{2}\left(a_1 + \frac{2}{a_1}\right) = \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2},$$

$$a_3 = \frac{1}{2}\left(a_2 + \frac{2}{a_2}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{3/2}\right) \approx 1.4166.$$

A SIMPLE COMPUTER CODE will produce as accurate an approximation as required (see next page).

Maple 16 - [Untitled (1) - [Server 1]]

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```
> sqrt(2)=sqrt(2.);
```

$$\sqrt{2} = 1.414213562$$

Sequence:

```
> a[1]:=1.;
  for n from 1 to 5 do
     a[n+1]:=(1/2)*(a[n]+2/a[n]);
  end;
```

$$a_1 := 1.$$

$$a_2 := 1.500000000$$

$$a_3 := 1.416666667$$

$$a_4 := 1.414215686$$

$$a_5 := 1.414213562$$

$$a_6 := 1.414213562$$

Here in a5 and a6, we have the same 10 decimal places.
This indicates that in a5, the first decimal places are correct!

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QUESTION: how we can deduce from the recursive rule

$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$ defining our sequence that it converges to $\sqrt{2}$??

DISCUSSION:

Assuming that the sequence $\{a_n\}$ converges to some number $L > 0$,

i.e. $\lim_{n\to\infty} a_n = L$,

let $n \to \infty$ in the recursive rule:

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$$

$$L = \frac{1}{2}\left(L + \frac{2}{L}\right)$$

so we have

$$2L = L + \frac{2}{L} \quad \Rightarrow \quad L = \frac{2}{L} \quad \Rightarrow \quad L^2 = 2 \quad \Rightarrow \quad L = \sqrt{2}.$$

CONCLUSION: if the sequence $\{a_n\}$ converges to some number $L > 0$,

then $L = \sqrt{2}$

EXERCISE: evaluate the first 4 terms in the $\sqrt{3}$ sequence.

Solution:

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{3}{a_n}\right)$$

So we get:

$$a_1 = 1,$$

$$a_2 =$$

$$a_3 =$$

$$a_4 =$$

EXAMPLE: the sequence is defined by the recursive rule

$$a_1 = 1, \qquad a_{n+1} = \frac{1}{4}\left(3a_n + \frac{33}{a_n^3}\right)$$

Assuming that the sequence $\{a_n\}$ converges to some number $L > 0$, find L.

Solution:

Answer: $L = \sqrt[4]{33}$

§3.3 Series

A series is formed when the terms of a sequence are added together.

For example,
$2, 5, 8, 11, 14, \cdots$ is an infinite sequence,
but $2 + 5 + 8 + 11 + 14 + \cdots$ is an infinite series.

The Sigma Notation

The Sigma Notation:

The Greek letter $\sum$ (pronounced sigma), which means the sum of, is generally used to express a series in a concise way.

Example: $$2+4+8+16+32 = 2+2^2+2^3+2^4+2^5 = \sum_{n=1}^{5} 2^n.$$

Note: 2^n is the nth term in the sequence.

Since we are summing the terms from 1 to 5 inclusive, the least value of n is placed below the $\sum$ sign and the greatest value of n is placed above.

NOTE: A finite series always ends with the last term even if several middle terms are omitted:

Example: $$\sum_{n=1}^{20} \frac{1}{2n+2} = \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots + \frac{1}{42}.$$

NOTE: An infinite series may also be written in sigma notation, where ∞ is used to indicate that there is no upper limit for n:

Example: $$2 + 4 + 6 + 8 + \cdots = \sum_{n=1}^{\infty} 2n.$$

EXERCISE: Write the following series using $\sum$ notation:

(i) $1 + 4 + 9 + 16 + \cdots + 81$

(ii) $1 - x + x^2 - x^3 + x^4 - \cdots$

(ii) $-1 + \frac{1}{2}x - \frac{1}{3}x^2 + \frac{1}{4}x^3 - \frac{1}{5}x^4 + \cdots$

NOTE: In a series where the sign alternates from positive to negative, $(-1)^n$ may be used for the sign.

$(-1)^n$ results in even terms being positive and odd terms being negative.

$(-1)^{n+1}$ results in even terms being negative and odd terms being positive.

