UNIVERSITY *of* LIMERICK

OLLSCOIL LUIMNIGH

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF MATHEMATICS & STATISTICS

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MA4402	SEMESTER: Autumn 2011
MODULE TITLE: Computer Mathematics	DURATION: 2 hours 30 minutes
LECTURER: Dr. Davide Cellai	GRADING SCHEME:
EXTERNAL EXAMINER: Prof. T. Myers	Examination 80%

INSTRUCTIONS TO CANDIDATES:

Full marks for correct answers to any 5 questions. Answers must be properly justified. All questions carry equal marks. The enclosed formulae sheet and calculators may be used.

Q 1 (a) Consider the function $f : X \subseteq \mathbb{R} \to (-\infty, 0]$, $f(x) = -\sqrt{x}$.

(i) Find the largest domain X in which f is well defined.

(ii) Find the range of the function and say if f is surjective.

(iii) Is f injective?

(iv) Is f bijective?

(b) Consider the function $f : \mathbb{R}^2 \to [0, +\infty)$, $f(x, y) = \sqrt{\frac{1}{2}(x^2 + y^2)}$.

(i) Find the range of the function and say if f is surjective.

(ii) Calculate $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

(iii) Prove that f is not injective by providing a counterexample.

Q 2 Consider the following objects

$$\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

(a) Calculate:

(i) $AB\mathbf{v}$ (ii) $BA\mathbf{v}$ (iii) $B^T A^T$

(b) Calculate the vector obtained by rotating $\mathbf{w}$ CLOCKWISE by $\pi/3$ radians about the origin.

Q 3 (a) Calculate the limits of the following sequences as $n \to +\infty$.

(i) $a_n = 3 + \frac{2}{n}$

(ii) $a_n = \frac{\sqrt{n}+1}{n^2+1}$

(iii) $a_n = \frac{2^n-1}{2^n+1}$

(iv) $a_n = \cos\left(\frac{7}{n}\right)$

(v) $a_n = \sqrt{\frac{2n^2+1}{n^2-n+1}}$

(vi) $a_n = 1 + \frac{(2n)!}{(2n+1)!}$

(b) Let $\{a_n\}_{n=0}^{\infty}$ be the recursively defined sequence where:

$$a_0 = 1, \qquad a_{n+1} = \sqrt{a_n + 6}.$$

Assuming that the sequence is convergent and that $a_n > 0$ for any $n \in \mathbb{N}$, find the limit of $\{a_n\}_{n=0}^{\infty}$ as $n \to +\infty$.

Q 4 The hyperbolic cosine $\cosh(x)$ is a function defined by the series:

$$\cosh(x) = \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!}$$

(a) Use the ratio test to show that this series is convergent for all $x \in \mathbb{R}$.

(b) Use the series to estimate $\cosh(0.5)$ correct to 3 decimal places, writing the partial sums in the following table.

n		a_n		s_n
0		.		.
1		.		.
2		.		.
3		.		.
4		.		.

Q 5 Consider the equation:

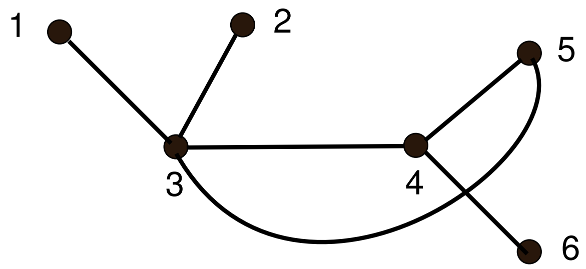
$$e^{x+2} - x^2 + 3x = 0$$

Use the Newton-Raphson method to find the solution, with a precision of 3 decimal places, taking as initial guess $x_0 = 0$.

Q 6 (a) (i) Draw the complete graph K_3 and the bipartite complete graph $K_{2,4}$.

(ii) Draw the graph K_4 as a planar graph and show that it satisfies Euler's formula for planar graphs.

(b) Consider the following graph:



(i) Is the graph simple?

(ii) Is it connected?

(iii) Is it planar?

(iv) Give a cycle in the graph.

(v) Write the adjacency matrix of the graph.

Formulae Sheet

Algebra

- Powers

  $a^p a^q = a^{p+q}$; $\frac{a^p}{a^q} = a^{p-q}$; $(a^p)^q = a^{pq}$; $a^0 = 1$;

  $a^{-p} = \frac{1}{a^p}$; $a^{\frac{1}{q}} = \sqrt[q]{a}$; $a^{\frac{p}{q}} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$; $(ab)^p = a^p b^p$;

  $\left(\frac{a}{b}\right)^p = \frac{a^p}{b^q}$

- Logarithms

  $\log_a(xy) = \log_a x + \log_a y$; $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$; $\log_a(x^q) = q\log_a x$;

  $\log_a(1) = 0$; $\log_a\left(\frac{1}{x}\right) = -\log_a x$; $a^x = y \Leftrightarrow \log_a(y) = x$;

  $\log_a a = 1$; $a^{\log_a x} = x$; $\log_a x = \frac{\log_b x}{\log_b a}$

- Roots of the quadratic equation $ax^2 + bx + c = 0$:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- Definition of absolute value of x:

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

Trigonometry

$$\sin^2\theta + \cos^2\theta = 1, \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta},$$

$$\sin(-\theta) = -\sin\theta, \qquad \cos(-\theta) = \cos\theta, \qquad \tan(-\theta) = -\tan\theta$$

$\tan\theta$ is not defined when $\cos\theta = 0$.
$\cot\theta$ is not defined when $\sin\theta = 0$.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\sin\theta$	0	1	0	-1	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	1	0	-1	0	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	0	-	0	-	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Compound angle formulae:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

Linear algebra

Vector: $\mathbf{v} = (v_1, \ldots, v_n)$. Norm:

$$||\mathbf{v}|| = \sqrt{\sum_{i=1}^{n} v_i^2}$$

Dot product:

$$\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^{n} v_i w_i = ||\mathbf{v}||\,||\mathbf{w}|| \cos\theta$$

Determinant of a 2×2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \det A = ad - bc$$

Derivatives

- Product rule:

$$h(x) = f(x)g(x) \quad \Rightarrow \quad h'(x) = f'(x)g(x) + f(x)g'(x)$$

- Quotient rule:

$$h(x) = \frac{N(x)}{D(x)} \quad \Rightarrow \quad h'(x) = \frac{N'(x)D(x) - N(x)D'(x)}{[D(x)]^2}$$

- Chain rule:

$$h(x) = f(g(x)) \quad \Rightarrow \quad h'(x) = f'(g(x))g'(x)$$

- Derivatives of simple functions

$f(x)$	$f'(x)$
x^p	px^{p-1}
$\ln x$	$\frac{1}{x}$
e^x	e^x
e^{ax}	ae^{ax}
a^x	$a^x \ln a$
$\cos x$	$-\sin x$
$\sin x$	$\cos x$
$\tan x$	$\frac{1}{\cos^2 x}$
$\cos^{-1} \frac{x}{a}$	$-\frac{1}{\sqrt{a^2-x^2}}$
$\sin^{-1} \frac{x}{a}$	$\frac{1}{\sqrt{a^2-x^2}}$
$\tan^{-1} \frac{x}{a}$	$\frac{a}{a^2+x^2}$

Newton-Raphson iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$