Infinite Series

Definition (Infinite Series)

Given a sequence $\{a_n\}$, the sum of all infinitely many terms in this sequence

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

is called an infinite series.

NOTE: it may be counterintuitive that we add infinitely many terms and still can get a finite number, but this may be the case.

EXAMPLE:

consider $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$.

Let us do the summation gradually: one term, then 2 terms, then 3 terms, etc.:

$S_1 = 1$ —this it the first term;

$S_2 = 1 + \frac{1}{2^2}$ —this it the sum of first 2 terms;

$S_3 = 1 + \frac{1}{2^2} + \frac{1}{3^2}$ —this it the sum of first 3 terms;

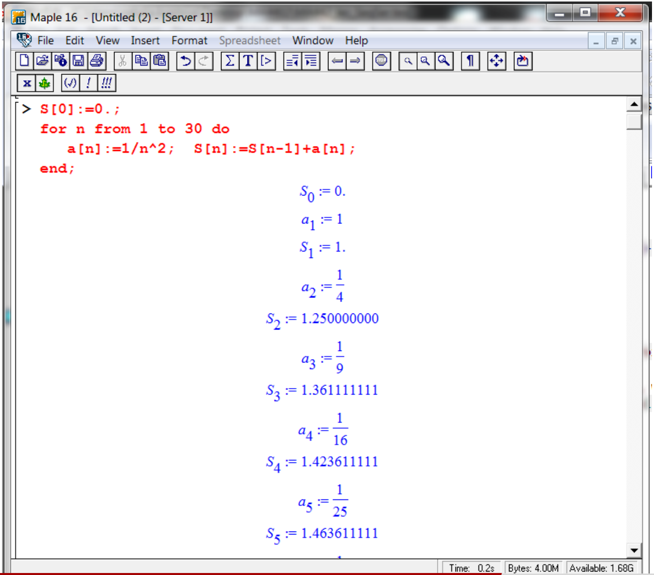
$S_4 = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}$ —this it the sum of first 4 terms;

In general, S_n is the sum of the first n terms in our sequence:

$$S_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots + \frac{1}{n^2}$$

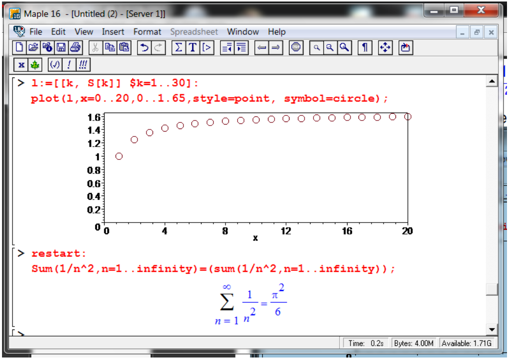
For the series $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$,

here are a few values of S_n obtained using a computer package:



For the series $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$,

plot a few values of S_n obtained using a computer package:



We see that as $n \to \infty$, the values S_n approach $\frac{\pi^2}{6} \approx 1.644934068$. This implies that if we add all infinitely many terms of the sequence $\frac{1}{n^2}$,

we get $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.644934068}$

Definition (Convergent Infinite Series)

Given a sequence $\{a_n\}$, let the sum of its first n terms

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

If $\lim_{n\to\infty} S_n = S$, where S is some real number, then we write

$\sum_{n=1}^{\infty} a_n = S$ and say that the series $\sum_{n=1}^{\infty} a_n$ converges to the sum S.

NOTE: if $\lim_{n\to\infty} S_n$ is $\pm\infty$ or does not exist,

then we say that the series $\sum_{n=1}^{\infty} a_n$ diverges.

EXAMPLE of a DIVERGENT series: Harmonic Series

consider $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$.

Let us do the summation gradually: one term, then 2 terms, then 3 terms, etc.:

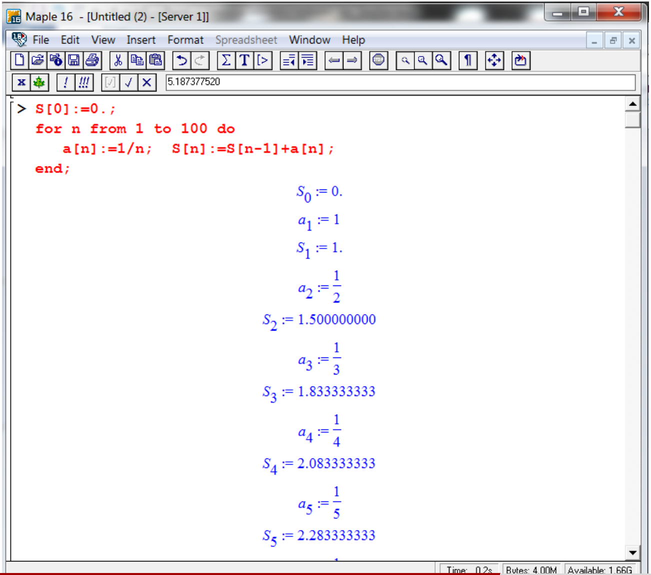
$S_1 = 1$ —this it the first term;
$S_2 = 1 + \frac{1}{2}$ —this it the sum of first 2 terms;
$S_3 = 1 + \frac{1}{2} + \frac{1}{3}$ —this it the sum of first 3 terms;
$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ —this it the sum of first 4 terms;

In general, S_n is the sum of the first n terms in our sequence:

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n}$$

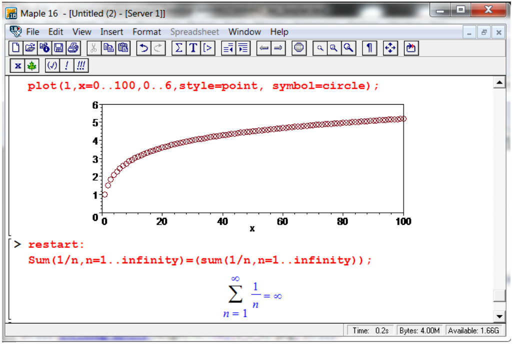
For the series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$,

here are a few values of S_n obtained using a computer package:



For the series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$,

plot a few values of S_n obtained using a computer package:



We see that as $n \to \infty$, the values S_n do not approach any finite number (no matter how large n we consider), but go to $+\infty$.

This implies that if we add all infinitely many terms of the sequence $\frac{1}{n}$,

we get $+\infty$ so $\boxed{\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}}$

Further EXAMPLES:

1. Telescoping Series $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$$
2. Series $$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \cdots$$
3. Geometric Series $$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$
4. More general Geometric Series $$\sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + x^3 + \cdots$$

Geometric Series

Geometric series: $\sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + x^3 + \cdots$

Convergence/Divergence of Geometric Series

$$\sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + x^3 + \cdots = \begin{cases} \frac{1}{1-x}, & \text{if } x \in (-1, 1), \\ \text{divergent}, & \text{otherwise} \end{cases}$$

Proof:

Case (i): If $x = 1$, then $\sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + x^3 + \cdots = 1 + 1 + 1 + \cdots$ is divergent (see previous page).

Case (ii): Let $x \neq 1$. Then one can show that

$$S_n = \sum_{k=1}^{n} x^{k-1} = 1 + x + x^2 + \cdots x^{n-1} = \frac{1 - x^n}{1 - x} \quad \text{(Exercise!)}$$

Now, if $|x| < 1$, then $\lim_{n\to+\infty} x^n = 0$, so

$$\lim_{n\to+\infty} s_n = \frac{1}{1-x}.$$

Otherwise, $\lim_{n\to+\infty} x^n$ does NOT exist, so the series is divergent.

. . . .

Example The rational number $0.323232323232\ldots$ can be represented as

$$0.32 + 0.0032 + 0.000032 + \cdots = \frac{32}{100} + \frac{32}{100^2} + \frac{32}{100^3} + \cdots$$

$$= \frac{32}{100}\left(1 + \frac{1}{100} + \frac{1}{100^2} + \frac{1}{100^3} + \cdots\right) = \frac{32}{100} \cdot \frac{1}{1 - \frac{1}{100}} = \frac{32}{99}.$$

§3.4 Tests for Divergent/Convergent Series

Theorem If the series $\sum_{n=0}^{\infty} a_n$ is convergent, then

$$\lim_{n\to+\infty} a_n = 0.$$

From this we have

Divergence Test

If $\lim_{n\to+\infty} a_n \neq 0$ then the series $\sum_{n=0}^{\infty} a_n$ is divergent.

EXAMPLES:

(i) The series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} n = 1 - 2 + 3 - 4 + \cdots$ is divergent as

$$\lim_{n\to\infty} (-1)^{n+1} n \neq 0;$$

(ii) The series $\displaystyle\sum_{n=0}^{\infty} (-1)^{n} = 1 - 1 + 1 - 1 + 1 + \cdots$ is divergent as

$$\lim_{n\to\infty} (-1)^{n} \neq 0;$$

EXAMPLES:

(iii) The series $\sum_{n=0}^{\infty} \frac{n}{2n-1} = 0 + 1 + \frac{2}{3} + \frac{3}{5} + \frac{4}{7} + \cdots$ is divergent as

$$\lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2} \neq 0.$$

NOTE: if $\lim_{n\to+\infty} a_n = 0$, the series may be divergent (see 3 above examples), or convergent, such as the divergent series

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$

(was considered earlier).

In other words, $\lim_{n\to+\infty} a_n = 0$ is a necessary condition for the convergence of $\sum_{n=0}^{\infty} a_n$, but NOT sufficient.

Ratio test

Ratio test

Let $\sum_n a_n$ a series such that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = p$$

If $p < 1$ the series is convergent.
If $p > 1$ (or infinite) the series is divergent.
If $p = 1$ the test is inconclusive.

NOTE: The ratio test is generally used to test for convergence or divergence a series in which

(i) the variable (generally n) appears in factorial form, e.g., $\sum_{n=0}^{\infty} \frac{3n!}{n+1}$

(ii) the variable appears as a power, e.g., $\sum_{n=0}^{\infty} \frac{2^n}{n^2}$

EXAMPLES:

1. $\sum_{n=1}^{\infty} \frac{1}{n!}$ is convergent ($p = 0 < 1$)
2. $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ is divergent ($p = 2 > 1$)
3. $\sum_{n=1}^{\infty} \frac{n^5}{3^n}$ is convergent ($p = \frac{1}{3} < 1$)
4. $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ is divergent ($p = 4 > 1$)
5. $\sum_{n=1}^{\infty} \frac{1}{n^q}$ (for some positive q) —test is inconclusive ($p = 1$);

   Recall: $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, while $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

6. $\displaystyle\sum_{n=0}^{\infty} \frac{10^n}{n!}$ is convergent (as $p = 0 < 1$)
NOTE: this series converges to e^{10}.

7. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is convergent for each x (as $p = 0 < 1$ for each fixed x)
(when we investigate convergence of the series, treat x as a fixed constant)
NOTE: this series converges to e^x for each x.

8. $\sum_{n=0}^{\infty} \frac{(2x)^n}{(n^2+1)3^n}$ is convergent for $|x| < \frac{3}{2}$ (as $p = \frac{|2x|}{3} < 1$)

9. $\sum_{n=0}^{\infty} n! x^n$
is divergent for each $x \neq 0$ (as $p = \infty > 1$ for each fixed $x \neq 0$)

10. $x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$ is convergent for each $|x| < 1$
(as $p = |x| < 1$ for each fixed $|x| < 1$)

NOTE: this series converges to $\ln(1+x)$ for each $|x| < 1$.

§3.5 Functions as Infinite Power Series

Power Series

A power series (about $x = 0$) is a series of the type:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots,$$

where x is a variable, and $c_0, c_1, \cdots$ are given constants.

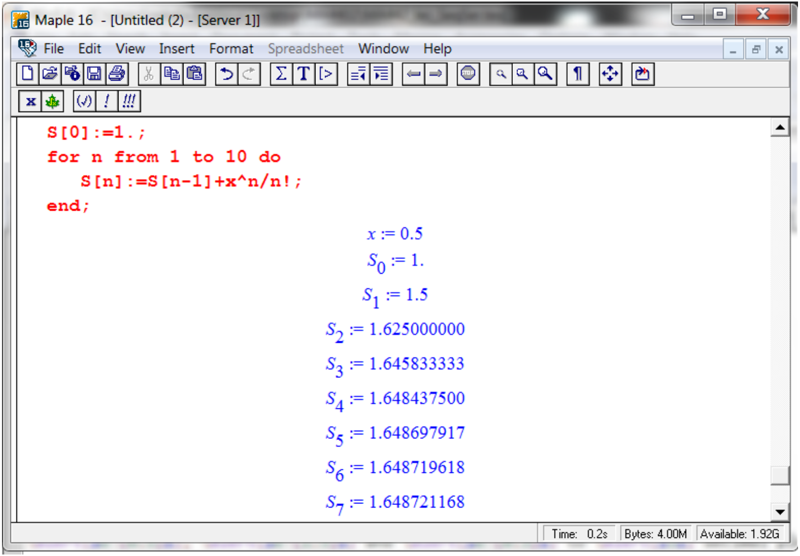
Exponential function: the power series representation is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

NOTE: the Ratio Test shows (§3.4) that this series is convergent for any real x.

APPLICATION: one can use the sum of sufficiently many first terms $S_N(x) = \sum_{n=0}^{N} \frac{x^n}{n!}$ to get an approximation of e^x.

NOTE: one can use computer languages/packages to evaluate the sum of sufficiently many first terms $S_N(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^N}{N!}$:



Consider the power series representation of e^x: $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

EXERCISE: Use the series and the following table to estimate the values of $e^{0.5}$, $e^{1.5}$ and $e^{0.3}$ to 5 decimal places.

	$x = 0.5$	$x = 1.5$	$x = 0.3$
$S_0 = 1$	1	1	
$S_1 = 1 + x$	1.5	2.5	
$S_2 = (1 + x) + \frac{x^2}{2!}$	1.625000000	3.625000000	
$S_3 = (1 + x + \frac{x^2}{2!}) + \frac{x^3}{3!}$	1.645833333	4.187500000	
$S_4 = S_3 + \frac{x^4}{4!}$	1.648437500	4.398437500	
$S_5 = S_4 + \frac{x^5}{5!}$	1.648697917	4.461718750	
$S_6 = S_5 + \frac{x^6}{6!}$	1.648719618	4.477539062	
$S_7 = S_6 + \frac{x^7}{7!}$	1.648721168	4.480929129	
$S_8 = S_7 + \frac{x^8}{8!}$	—	4.481564767	
$S_9 = S_8 + \frac{x^9}{9!}$	—	4.481670707	
$S_{10} = S_9 + \frac{x^{10}}{10!}$	—	4.481686598	

How one can get a power series representation??

QUESTION:
How one can get a power series representation for any given function??

ANSWER:
There is a recipe called the Taylor series formula, which yields a power series representation for any function (no further details in this course).

Then one can use the Ratio test, to check where this power series is convergent (i.e. where the power series is equal to the function of interest).

Further examples of power series representations

Trigonometric functions:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

NOTE: the Ratio Test shows (§3.4) that these series are convergent for any real x.

Important NOTE: x must be measured in radians, not in degrees (otherwise it doesn't work).

Example: use $\cos(\frac{\pi}{3}) = 1 - \frac{(\pi/3)^2}{2!} + \frac{(\pi/3)^4}{4!} + \ldots$ to evaluate $\cos(\frac{\pi}{3})$ to 3 decimal places.

EXAMPLE: Evaluate $\sin(10^o)$ to 5 decimal places.

SOLUTION:

The angle should be represented in radians:

$$10^o = 10^o \frac{\pi}{180^0} = \frac{\pi}{18} \approx 0.1745329252 \text{ radians}.$$

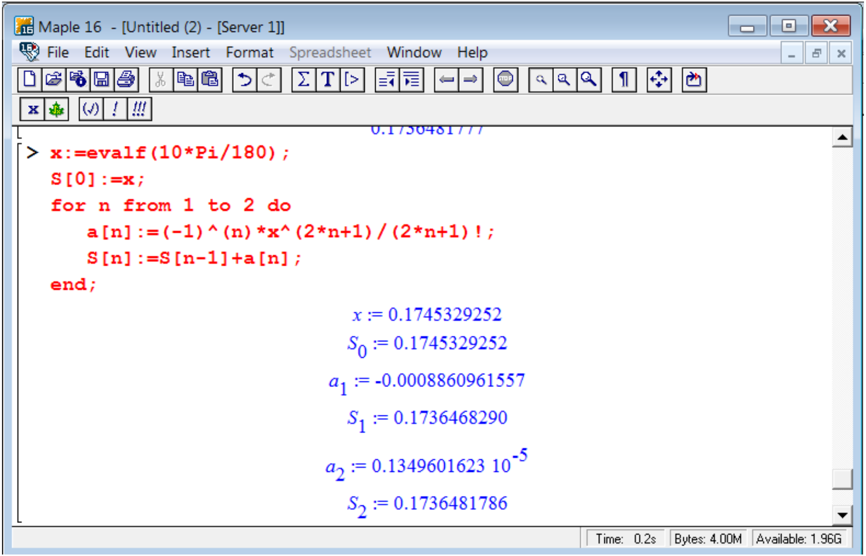
Use the series representation: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$ with $x = 0.1745329252$.

Note: it is convenient to fill in a table of type:

n	a_n									S_n								
0		.									.							
1		.									.							
2		.									.							
3		.									.							
4		.									.							

Answer: 0.17364 (see next page)

Intermediate RESULTS for the previous example using a simple loop:



Hyperbolic Functions:

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

NOTE: the Ratio Test shows (§3.4) that these series are convergent for any real x.

QUESTION (Final Exam 2011):

The hyperbolic cosine $\cosh(x)$ is a function defined by the series:

$$\cosh(x) = \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!}$$

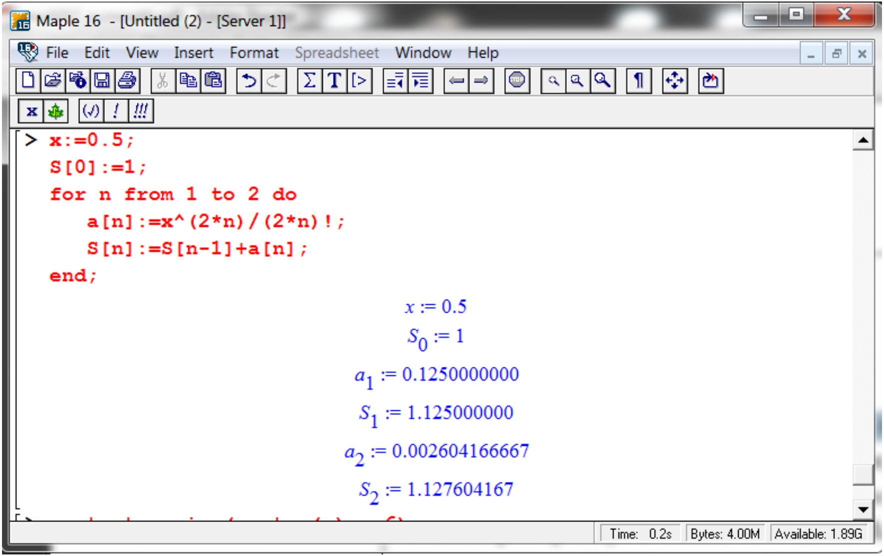
1. Use the ratio test to show that this series is convergent for all x.
2. Use the series to estimate $\cosh(0.5)$ correct to 3 decimal places, writing the partial sums in the following table.

SOLUTION: (1) Ratio Test; (2) The Table:

n	a_n									S_n								
0		.									.							
1		.									.							
2		.									.							
3		.									.							
4		.									.							

Answer: 1.12 (see next page)

Intermediate RESULTS for the previous example using a simple loop:



Log function in base $e \simeq 2.7182818$:

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Inverse Tangent function:

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

NOTE: the Ratio Test shows (§3.4) that these two series are convergent for $|x| < 1$.

How many terms??

QUESTION: How many terms in the power series representation of a function one needs, to get certain accuracy / a certain number of decimal places correct??

- Theoretical Approach:
  There is a theoretical estimate of the accuracy...
  Reference: the Lagrange remainder of the Taylor series...

  ....

- Heuristic Approach:
  If the addition of the next term in the series does NOT change the decimal places of interest, one can STOP.

NOTE: how many terms one needs to compute an approximate value of $f(x)$ depends on the

(i) required accuracy; (ii) function f; (iii) value x.